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ELLIPTIC EQUATIONS AND SYSTEMS WITH CRITICAL TRUDINGER-MOSER NONLINEARITIES

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Dedicated to Louis Nirenberg on the occasion of his 85th birthday

Abstract. In this article we give first a survey on recent results on some Trudinger-Moser type inequalities, and their importance in the study of nonlinear elliptic equations with nonlinearities which have critical growth in the sense of Trudinger-Moser. Furthermore, recent results concerning systems of such equations will be discussed.

1. **Introduction.** Elliptic equations and systems with critical growth nonlinearities have been widely studied in recent years. In dimension $N \geq 3$ the critical growth is given by the Sobolev embeddings. While equations with subcritical growth are solved by standard variational methods, equations with critical growth need more specific methods due to the loss of compactness. Indeed, the existence results become very subtle: the equation

$$-\Delta u = u^{\frac{N+2}{N-2}} \text{ in } \Omega , \quad u|_{\partial\Omega} = 0 , \ \Omega \text{ starshaped} ,$$

has no nontrivial solution due to Pohozaev's identity [50], while the perturbed equation

$$-\Delta u = \lambda u + u^{\frac{N+2}{N-2}} \text{ in } \Omega , \quad u|_{\partial\Omega} = 0 \tag{1.1}$$

has positive solutions for $0 < \lambda < \lambda_1$ ($N \geq 4$) due to the famous result by Brezis-Nirenberg [13]. While the situation in $N \geq 3$ is by now well-understood [13, 22], the case $N = 2$ is quite different, and there are less results available. The critical growth is of exponential type, and is governed by the Trudinger-Moser inequality.

In this paper we give an overview of recent results concerning Trudinger-Moser type inequalities and on related equations and systems of equations with critical Trudinger-Moser nonlinearities in domains in $\mathbb{R}^2$.

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2. **Trudinger-Moser type inequalities.**

2.1. **The Trudinger-Moser inequality.**

Let $\Omega \subset \mathbb{R}^N$ be a smooth and bounded domain. The well-known Sobolev embedding theorems say that for $p < N$

$$W_0^{1,p}(\Omega) \subset L^q(\Omega)$$

if

$$1 \le q \le \frac{pN}{N-p} \ ,$$

where $W_0^{1,p}(\Omega)$ is the standard Sobolev space of L^p-functions whose weak derivatives belong also to L^p. The Trudinger-Moser inequalities concern the borderline cases

$$p = N \ .$$

Indeed, since (formally) $\frac{pN}{N-p} \sim +\infty$, one may ask if in this case the embedding of $W_0^{1,N}$ goes into L^∞. This is not the case, as simple examples show. So, it was expected that there must be a maximal growth function $g(t)$ such that $u \in W^{1,N}(\Omega)$ implies that $\int_\Omega g(u)dx < \infty$. Indeed, S. Pohozaev [49] and N. Trudinger [60] showed that this maximal growth is of exponential type. More precisely, let $p = N$, then

$$u \in W_0^{1,N}(\Omega) \implies \int_\Omega e^{|u|^{N'}} \, \mathrm{d}x < \infty \ ,$$

where

$$N' = \frac{N}{N-1} \ .$$

The inequalities are sharp in the sense that for any higher growth one finds functions for which the corresponding integral becomes infinite. The proofs of Pohozaev and Trudinger use the same idea, namely developing the exponential in a power series, which reduces the problem to show that a series of L^m-norms ($m \in \mathbb{N}$) remains finite. By controlling the embedding constants of $W^{1,N} \subset L^m$ one obtains the result.

The above assertions were sharpened by J. Moser. He proved

Theorem 2.1. (see [48])
Denote by $\|\nabla u\|_N$ the (equivalent to the standard) norm in $W_0^{1,N}(\Omega)$. Then

$$\sup_{\|\nabla u\|_N \le 1} \int_\Omega e^{\alpha\, |u|^{N'}} \, \mathrm{d}x \quad \begin{cases} \le c\,|\Omega| \ , & \text{if } \ \alpha \le \alpha_N \\ = \infty \ , & \text{if } \ \alpha > \alpha_N \end{cases} \ , \tag{2.2}$$

where $\alpha_N = N\omega_{N-1}^{1/(N-1)}$, and ω_{N-1} is the measure of the unit sphere in $\mathbb{R}^N$.

Inequality (2.2) is now called the *Trudinger-Moser inequality.*

Proof. (Sketch) To prove this result, J. Moser used *symmetrization*; that is, to every function u is associated a radially symmetric function u^* such that the sublevel-sets of u^* are balls with the same area as the corresponding sublevel-sets of u, i.e. $|\{x \in \mathbb{R}^N \ : u^*(x) < d\}| = |\{x \in \Omega \ : u(x) < d\}|$, where $|A|$ denotes the Lebesgue measure of the set A. Then u^* is a positive and non-increasing function defined on $B_R(0)$ with $|B_R| = |\Omega|$. By construction, one has the following property:

Let $f \in C(\mathbb{R})$, then

$$\int_{B_R} f(u^*) \, \mathrm{d}x = \int_\Omega f(u) \, \mathrm{d}x \ ;$$

furthermore, the well-known Pólya-Szegö inequality asserts that

$$\int_{B_R} |\nabla u^*|^N \, \mathrm{d}x \le \int_\Omega |\nabla u|^N \, \mathrm{d}x$$

From this we clearly deduce that

$$\sup_{\|\nabla u\|_N \le 1} \int_\Omega e^{\alpha |u|^{N'}} \, \mathrm{d}x \le \sup_{\|\nabla u^*\|_N \le 1} \int_{B_R} e^{\alpha |u^*|^{N'}} \, \mathrm{d}x \, ,$$

and hence it is sufficient to consider the radial case.

Next, we perform a change of variables: set

$$r = |x| = Re^{-t/N} \quad \text{and} \quad w(t) = N^{\frac{N-1}{N}} \omega_{N-1}^{\frac{1}{N}} u^*(r) \, .$$

One checks that

$$\int_{B_R} |\nabla u^*|^N \, \mathrm{d}x = \int_0^\infty |w'(t)|^N \, \mathrm{d}t \, , \quad \int_{B_R} e^{\alpha |u^*|^{N'}} \, \mathrm{d}x = |B_R| \int_0^\infty e^{\frac{\alpha}{\alpha_N} |w(t)|^{N'} - t} \, \mathrm{d}t$$

Thus, we have reduced the problem to

$$\sup_{\int_0^\infty |w'|^N \, \mathrm{d}t \le 1} |B_R| \int_0^\infty e^{\frac{\alpha}{\alpha_N} |w(t)|^{N'} - t} \, \mathrm{d}t \, . \tag{2.3}$$

Assume now that $\alpha < \alpha_N$; assuming that $w \in C^1$ we have

$$w(t) = \int_0^t w'(s) \, \mathrm{d}s \le t^{1/N'} \left(\int_0^t |w'(t)|^N \, \mathrm{d}t \right)^{1/N} \le t^{1/N'} \, ,$$

by assumption. Inserting this in (2.3) we find

$$\int_0^\infty e^{\frac{\alpha}{\alpha_N} |w(t)|^{N'} - t} \, \mathrm{d}t \le \int_0^\infty e^{\frac{\alpha}{\alpha_N} t - t} \, \mathrm{d}t < \infty \, .$$

The case $\alpha = \alpha_N$ is more delicate, we refer to [48].

