AMERICAN JOURNAL of PHYSICS

Force measurement and force transformation in special relativity

Roberto de A. Martins

Citation: American Journal of Physics **50**, 1008 (1982); doi: 10.1119/1.12954
View online: http://dx.doi.org/10.1119/1.12954
View Table of Contents: http://scitation.aip.org/content/aapt/journal/ajp/50/11?ver=pdfcov
Published by the American Association of Physics Teachers

Force measurement and force transformation in special relativity

Roberto de A. Martins
Centro de Lógica, Epistemologia e História da Ciência, Universidade Estadual de Campinas (Unicamp), Caixa Postal 6133, 13.100 Campinas, SP, Brazil

(Received 6 January 1981; accepted for publication 20 November 1981)

Relativistic dynamics has been historically built without an immediate connection to empirical operations. In order to test dynamic laws by experiment, it is necessary that at least one dynamic parameter be made measurable for moving systems. A method of force measurements applicable to moving systems is proposed. It is shown with the aid of thought experiments that the method leads to the accepted relativistic force transformations.

I. INTRODUCTION

The transformation of forces in special relativity is well known[1]: if a force acts upon a body and the motion of this body relative to several reference systems is parallel to the force direction, then the force has an invariant value. If the velocity of the body is perpendicular to the direction of the force, then the force has its value reduced by the same factor appearing in Lorentz contraction. If v is the velocity of the body relative to S' parallel to the x axis, and F_x and F_y, respectively, the components of the force relative to the instantaneous inertial proper frame of the body, then these components, relative to S', transform according to

$$F'_x = F_x, \tag{1}$$

$$F'_y = F_y(1-v^2/c^2)^{1/2}. \tag{2}$$

The derivation of these relations is sometimes grounded on electrodynamic considerations.[2] But in any elementary course on special relativity a strictly mechanical derivation is desirable. Tolman endeavored to use the lever law to derive force transformations[3] but he arrived at a result opposite to the accepted one. This attempt gave rise to one of the most famous relativistic problems: the "lever paradox."[4]

Deductions clothed in sophisticated symbolism are usual in advanced textbooks.[5] But these give no insight into the meaning of the relations, and do not elucidate the empirical meaning of force. Some modern elementary textbooks introduce relativistic dynamics with the study of momentum conservation in collisions and a simple derivation of mass transformation.[6] In this approach, force is defined as dp/dt, and the transformation formulas for force components are derived from the transformations of momentum and time. This approach is also due to Tolman.[7] It does not elucidate the empirical meaning of forces, either.

It is interesting to see how Einstein managed the issue. In his first paper on special relativity,[8] he emphasized the operational meaning of space and time intervals. He worked out in detail the methods of ascertaining the length of a moving stick and the simultaneity of distant events. From this analysis and the postulates of special relativity, the known kinematical transformations were derived. The success of this kind of approach has stimulated the development of Bridgman's operationalism,[9] and has sometimes been regarded as one of the basic components of special relativity.[10]

It has not been made so conspicuous that Einstein's dynamics did not follow a similar method. He takes force as the fundamental parameter of his dynamics.[8] Electromagnetic and other dynamic quantities (mass,work) are directly or indirectly related to force. Being derived parameters, their measurement depends on the measurement of forces. At several places he states that forces are to be measured by a spring meter. This provides a crude operationalization of force. But while the relation between force and elastic deformation of bodies at rest was well known, the behavior of a moving spring had not been studied, and therefore the properties of the force produced by a moving spring attached to a moving body were not clear. Was this force to be considered constant (invariant)? Perhaps Einstein though so, at that time, but there was no ground for such a supposition: moving bodies undergo Lorentz contraction; with the resulting change of atomic distances, the elastic constant of moving materials could also change.

Based on unanalyzed force measurements, relativistic dynamics was therefore built upon a weaker empirical foundation than relativistic kinematics. In a later approach to relativistic dynamics,[11] Einstein makes no reference to measurements, and contents himself with the derivation of the known formulas with the aid of conservation laws. Since that time, the concept of force in special relativity has been relegated to the background, and no operationalization or direct derivation of its transformation properties has been undertaken.[12]

It would be highly desirable to produce derivations of relativistic transformations of the main physical parameters that resulted as direct and simple as the elementary derivations of time and length transformations. Besides, it would also seem desirable to give to the main physical parameters an operational elucidation such as those developed by Einstein for space and time measurements. It is not necessary that each parameter of a physical theory be made directly measurable, but the empirical elucidation of physical parameters is desirable whenever possible. The possibility of independent measurement of new parameters of any theory increases its empirical content and testability. In an attempt to satisfy those demands, this paper presents an empirical elucidation of the force concept that is very simple, independent of electrodynamics, and compatible with special relativity. With the use of this elucidation a simple derivation of force transformations is obtained.

II. FORCES ACTING UPON MOVING BODIES

Elementary textbooks on classical mechanics usually describe a method for producing and measuring forces with the aid of springs and dynamometers.[13] This is an acceptable method when the bodies and springs are at rest relative to the reference system. But the main problem, in relativis-

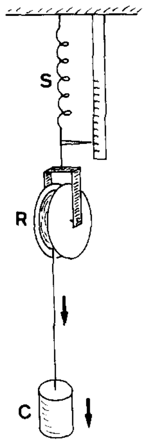

Fig. 1. Bouasse's suggested experiment for testing the constancy of the weight of a moving body.

tic dynamics, is to provide an empirical meaning to forces produced by *moving* elastic bodies. How can this force be compared with that produced by a still spring?[14] It would be necessary to submit a body to the forces of the two springs, at the same time, in order to compare them. But if the springs are in relative motion, how can this be done? Here is the main problem of relativistic measurement of forces.

The measurement of forces acting upon moving bodies has puzzled prerelativistic physicists. Descartes had strong doubts about the constancy of the weight of a falling body.[15] Why should this force be constant? Could not this force depend on the speed of the body? He discussed these conjectures, but proposed no method for deciding this question by experiment. It appears to me that the earliest solution was proposed by Bouasse in 1895.[16]

Let us suppose that a body C is attached to a thread of very small mass wound around a reel R (Fig. 1). The reel is connected to a spring S. The elongation of the spring is measured by a scale that indicates the force produced by the spring upon the reel-and-body system. If C is at rest or in uniform motion, the resultant force acting upon it must be null. Hence, the force produced by the spring upon the system will in these cases be equal to its weight if the air resistance is negligible. It is possible to give to the reel a uniform speed and test whether the elongation of the spring changes. If it does not change, then the weight of the body C does not depend upon its speed, when in uniform motion.

Bouasse's ingenious suggestion shows a way of attaching a *still* spring to a *moving* body. An adaptation of this method may be used to compare forces produced by still and moving springs, whenever these forces are parallel to the velocity of the moving spring (longitudinal motion).

Suppose two threads of negligible mass are wound around a system R of coupled reels (Fig. 2). These reels unwind the threads at a constant speed u. At the extremity of each thread a dynamometer is attached. One of them (A) is at rest relative to a reference system S. The other one (B) recedes from A with a constant speed v relative to this system. The system of reels R is acted upon by the two opposite forces produced by A and B. It moves with a constant speed u relative to S. The threads transmit the forces from A and B to R without change, since they have very small masses. Hence we may talk as if the dynamometers acted directly upon R.