Finally, we show that the exponent α_N is optimal. Indeed, suppose that $\alpha > \alpha_N$, and define the so-called Moser-sequence

$$w_n(t) := \begin{cases} \frac{t}{n^{1/N}} \, , & 0 \le t \le n \\ n^{1/N'} \, , & n \le t \end{cases}$$

Then clearly $\int_0^\infty |w_n'(t)|^N \, \mathrm{d}t = 1$, and

$$\int_0^\infty e^{\frac{\alpha}{\alpha_N} |w_n|^{N'} - t} \, \mathrm{d}t \ge \int_n^\infty e^{\frac{\alpha}{\alpha_N} n - t} \, \mathrm{d}t = e^{(\frac{\alpha}{\alpha_N} - 1)n} \to \infty$$

□

2.2. **Best constant attained.** It is well-known that in the case of the Sobolev-embedding $W_0^{1,2}(\Omega) \subset L^{2^*}(\Omega)$, the best embedding constant

$$S_N^{2^*} = \sup_{\|\nabla u\|_2 = 1} \int_\Omega |u|^{2^*} \, \mathrm{d}x \tag{2.4}$$

is not attained if $\Omega \ne \mathbb{R}^N$. Note that (2.4) is in an analogous form to (2.2).

By contrast, in the situation of Trudinger-Moser (2.2), one has the following surprising result by L. Carleson and A. Chang.

Theorem 2.2. (see [16])
If $\Omega \subset \mathbb{R}^N$ is the ball $B_1(0)$, then the supremum in (2.2) is achieved when $\alpha \le \alpha_N$.

Proof. (Idea of the proof in the case $N = 2$) By symmetrization one may restrict to radially symmetric functions. Carleson-Chang proceed as follows: Assuming that the supremum in (2.2) is not attained, they conclude that any maximizing sequence must concentrate in the origin. Using this information, they then succeed to determine the limit value of the integral

$$\lim_{n\to\infty} \int_{B_1(0)} e^{4\pi u_n^2}\, \mathrm{d}x = (1+e)\,|B_1|$$

along any concentrating maximizing sequence. Thus, this level represents a *non-compactness level*. Finally, they exhibit an explicit function for which the integral (2.2) is above this value, thus reaching a contradiction. □

M. Struwe [57] showed that this result continues to hold for small perturbations of the ball in $\mathbb{R}^2$, and M. Flucher [35] extended the result to any bounded domain in $\mathbb{R}^2$. Lin [45] generalized the result to any dimension.

In [26] an explicit concentrating sequence was constructed, along which the integral (2.2) converges to the non-compactness level found by Carleson-Chang. Furthermore, it was shown (by an asymptotic analysis) that along this sequence the integral (2.2) converges from above to this value. This gives a new proof of the result of Carleson-Chang, and it also shows that that there is a strong analogy with the famous result of Brezis-Nirenberg [13] concerning perturbations of the Sobolev case.

We remark that the following differential equation is associated to the problem (2.2) in the case $N = 2$:

$$\begin{cases} -\Delta u &= \lambda u e^{u^2}\,, \qquad \text{in } \ \Omega \\ \quad\ u &= 0\,, \qquad\quad \text{on } \ \partial\Omega \end{cases}, \quad \text{where } \ \lambda = \frac{1}{\int_\Omega u^2 e^{u^2}\,\mathrm{d}x}\,, \tag{2.5}$$

and the fact that the supremum in (2.2) is attained yields a (positive) solution to this equation.

2.3. **Higher order Sobolev spaces: The inequalities of D.R. Adams.** The Trudinger-Moser inequality was generalized to higher order Sobolev spaces by D.R. Adams. Consider $W_0^{k,\frac{N}{k}}(\Omega)$, with $\frac{N}{k} > 1$ where

$$W_0^{k,\frac{N}{k}}(\Omega) = cl\Big\{u \in C_0^\infty(\Omega) \ : \ \|u\|_{W^{k,\frac{N}{k}}} < \infty\Big\}$$

Note that for

$$\left.\begin{aligned} &k \text{ even}: \ \|\nabla^k u\|_{L^{\frac{N}{k}}} := \|\Delta^{\frac{k}{2}} u\|_{L^{\frac{N}{k}}} \\ &k \text{ odd}: \ \|\nabla^k u\|_{L^{\frac{N}{k}}} := \|\nabla \Delta^{\frac{k-1}{2}} u\|_{L^{\frac{N}{k}}} \end{aligned}\right\} \text{equivalent norms on } \ W_0^{k,\frac{N}{k}}(\Omega)$$

Then one has (see D.R. Adams [2]):

Theorem 2.3.

$$\sup_{\|\nabla^k u\|_{L^{\frac{N}{k}}} \le 1} \int_\Omega e^{\beta\,|u|^{\frac{N}{N-k}}}\,\mathrm{d}x \begin{cases} \le c, & \textit{if } \ \beta \le \beta_{k,N} \\ = +\infty\,, & \textit{if } \ \beta > \beta_{k,N} \end{cases} \tag{2.6}$$

where $\beta_{k,N}$ are explicit.

Proof. (Idea) We recall that J. Moser used symmetrization for proving his result (2.2), thereby reducing the problem to the inequality (2.3) for functions of one variable. This inequality is equivalent to the following *one-dimensional calculus inequality*:

For any measurable function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\int_0^\infty (\phi(t))^N \, \mathrm{d}t \le 1$$

holds

$$\int_0^\infty e^{-F(t)} \, \mathrm{d}t \le c_0 \ , \quad \text{where} \ \ F(t) = t - \Big(\int_0^t \phi(s) \, \mathrm{d}s \Big)^{N/(N-1)} \tag{2.7}$$

For the extension to higher order derivatives, the method of symmetrization is not available. But working with Riesz potentials, D.R. Adams [2] was again able to reduce the problem to a one-dimensional calculus inequality, namely to

Adams' inequality: *Let $a : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$ be a measurable function such that*

$$a(s,t) \le 1, \ \textit{if} \ \ 0 < s < t, \quad \textit{and} \ \ \sup_{t>0} \Big(\int_{-\infty}^0 + \int_t^\infty a(s,t)^{p'} \, \mathrm{d}s \Big)^{1/p'} = b < \infty$$

Then there exists a constant $c_0(p,b)$ such that for $\phi : \mathbb{R} \to \mathbb{R}^+$ satisfying

$$\int_{-\infty}^\infty \phi(s)^p \, \mathrm{d}s \le 1$$

holds

$$\int_0^\infty e^{-F(t)} \, \mathrm{d}t \le c_0 \ , \quad \textit{where} \ \ F(t) = t - \Big(\int_{-\infty}^\infty a(s,t)\phi(s) \, \mathrm{d}s \Big)^{p'} \tag{2.8}$$

Notice that the above one-dimensional inequality of J. Moser corresponds to the case

$$a(s,t) = 1 \ , \quad \text{if} \ \ 0 < s < t \ ; a(s,t) = 0 \ \ \text{otherwise}$$

in Adams' inequality. The inequalities (2.6) follow from Adams' inequality in much the same way as the Moser result from the above simpler calculus inequality (2.7). □

3. **Unbounded domains.** From (2.2) one sees that these inequalities are valid only for bounded domains, and therefore the Trudinger-Moser inequality is not available for unbounded domains. Related inequalities for unbounded domains have been first considered by Cao [5] in the case $N = 2$ and for any dimension by J.M. do Ó [32] and Adachi-Tanaka [1]. All these results have been proved in the case of the unit ball and the supremum taken with respect to the Dirichlet norm $\int_\Omega |\nabla u|^N \, \mathrm{d}x$, and assuming in some sense a subcritical growth: $e^{\alpha |u|^{N/(N-1)}}$ with $\alpha < \alpha_N$. In [53] it was shown that in the case $N = 2$ the result of J. Moser (2.2) can be fully extended to unbounded domains (and thus to all of $\mathbb{R}^2$) if the Dirichlet norm $\int_\Omega |\nabla u|^2 \, \mathrm{d}x$ is replaced by the full H^1-norm $\|u\|_1^2 = \int_\Omega (|\nabla u|^2 + |u|^2) \, \mathrm{d}x$. More precisely, one has:

Theorem 3.1. *Let Ω be a domain in $\mathbb{R}^2$, and consider the Sobolev space $H_0^1(\Omega)$ with $\|u\|_1 = \left(\int_\Omega (|\nabla u|^2 + |u|^2)\,\mathrm{d}x\right)^{1/2}$. Then there is a constant d independent of Ω such that*

$$\sup_{\|u\|_1 \le 1} \int_\Omega \left(e^{4\pi u^2} - 1\right)\,\mathrm{d}x \le d \tag{3.9}$$

Proof. It is sufficient to consider the case $\Omega = \mathbb{R}^2$. By symmetrization we can reduce to the radial case, and thus we may assume that u is radial and non-increasing. Dividing the integral into two parts

$$\int_{\mathbb{R}^2} \left(e^{4\pi u^2} - 1\right)\,\mathrm{d}x = \int_{|x|\le r_0} \left(e^{4\pi u^2} - 1\right)\,\mathrm{d}x + \int_{|x|\ge r_0} \left(e^{4\pi u^2} - 1\right)\,\mathrm{d}x\ ,$$

one writes the second integral as a series

$$\int_{|x|\ge r_0} \left(e^{4\pi u^2} - 1\right)\,\mathrm{d}x = \sum_{k=1}^{\infty} \int_{|x|\ge r_0} \frac{(4\pi)^k\,|u|^{2k}}{k!}\,\mathrm{d}x\ . \tag{3.10}$$

Using a "radial lemma" (see [12, 56]) one estimates

$$|u(r)| \le \frac{1}{\sqrt{\pi}}\ \|u\|_{L^2}\frac{1}{r}\ ,\quad \text{for all } r > 0\ ;$$

from this one obtains easily that the series in (3.10) converges for r_0 sufficiently large, and hence

$$\int_{|x|\ge r_0} \left(e^{4\pi u^2} - 1\right)\,\mathrm{d}x \le c(r_0)$$

For the first integral, once writes $u(r) = u(r) - u(r_0) + u(r_0) =: v(r) + u(r_0)$ and estimates

$$u(r) \le v(r)\left(1 + \frac{1}{\pi r_0^2}\|u\|_{L^2}^2\right)^{1/2} + d(r_0) =: w(r) + d(r_0)\ , \tag{3.11}$$

where we have used again the radial lemma. Hence

$$\int_{B_{r_0}} e^{4\pi u^2}\,\mathrm{d}x \le c\int_{B_{r_0}} e^{4\pi w^2}\,\mathrm{d}x \le d$$

by Moser's inequality, since $w \in H_0^1(B_{r_0})$ with

$$\begin{aligned} &\int_{B_{r_0}} |\nabla w|^2\,\mathrm{d}x \\ =\ &\int_{B_{r_0}} |\nabla v|^2\,\mathrm{d}x\left(1 + \frac{1}{\pi r_0^2}\|u\|_{L^2}^2\right) = \left(1 + \frac{1}{\pi r_0^2}\|u\|_{L^2}^2\right)\int_{B_{r_0}} |\nabla u|^2\,\mathrm{d}x \\ \le\ &\left(1 + \tfrac{1}{\pi r_0^2}\|u\|_{L^2}^2\right)\left(1 - \|u\|_{L^2}^2\right) \le 1\ , \end{aligned}$$

provided that $\pi r_0^2 \ge 1$. □

Again, one can ask the question whether the supremum in (3.9) is attained.

Theorem 3.2. *(see [53])*

For $\Omega = B_R(0)$, the ball of radius R, and for $\Omega = \mathbb{R}^2$ the supremum in (3.1) is attained.

Proof. (Idea) As in the proof of Carleson and Chang, the proof relies on the determination of the limit of $\int_{B_R(0)} \left(e^{4\pi u^2} - 1\right) \mathrm{d}x$ along a concentrating maximizing sequence. Indeed, one shows that for any concentrating maximizing sequence (u_n)

$$\lim_{n\to\infty} \int_{B_R(0)} \left(e^{4\pi u_n^2} - 1\right) \mathrm{d}x = \pi e^{1-D(R)}$$

where

$$D(R) = 2K_0(R)\left(2RK_1(R) - \frac{1}{I_0(R)}\right) \ ;$$

here K_0, K_1, I_0 are modified Bessel functions, i.e. solutions of (for $k = 0, 1$)

$$-x^2u''(x) - xu'(x) + (x^2 + k^2)u(x) = 0$$

One shows that

$$D(R) \sim -2\log R \ , \quad \text{for} \quad R \ \text{ small}$$

$$D(R) \sim \tfrac{1}{Re^R} \ , \quad \text{for} \quad R \ \text{ large}$$

and hence one obtains for $\Omega = \mathbb{R}^2$ that for concentrating maximizing sequences

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} \left(e^{4\pi u_n^2} - 1\right) = \pi e$$

Finally, one shows by constructing an explicit concentrating maximizing sequence that the convergence is from above,i.e. there exists n_0 such that for all $n \geq n_0$

$$\int_{B_R(0)} \left(e^{4\pi u_n^2} - 1\right) > \pi e^{1-D(R)}$$

and hence

$$\sup_{\|\nabla u\|_2^2 + \|u\|_2^2 \leq 1} \int_{B_R(0)} \left(e^{4\pi u^2} - 1\right) > \pi e^{1-D(R)}$$

From this it follows immediately that the supremum in (3.9) is attained. □

4. **Generalized Trudinger-Moser inequalities.** Numerous generalizations, extensions and applications of the Trudinger-Moser (TM) inequality have been given in recent years:

TM-type inequalities involving higher order derivatives were given by D.R. Adams [2]. For extensions of the TM-inequality to manifolds, see P. Cherrier [18], L. Fontana [36], Y. Li [43, 44], Y. Yang [61].

Recently, Adimurthi - O. Druet [5] have given an improved TM-inequality with remainder term; they proved the following result

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, and set*

$$C_\alpha = \sup_{u\in H_0^1(\Omega), \|\nabla u\|_2 \leq 1} \int_\Omega e^{4\pi u^2(1+\alpha)\|u\|^2} \, \mathrm{d}x$$

Let λ_1 denote the first eigenvalue of the Laplacian in $H_0^1(\Omega)$. Then

$$C_\alpha < \infty \ , \quad \textit{if} \ \ \alpha < \lambda_1$$

$$C_\alpha = +\infty \ , \quad \textit{if} \ \ \alpha \geq \lambda_1$$

The sharpness is obtained by suitably modified Moser-type sequences, while the proof of the convergence is inspired by the following *concentration-compactness* result by P.-L. Lions [46]:

Theorem 4.2. *Let $(u_\epsilon)_{\epsilon>0}$ be a sequence of functions in $H_0^1(\Omega)$ with $\|\nabla u_\epsilon\| = 1$ such that $u_\epsilon \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$. For any $p < 1/(1-\|u_0\|_2^2)$,*

$$\limsup_{\epsilon\to 0} \int_\Omega e^{4\pi\, p\, u^2}\, \mathrm{d}x < +\infty$$

This result gives more precise information than the Trudinger-Moser inequality (2.2) in the case when $u_\epsilon \rightharpoonup u_0$ in $H_0^1(\Omega)$ with $u_0 \neq 0$.

Adimurthi-Druet extend the result of Lions, giving extra information even when $u_\epsilon \rightharpoonup 0$ in $H_0^1(\Omega)$. They obtain this extra information by doing a careful blow-up analysis of sequences of solutions to approximate elliptic equations with near critical Trudinger-Moser growth.

We also mention that recently TM-inequalities with other boundary data, and Trudinger-Moser trace inequalities have been obtained, see A. Cianchi [20], [21].

Finally, we mention TM-type inequalities in other function spaces, in particular in Orlicz spaces, Zygmund spaces, Lorentz spaces, Besov spaces etc., see e.g. A. Cianchi [19], N. Fusco - P.-L. Lions - C. Sbordone [37], A. Alvino - V. Ferone - G. Trombetti [7], D.E. Edmunds - P. Gurka - B. Opic [34], S. Hencl [40], H. Brezis - S. Wainger [14].

In particular, we recall here some recent results for embeddings of Lorentz-Sobolev spaces into Orlicz spaces and the related TM-inequalities.

We first recall the definition of Lorentz spaces:

4.1. **Sobolev-Lorentz spaces.** Lorentz spaces $L^{p,q}$ are scales of interpolation spaces between the Lebesgues spaces L^p, and are obtained via spherically decreasing rearrangement: for a measurable function $u : \Omega \to \mathbb{R}$ let $u^*(s)$ denote its *decreasing rearrangement*. Then the function u belongs to the Lorentz space $L^{p,q}(\Omega)$ if

$$\|u\|_{p,q} = \left(\int_0^\infty [u^*(t)\, t^{1/p}]^q\, \frac{dt}{t}\right)^{1/q} < +\infty\ .$$

We refer to [3] for the precise definitions; we recall here only that, for $\Omega \subset \mathbb{R}^N$ of finite measure,

$$\begin{aligned} &L^{p,p} = L^p\ ,\ L^{p,q_1} \subset L^{p,q_2}\ , && \text{if}\ \ q_1 < q_2\ , \\ &L^r \subset L^{p,q} \subset L^s\ , && \text{if}\ \ 1 < s < p < r\ ,\ \ \text{for all}\ \ 1 \le q \le \infty \end{aligned}$$

We denote the norm in $L^{p,q}$ by $\|u\|_{p,q}$. The following Hölder inequality holds:

$$\left|\int_\Omega fg\, \mathrm{d}x\right| \le \|f\|_{p,q}\|g\|_{p',q'}\ ,\ \ \text{where}\ \ p' = \frac{p}{p-1}, q' = \frac{q}{q-1} \tag{4.12}$$

First, we recall that the standard Sobolev embeddings can be sharpened by the use of Lorentz spaces, see e.g. [8]. Denoting

$$W_0^1 L^{p,q}(\Omega) = cl\big\{u \in C_0^\infty(\Omega)\ :\ \|\nabla u\|_{p,q} < \infty\big\}\ ,\ \Omega \subset \mathbb{R}^N\ \ \text{bounded}$$

one has the following

Sobolev-Lorentz embedding: Suppose that $1 \le p < N$; then

$$W_0^1 L^{p,q} \subset L^{p^*,q} \ , \quad \text{where} \quad p^* = \frac{pN}{N-p} \ ,$$

and hence in particular, since $p < p^*$

$$W_0^{1,p} = W_0^1 L^{p,p} \subset L^{p^*,p} \underset{\neq}{\subset} L^{p^*,p^*} = L^{p^*} \ .$$

For the limiting case $p = N$, once has the following important refinement of the Trudinger embedding, see Brezis-Wainger [14] and A. Alvino, V. Ferone and G. Trombetti [7]:

Theorem 4.3. *If* $u \in W_0^1 L^{2,q}(\Omega)$, *then* $e^{|u|^{q'}} \in L^1(\Omega)$, *where* $q' = \frac{q}{q-1}$, *and following corresponding Moser-type inequality holds:*

Brezis-Wainger inequality: *There exist numbers* $\beta_q > 0$ *such that*

$$\sup_{\{\|\nabla u\|_{N,q} \le 1\}} \int_\Omega e^{\beta |u(x)|^{\frac{q}{q-1}}} \, dx \ \begin{cases} \le C(N,q)|\,\Omega| \ , & \textit{for} \ \ \beta \le \beta_q \\ = +\infty \ , & \textit{for} \ \ \beta > \beta_q \end{cases} \tag{4.13}$$

The Trudinger-Moser inequality corresponds to the case $W_0^{1,N}(\Omega) = W_0^1 L^{N,N}(\Omega)$. It is remarkable that in (4.13) the exponent depends only on the second index q of the Lorentz space, and is independent of N.

Note that the inequalities (2.2) and (4.13) are sharp with respect to the coefficients α, resp. β, in the exponents. In fact, considering for simplicity the inequality (2.2) in the case $N = 2$, one notes that if $\alpha = \alpha_2 = 4\pi$, then any unbounded lower order perturbation $f(s)$ in the exponent (i.e. $f(s)$ with $\lim_{|s|\to\infty} f(s) = +\infty$ and $\lim_{s\to\infty} \frac{f(s)}{s^2} = 0$) will yield

$$\sup_{\|\nabla u\|_2 \le 1} \int_\Omega e^{4\pi |u(x)|^2 + f(u(x))} \, dx = +\infty \ .$$

In [54] the TM-inequality (2.2) and the more general Brezis-Wainger inequality (4.13) were generalized with regard to such lower order perturbations. More precisely, concerning inequality (2.2) (with $N = 2$) it was asked: in the limiting case $\alpha = \alpha_2 = 4\pi$, and given an unbounded lower order perturbation function $f(s)$, can we characterize a largest space $\Lambda(g)$ of Lorentz type such that

$$\sup_{\|\nabla u\|_{\Lambda(g)} \le 1} \int_\Omega e^{4\pi |u(x)|^2 + f(u(x))} \, dx < +\infty \ . \tag{4.14}$$

Thus, we reverse the question: given an unbounded perturbation of the Trudinger-Moser nonlinearity, what integrability condition must be imposed on the gradient of u in order to have a bounded integral of u with respect to this perturbed TM-nonlinearity.

This is a subtle question: note that if we replace in (2.2) the condition $\|\nabla u\|_2 \le 1$ by $\|\nabla u\|_2 \le 1-\delta$, for an arbitrary $\delta > 0$, then $\sup_{\{\|\nabla u\|_2 \le 1-\delta\}} \int_\Omega e^{4\pi (\frac{1}{1-\delta}|u(x)|)^2} \, dx \le c$, and hence for any subquadratic perturbation $f(u)$ we get

$$\sup_{\{\|\nabla u\|_2 \le 1-\delta\}} \int_\Omega e^{4\pi |u(x)|^2 + f(u(x))} \, dx \le c \ .$$

The adequate class of Lorentz spaces for this problem are *weighted Lorentz spaces*, which were proposed by G.G. Lorentz [47] already in his original paper "On the Theory of Spaces". Weighted Lorentz spaces are defined as follows: Let $\phi : \Omega \to \mathbb{R}^+$ be a measurable function, and let $\phi^*(s)$ denote its decreasing rearrangement. Furthermore, let $w(t) : \mathbb{R} \to \mathbb{R}^+$ a nonnegative integrable function, such that $\int_0^t w(s)ds < +\infty$ for all $t > 0$. The weighted Lorentz space $\Lambda_p(w)$ is defined as follows: $\phi \in \Lambda^p(w)$, $1 \le p < +\infty$, if

$$\|\phi\|_{\Lambda_p(w)} = \left(\int_0^{+\infty} (\phi^*(t))^p \, w(t) \, \mathrm{d}t \right)^{1/p} < +\infty. \tag{4.15}$$

Surprisingly, one can establish a precise relation between a weight $w(s)$ and the corresponding lower order perturbation function $f(u)$ to obtain sharp TM-type inequalities. To formulate the result, let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function (the "weight function") such that

(H_1) $\lim_{t \to +\infty} \varphi(t) = 0$

(H_2) $\int_0^{+\infty} \varphi(t) = +\infty$

(H_3) $\varphi(t)$ is non increasing as $t \to +\infty$

Then one has the following *optimal* Moser type inequality:

Theorem 4.4. *Let Ω be an open subset of $\mathbb{R}^N$, of finite measure, and let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying (H_1)–(H_3). Let $f(t) \in \mathcal{C}^1(\mathbb{R}^+)$ be defined by*

$$f(t) = \int_0^{\alpha_N t^{\frac{N}{N-1}}} \frac{\varphi(s)}{1+\varphi(s)} \, \mathrm{d}s \tag{4.16}$$

where $\alpha_N = N\omega_{N-1}^{1/(N-1)}$ and ω_{N-1} denotes the $(N-1)$-dimensional surface of the unit ball in $\mathbb{R}^N$, $N \ge 2$. Then

$$\sup_{\{u \in C_0^1(\Omega), \|\nabla u\|_{\Lambda_N,\varphi} \le 1\}} \int_\Omega e^{\alpha_N |u|^{\frac{N}{N-1}} + f(u)} \, \mathrm{d}x \le C\,|\Omega| \ , \tag{4.17}$$

where

$$\|v\|^N_{\Lambda_{N,\varphi}} = \int_0^{+\infty} \big(v^*(s)\big)^N \Big\{ 1 + \varphi\Big(\big| \log\big(\frac{s}{|\Omega|}\big) \big| \Big) \Big\}^{N-1} \mathrm{d}s.$$

and $C = C(\|\varphi\|_\infty)$ is a positive constant that depends only on $\|\varphi\|_\infty$.

Furthermore, the result is sharp.

Examples:

1) Let $\varphi_1(s) = \frac{1}{2\sqrt{4\pi\,(s+1)} - 1}$, then $f(s) = s$, i.e.

$$\sup_{\|\nabla u\|_{\Lambda_{2,\varphi_1}} \le 1} \int_\Omega e^{4\pi u^2 + u} \, \mathrm{d}x \le C\,|\Omega|$$

2) Let $\varphi_2(s) = \frac{\sqrt{\pi}\, p}{s + 4\pi\sqrt{s+p}}$, then $f(s) = p\log(1+|u|)$, i.e.

$$\sup_{\|\nabla u\|_{\Lambda_{2,\varphi_2}} \le 1} \int_\Omega (1+|u|)^p \, e^{4\pi u^2} \, \mathrm{d}x \le C\,|\Omega|$$

Proof. (Idea) The proof of Theorem 4.4 relies on a generalization of Adams' inequality (2.8)

Lemma 4.1. *Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function satisfying hypotheses $(H1), (H2)$, and let $f(t)$ be defined by (4.16). Let $a(s,t)$ be a non-negative measurable function on $\mathbb{R} \times [0, +\infty)$ such that*

$$a(s,t) \le 1, \qquad \text{for a.e.} \qquad 0 < s < t \tag{4.18}$$

$$\sup_{t>0} \left(\int_{-\infty}^{0} + \int_{t}^{+\infty} \frac{a^{\frac{N}{N-1}}(s,t)}{1+\varphi(s)} \, \mathrm{d}s \right)^{\frac{N-1}{N}} = \gamma < \infty \tag{4.19}$$

Then there exists a constant $c_0 = c_0(\|\varphi\|_\infty, \gamma)$ such that for $\phi \ge 0$ with

$$\int_{-\infty}^{+\infty} \phi^N(s) \left(1+\varphi(s)\right)^{N-1} \, \mathrm{d}s \le 1 \tag{4.20}$$

one has

$$\int_0^{+\infty} e^{-\Psi(t)} \, \mathrm{d}t \le c_0 \ , \tag{4.21}$$

where

$$\Psi(t) = t - \left\{ \left(\int_{-\infty}^{+\infty} a(s,t)\phi(s) \, \mathrm{d}s \right)^{\frac{N}{N-1}} + f\left(\frac{1}{\alpha_N^{\frac{N-1}{N}}} \int_{-\infty}^{+\infty} a(s,t)\phi(s) \, \mathrm{d}s \right) \right\} \tag{4.22}$$

Note that for $\varphi(s) \equiv 0$ we have $f(t) \equiv 0$, and hence $\Psi(t) = F(t)$ in Adams' inequality.

Again, Theorem 4.4 follows directly from the above inequality. □

5. **Equations with critical Trudinger-Moser growth.** Instead of equation (2.5) one can consider the related (but not equivalent) equation

$$\begin{cases} -\Delta u &= \lambda u e^{u^2} \ , \quad \text{in } \ \Omega \\ \quad u &= 0 \ , \qquad \text{on } \ \partial\Omega \end{cases} \tag{5.23}$$

where $\lambda > 0$ is now a free parameter. This equation is the Euler-Lagrange equation to the free functional

$$\begin{aligned} &J \ : \ H_0^1(\Omega) \to \mathbb{R} \\ &J_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, \mathrm{d}x - \lambda \int_\Omega e^{4\pi u^2} \, \mathrm{d}x \end{aligned} \tag{5.24}$$

with a nonlinearity of critical growth, which manifests many of the characteristics of such equations. We first discuss some existence results.

5.1. **Existence of solutions.** Existence results for equation (5.23) have been considered by Adimurthi [4], and then by de Figueiredo-Miyagaki-Ruf [28], in the more general form

$$\begin{cases} -\Delta u &= f(u) \ , \quad \text{in } \ \Omega \\ \quad u &= 0 \ , \qquad \text{on } \ \partial\Omega \end{cases} \ , \tag{5.25}$$

We first introduce the notion of critical Trudinger-Moser growth (TM-growth): Let $\alpha_0 > 0$ be given. Then the function $f \in C(\mathbb{R})$ has critical TM-growth α_0 if

$$\begin{aligned} &\limsup_{|t|\to\infty} \frac{f(t)}{e^{\alpha u^2}} = 0 \ , \quad \forall\ \alpha > \alpha_0 \\ &\liminf_{|t|\to\infty} \frac{f(t)}{e^{\alpha u^2}} = +\infty \ , \quad \forall\ \alpha < \alpha_0 \ , \end{aligned} \tag{5.26}$$

while we say that f has subcritical TM-growth if

$$\lim_{|t|\to\infty} \frac{f(t)}{e^{\alpha t^2}} = 0 \ , \ \forall\ \alpha > 0 \tag{5.27}$$

In [28] the following theorem was proved (which refines and generalizes the result of Adimurthi [4]).

Theorem 5.1. *(*see [28])
Suppose that $f \in C(\mathbb{R})$ has the form

$$f(s) = h(s)e^{\alpha_0 s^2} \ , \quad \textit{where} \quad h(s) \quad \textit{has subcritical TM-growth}$$

Suppose in addition that $f(0) = 0$, $f(s) = \lambda s + o(s)$ for s near zero, $\lambda \in [0, \lambda_1)$, and

f1) $0 \le F(s) := \int_0^s f(t)dt \le Mf(s)$, $\forall\ s \in \mathbb{R}$, *for* $|s| \ge s_0$

f2) $0 < F(s) \le \frac{1}{2} f(s)s$, $\forall\ s \in \mathbb{R} \setminus \{0\}$

Then equation (5.23) *has a nontrivial solution provided that*

$$\liminf_{|s|\to\infty} h(s)s > \frac{2}{d^2\alpha_0} \ , \tag{5.28}$$

where d is the radius of the largest ball contained in Ω.

Proof. (Sketch) The proof uses the mountain-pass theorem of Ambrosetti-Rabinowitz [10]. However, due to the critical growth, there is a loss of compactness, and one cannot conclude directly. One proceeds similarly as in the proof by Brezis-Nirenberg [13]:
a) One determines the level of non-compactness for the functional, or more precisely, one shows that the functional

$$I(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, \mathrm{d}x - \int_\Omega F(u) \, \mathrm{d}x$$

satisfies the Palais-Smale (PS) condition in the interval $(0, \frac{2\pi}{\alpha_0})$. Indeed, let u_n be a (PS)-sequence, i.e. satisfying

i) $I(u_n) \to c > 0$ and

ii) $|I'(u_n)[v]| = |\int_\Omega \nabla u_n \nabla v - f(u_n)v| \le \varepsilon_n \|u_n\|$, $\forall\ v \in H_0^1(\Omega)$, with $\varepsilon_0 \to 0$, as $n \to \infty$.

Combining these two conditions, one easily obtains that the PS-sequences ar bounded, i.e. $\|u_n\|^2 = \int_\Omega |\nabla u_n|^2 \, \mathrm{d}x \le const.$, and hence we have for a subsequence:

$$u_n \rightharpoonup u \ \text{ in } \ H_0^1 \ , \quad \text{and} \quad u_n \to u \ \text{ in } L^q, \quad \text{for every} \quad q > 1 \ .$$

We want to exclude that $u = 0$. Suppose by contradiction that $u = 0$, then $u_n \to 0$, for all $q > 1$. One shows that then also $\int_\Omega F(u_n) \to \int_\Omega F(u) = 0$; this follows from the fact that $\int_\Omega f(u_n)u_n \, dx \le c$ by ii), and by assumption $f1$). Note that then i) implies

$$\int_\Omega |\nabla u_n|^2 \, dx \to 2c > 0$$

while ii) yields

$$\left| \int_\Omega |\nabla u_n|^2 \, dx - \int_\Omega f(u_n)u_n \, dx \right| \le \varepsilon_n \|u_n\| \le \varepsilon_n C$$

Thus, if we can show that $\int_\Omega f(u_n)u_n \to 0$, then we have a contradiction. We claim that for $c < \frac{2\pi}{\alpha_0}$ this is indeed the case. In fact, we can estimate with Hölder

$$0 \le \int_\Omega f(u_n)u_n \, dx \le \Big(\int_\Omega |f(u_n)|^q \, dx \Big)^{1/q} \|u_n\|_{L^{q'}}^{1/q'}$$

Since the last factor on the righthand side tends to zero, it is sufficient to show that the first factor is bounded. Indeed, we can estimate

$$\int_\Omega |f(u_n)|^q \, dx \le c(\delta) \int_\Omega e^{q(\alpha_0+\delta)u_n^2} \, dx = c(\delta) \int_\Omega e^{q(\alpha_0+\delta)\|u_n\|^2 (\frac{u_n}{\|u_n\|})^2} \, dx$$

The last integral is bounded by Moser's inequality (2.2) if

$$q(\alpha_0 + \delta)\|u_n\|^2 \le 4\pi$$

By i) we have

$$q(\alpha_0 + \delta)\|u_n\|^2 \le q(\alpha_0 + \delta)(2c + \varepsilon) < q(\alpha_0 + \delta)\frac{4\pi}{\alpha_0} \ ,$$

by our assumption on c (for ε sufficiently small). Hence we obtain the desired estimate choosing q near 1 and δ sufficiently small.

b) Next, one uses the condition (5.28) to show that there exists a minimax level c at level $\frac{2\pi}{\alpha_0}$.

Since $f(s) = \lambda s + o(s)$ near zero, one easily sees that $I(u)$ has a local minimum in zero. One shows that there exists a $w \in H_0^1(\Omega)$ with $\|w\| = 1$ and such that

$$\lim_{t\to\infty} I(tw) = -\infty \ , \quad \text{and} \quad \max_{t\ge 0} I(tw) < \frac{2\pi}{\alpha_0}$$

For simplicity, assume that $B_1(0) \subset \Omega$ is the largest ball contained in Ω; define the function $w(x)$ identically zero in $\Omega \setminus B_1(0)$, and radial in $B_1(0)$ by the *Moser-sequence*

$$w_n(r) = \frac{1}{\sqrt{2\pi}} \begin{cases} (\log n)^{1/2} \ , \ 0 \le r < \frac{1}{n} \\ \frac{\log 1/r}{(\log n)^{1/2}} \ , \ \frac{1}{n} \le r \le 1 \end{cases}$$

One checks that $w_n \in H_0^1(B_1)$ and $\|w_n\| = 1$, and one shows that there exists a $n > 0$ such that

$$I(t_n w_n) := \max\{I(tw_n) \ , \ t \ge 0\} < \frac{2\pi}{\alpha_0}$$

Indeed, assuming to the contrary that

$$\frac{t_n^2}{2} \int_\Omega |\nabla w_n|^2 \, dx - \int_\Omega F(t_n w_n) \, dx = I(t_n w_n) \ge \frac{2\pi}{\alpha_0}$$

one gets that

$$t_n^2 \ge \frac{4\pi}{\alpha_0}$$

On the other hand, at t_n we have

$$\frac{d}{dt}I(t_n w_n) = 0 \quad \text{iff} \quad t_n^2 = \int_\Omega f(t_n w_n) t w_n \, \mathrm{d}x \ ,$$

and hence

$$t_n^2 \geq \int_{B_{1/n}} h(t_n w_n) t_n w_n e^{\alpha_0 t_n^2 w_n^2} \, \mathrm{d}x$$

Since by assumption $h(t_n w_n) t_n w_n \geq K - \varepsilon$, for $t_n w_n > r_\varepsilon$, we get

$$t_n^2 \geq (K-\varepsilon)\frac{\pi}{n^2} e^{\alpha_0 t_n^2 w_n^2} = (K-\varepsilon)\pi e^{2\log n\,(\alpha_0 \frac{t_n^2}{4\pi} - 1)}$$

One concludes first that $t_n^2 \to \frac{4\pi}{\alpha_0}$, and then

$$\frac{4\pi}{\alpha_0} \geq (K-\varepsilon)\pi, \ \forall \, \varepsilon > 0$$

Thus, if $K > \frac{4}{\alpha_0}$ one obtains a contradiction. Refining the estimates, one improves the condition $K > \frac{4}{\alpha_0}$ to $K > \frac{2}{\alpha_0}$. □

This result was generalized to the corresponding $N-$Laplacian equation in $\mathbb{R}^N$ by do Ó [31].

5.2. **Non-existence of solutions.** Theorem 5.1 is almost sharp; indeed, one has the following non-existence result for positive solutions if Ω is the ball.

Theorem 5.2. (see [29])

Suppose that $f \in C(\mathbb{R})$ is of the form $f(s) = h(s)e^{\alpha_0 s^2}$, where $h \in C^2$ satisfies the following conditions: there exists $r_1 > 0$ and $\sigma > 0$ such that

h_1) $h(r) = \frac{K}{r}$, for $r \geq r_0$

h_2) $0 \leq h(r) \leq cKr^{1+\sigma}$, for $0 \leq r \leq r_1$

Then there exists K_0 such that for $0 < K < K_0$ equation (5.23) has no positive radial solution.

The theorem gives non-existence of positive radial solutions. But by the result of Gidas-Ni-Nirenberg [38] one knows that any positive solution of (5.23) is radial; thus, Theorem 5.2 gives in fact non-existence of positive solutions.

Proof. (Idea) The radial equation (5.23) takes the form

$$u_{rr} + \frac{1}{r}u_r + h(u)e^{4\pi u^2} \ , \ \text{ in } \ (0,1) \ ; \ u'(0) = u(1) = 0$$

Using the *Emden-Fowler* transformation $t = -2\log r/2$ and setting $y(t) = u(r)$, this equations is transformed into

$$-\,y'' = h(y)e^{4\pi y^2 - t} \ , \ \text{ for } \ t > 2\log 2 \ ; \ y(2\log 2) = y'(+\infty) = 0 \ . \qquad (5.29)$$

One now uses the shooting method: that is, one considers solutions $y(t)$ of (5.29) with $y'(+\infty) = 0$ and $y(+\infty) = \gamma$, i.e. one shoots horizontally from infinity, and tries to adapt the height γ in order to land at the point $2\log 2$. By refining the delicate estimates of Atkinson-Peletier [9] one proves that this is impossible under the given assumptions, provided the constant K is sufficiently small. □

Remark 5.3. In the case $N \geq 3$ the non-existence results come from Pohozaev's identity. In $N = 2$ no corresponding inequality is known to this date.

5.3. **The effect of topology.** Theorem 5.1 and Theorem 5.2 show that the line between existence and non-existence is quite delicate. This phenomenon is typical for problems with critical growth, and it is also seen in the Brezis-Nirenberg result, where the addition of a lower order term to an equation with critical growth is sufficient to produce a solution.

In the pioneering works of Coron [23] and Bahri-Coron [11] it was shown that also geometric and topological properties of the domain are important in the question of solvability of critical growth equations. In [59] M. Struwe showed that positive solutions to the critical boundary value problem (5.23) exist for a large class of critical growth nonlinearities on suitable *non-contractible domains*:.

Theorem 5.4. (see [59])

Suppose that $f(s)$ has the form $f(s) = s\, e^{\varphi(s)}$, where φ is a smooth function satisfying

(φ_1) $\varphi(0) = 0, \varphi(s) \leq 1$ *for* $s \leq 0, \varphi(s) \leq 4\pi s^2$ *for* $s \geq 0$

(φ_2) $-1 \leq \varphi'(s)s \leq 8\pi s^2$, $\lim_{s\to\infty} \varphi'(s)/s = 8\pi, \varphi''(s) \leq 8\pi$

Then, for suitable numbers $R_0 > R_1 > R_2$, problem (5.23) admits a positive solution on any domain $\Omega \subset B_{R_0}(0)$ containing the annulus $R_2 \leq |x| \leq R_1$ and such that $0 \notin \Omega$.

Proof. (Idea) The proof uses an approximation strategy developed by Sacks-Uhlenbeck [55], which consists in considering the following approximate problem

$$- \operatorname{div}\Big[\big(1 + |\nabla u|^2\big)^{\alpha-1} \nabla u\Big] = f(u) \ \text{ in } \ \Omega \ ; \ u|_{\partial\Omega} = 0 \ , \ \alpha > 1 \ , \tag{5.30}$$

with associated energy functional

$$E_\alpha(u) = \frac{1}{2\alpha} \int_\Omega \Big[\big(1 + |\nabla u|^2\big)^\alpha - 1\Big] \,\mathrm{d}x - \int_\Omega F(u) \,\mathrm{d}x \ ,$$

whose critical points $u_\alpha \in W_0^{1,2\alpha}(\Omega)$ are weak solutions of equation (5.30).

One easily verifies that (5.30) admits a positive solution for any small $\alpha > 1$ on sufficiently small domains Ω. However, these solutions may degenerate as $\alpha \to 1$. If one now assumes by way of contradiction that the original problem (5.23) does not admit a solution $u > 0$ with $E(u) \leq \frac{1}{2}$, then for a sufficiently small, non-contractible domain Ω Coron's method may be applied to show that equation (5.23) admits for sufficiently small $\alpha > 1$ also solutions u_α of saddle type, whose energies $E_\alpha(u_\alpha)$ decrease monotonically to a limit $\beta > \frac{1}{2}$ as $\alpha \to 1$. Restricting the shape of Ω slightly more, one also gets the upper bound $\beta < 1$. By a careful blow-up analysis one now obtains the result. ☐

5.4. **Blow-up techniques.** As in the Sobolev case, non-compactness is related to the phenomenon of blow-up solutions. We recall the famous quantization result by M. Struwe [58], who showed that Palais-Smale sequences $\{u_n\}$ for the Brezis-Nirenberg functional associated to (1.1)

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \frac{\lambda}{2}\int_\Omega |u|^2 \,\mathrm{d}x - \frac{1}{2^*}\int_\Omega |u|^{2^*} \,\mathrm{d}x$$

have the following property:

$$I_\lambda(u_n) = I_\lambda(u_0) + kS_N + o(1)$$

where u_0 is a critical point of $I_\lambda(u)$, $k \geq 0$ is some integer, and S_N is a fixed constant. The interpretation is that compactness can only be lost through the formation of "standard bubbles".

An analogous result was recently proved by O. Druet [33] for the Trudinger-Moser equation (5.23) and the corresponding functional (5.24):

Theorem 5.5. *Suppose that $\{u_n\}$ is a sequence of critical points of $J_{\lambda_n}(u)$. Then*

$$J_{\lambda_n}(u_n) = k4\pi + o(1)\ , \quad \textit{as} \quad \lambda_n \to 0\ , \quad \textit{for some} \quad k \in \mathbb{N}\ .$$

The interpretation is the same as in the Sobolev case: compactness can only be lost by the formation of standard bubbles each of which adds an energy quantum of 4π.

The paper [30] by M. del Pino, M. Musso and B. Ruf is a counter part to the paper of O. Druet: it gives sufficient conditions under which such blow-up solutions actually exist.

To state the result, let $G(x, y)$ be Green's function of the problem

$$-\Delta_x G(x, y) = 8\pi\delta_y(x) \quad \text{and let} \quad H(x, y) = 4\log\frac{1}{|x-y|} - G(x, y).$$

Then $H(\xi, \xi)$ is called Robin's function of Ω.

For $\xi_1, \ldots, \xi_k \in \Omega$ and $m_1, \ldots, m_k \in \mathbb{R}^+$ consider now the function

$$\varphi_k(\xi, m) = \sum_{j=1}^{k} 2m_j^2(b + \log m_j^2) + m_j^2 H(\xi_j, \xi_j) - \sum_{i \neq j} m_i m_j \, G(\xi_j, \xi_j).$$

We make the following

Definition 5.6. *We say that φ_k has a* stable critical point situation (SCS) *if for some region Λ compactly contained in its domain, any small $C^1(\bar{\Lambda})$-perturbation of φ_k has a critical point in Λ.*

One has the following important facts:

- $\varphi_1(\xi, m)$ satisfies (SCS), with Λ a neighborhood of its minimum set, where
$$\varphi_1(\xi, m) = 2m^2(b + \log m^2) + m^2 H(\xi, \xi)$$
- $\varphi_2(\xi, m)$ satisfies (SCS) whenever Ω is not simply connected, where
$$\varphi_2(\xi, m) = \sum_{j=1}^{2} 2m_j^2(b + \log m_j^2) + m_j^2 H(\xi_j, \xi_j) - 2m_1 m_2 \, G(\xi_1, \xi_2).$$

One conjectures that (SCS) holds for any $k \geq 2$ (Ω not simply connected).

The stable critical points of φ_k determine the existence and the location of bubbling solutions:

Theorem 5.7. (see [30])

Assume that $\varphi_k(\xi, m)$ has (SCS). Then there exist solutions u_λ of (5.23) which

- *blow up around k points ξ_j as $\lambda \to 0$, where $\nabla \varphi_k(\xi, m) \to 0$*
- *away from the points ξ_j the solutions u_λ take the form*

$$u_\lambda(x) \ = \ \sqrt{\lambda} \sum_{j=1}^{k} m_j \ [\, G(x, \xi_j) \ + \ o(1) \,]$$

Furthermore

$$J_\lambda(u_\lambda) \ = \ 4\pi k \ + \ \lambda \, [\, -|\Omega| \ + \ 8\pi \, \varphi_k(\xi, m) \ + \ o(1) \,] \ .$$

Note that this result yields in view of the "facts" stated above for:

$k = 1$: a bubbling solution near the minimizer of $H(\xi, \xi)$

$k = 2$: a solution with two bubbles, provided that Ω is not simply connected.

Proof. (Idea) The proof follows the following lines:

- One first constructs an approximate solution, based on the "standard bubble", where the standard bubble is derived from the explicit solutions of the limiting *Liouville equation*

$$-\Delta u = e^u \ \text{ in } \ \mathbb{R}^2 \ ; \quad u(x) = u_\mu(x) = \log \frac{8\mu^2}{(\mu^2 + |x|^2)^2} \ , \ \mu > 0$$

- Then one linearizes equation (5.23) in this approximate solution, for given $\xi = (\xi_1, \dots \xi_k) \in \Omega^k$ and parameters $m = (m_1, \dots, m_k) \in \mathbb{R}_+^k$

- Then one does a finite dimensional variational reduction via a Lyapunov-Schmidt procedure, to reduce the problem to a finite-dimensional functional $f_k(\xi, m)$ which is C^1-close to $\varphi_k(\xi, m)$

- The critical points of f_k (which has critical points by the (SCS)-property of φ_k) yield the k-bubble solutions, and the information on the location of the bubbles. □

6. **Systems of equations with exponential type nonlinearities.** In this section we consider systems of elliptic equations with nonlinearities which are of exponential type. More precisely, the systems we consider are Hamiltonian systems of the following form:

$$\begin{cases} -\Delta u = g(v) \ , \quad \text{on} \ \ \Omega \\ -\Delta v = f(u) \ , \quad \text{on} \ \ \Omega \\ \quad u = v = 0 \ , \quad \text{on} \ \ \partial\Omega \\ \quad u > 0 \ , \ v > 0 \ , \quad \text{on} \ \ \Omega \end{cases} \ , \tag{6.31}$$

where $f, g \in C^1(\mathbb{R})$, and Ω is a bounded domain in $\mathbb{R}^2$. Such systems have been widely studied in recent years for domains in $\mathbb{R}^N, N \geq 3$. The functional associated to system (6.31) is given by

$$\begin{aligned} &I : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R} \\ &I(u,v) := \int_\Omega \nabla u \nabla v \, \mathrm{d}x - \int_\Omega F(u) \, \mathrm{d}x - \int_\Omega G(v) \, \mathrm{d}x \end{aligned} , \tag{6.32}$$

where F and G are as before the primitives to f and g, respectively.

In dimension $N \geq 3$, criticality for the functional (6.32) is given by polynomial growth conditions which involve both nonlinearities, namely by the so-called *critical hyperbola*:

Let $|F(s)| \leq c|s|^p + c_1$, $|G(s)| \leq d|s|^q + d_1$, with

$$\frac{1}{p} + \frac{1}{q} = 1 - \frac{2}{N}$$

One knows that system (6.31) has a solution for subcritical growth (i.e. for $\frac{1}{p} + \frac{1}{q} > 1 - \frac{2}{N}$) (cf. de Figueiredo-Felmer [25] and Hulshof-Van der Vorst [41]), and for perturbed systems with critical growth (cf. Hulshof-Mitidieri-Van der Vorst [42]).

We consider here such systems for the two-dimensional case.

6.1. **A coupled system with critical TM-nonlinearities.** The first results for coupled systems with subcritical of critical Trudinger-Moser growth are the following:

Theorem 6.1. (see [27])
Suppose that f has subcritical TM-growth and g has at most critical TM-growth (in the sense of conditions (5.27) *and* (5.26)*), and that f and g satisfy the other conditions of Theorem 5.1. Then system* (6.31) *has a nontrivial solution.*

Theorem 6.2. (see [27])
Suppose that both f and g have critical TM-growth with the same exponent α_0, and that they satisfy the other hypotheses of Theorem 5.1. Assume furthermore that

$$\lim_{t\to\infty} \frac{f(t)\,t}{e^{\alpha_0 t^2}} > \frac{4}{\alpha_0 d^2} \quad , \quad \lim_{t\to\infty} \frac{g(t)\,t}{e^{\alpha_0 t^2}} > \frac{4}{\alpha_0 d^2} , \tag{6.33}$$

where d is the radius of the largest ball contained in Ω. Then system (6.31) *has a nontrivial solution.*

Proof. (Idea) A main difficulty in the search for critical points of the functional (6.32) is its strong indefiniteness: in fact, it is easily seen that $I(u,v)$ is unbounded above and below on infinite-dimensional subspaces.

Note that every element $(u,v) \in H_0^1 \times H_0^1 =: E$ can be written in a unique way as

$$(u,v) = (y,y) + (v,-v) \ , \ y, v \in H_0^1(\Omega) \ ;$$

denoting $E^+ := \{(y,y) \ ; \ y \in H_0^1(\Omega)\}$ and $E^- := \{(v,-v) \ ; \ v \in H_0^1(\Omega)\}$ we get $E = E^+ \oplus E^-$, and the functional I may be written as

$$I(y,v) = \int_\Omega |\nabla y|^2 \, \mathrm{d}x - \int_\Omega |\nabla v|^2 \, \mathrm{d}x - \int_\Omega F(y+v) \, \mathrm{d}x - \int_\Omega G(y-v) \, \mathrm{d}x$$

One now proceeds by finite-dimensional approximation: one considers the functionals

$$I_n : E^+ \times E_n^- \to \mathbb{R} \ ,$$

where $E_n^- := \big\{(v,-v) \ ; \ v \in \mathrm{span}\{e_1,\dots,e_n\}\big\}$, $e_i = i$–th eigenfunction of the Laplacian. For I_n the classical Linking Theorem by P. Rabinowitz [51] applies.

In the case of subcritical growth, one obtains easily critical points (u_n, v_n) for I_n, $n \in \mathbb{N}$, and using compactness one proves that $(u_n, v_n) \to (u, v)$, where (u, v) is a solution to problem (6.31).

The proof of Theorem 6.2 is more delicate: one starts again by proving the existence of a sequence of approximate solutions (i.e. critical points u_n of I_n). One then identifies the non-compactness level for the functional I on E, and shows that all the critical levels $I_n(u_n)$ stay uniformly below the non-compactness level (for this, the conditions (6.33) are used). This allows to show that the weak limit u of the sequence $\{u_n\}$ is not trivial, and hence it is a weak solution of the system (6.31). □

6.2. **A critical hyperbola for elliptic systems in two dimensions.** As in the case $N \geq 3$, one would like to admit nonlinearities with different growth, possibly one with higher than critical TM-growth, provided the other nonlinearity has a suitably lower growth. The following theorem gives a result in this direction:

Theorem 6.3. (see [52])
Suppose that f and g satisfy the hypotheses

$$\lim_{s\to\infty} \frac{f(s)}{e^{s^q}} = 0 \quad , \quad \lim_{s\to\infty} \frac{g(s)}{e^{s^p}} = 0 \ ,$$

where

$$\frac{1}{p} + \frac{1}{q} = 1 \ .$$

If f and g satisfy the other hypotheses of Theorem 5.1, then system (6.31) *has a nontrivial solution.*

The curve of the exponents (p, q) can be viewed as a *critical hyperbola* for exponential growth. The theorem above gives existence for subcritical growth with respect to this hyperbola.

The existence of solutions for the corresponding critical problem has been solved for the case $p = q = 2$ in [27]. The general case remains an open problem.

Proof. (Idea) The proof relies again on the notion of Lorentz space $L^{p,q}(\Omega)$ and Sobolev-Lorentz space, see section 4.1.

One considers the functional $I(u, v)$, instead on $H_0^1(\Omega) \times H_0^1(\Omega)$, on the space

$$I(u,v) : W_0^1 L^{2,p}(\Omega) \times W_0^1 L^{2,q}(\Omega) \to \mathbb{R} \ , \quad \text{with} \ \ \frac{1}{p} + \frac{1}{q} = 1$$

The first term is well-defined on this space by Hölder's inequality (4.12) given above, while the second and third term are well defined provided $F(u) \leq c e^{|u|^{p'}}$ and $G(v) \leq c e^{|v|^{q'}}$, with $p' = q$ and $q' = p$, by the Brezis-Wainger theorem which gives:

$$u \in W_0^1 L^{2,p} \ \Rightarrow \ \int_\Omega e^{|u|^{p'}} \, \mathrm{d}x < \infty$$

$$v \in W_0^1 L^{2,q} \ \Rightarrow \ \int_\Omega e^{|v|^{q'}} \, \mathrm{d}x < \infty$$

Furthermore, one has compactness if F and G are subcritical relative to these growths, which allows to prove the (PS)-condition. The proof proceeds then similar to the previous theorem 6.1: one does a finite-dimensional approximation, obtains a sequence (u_n, v_n) of approximate solutions, and does a limit $n \to \infty$. □

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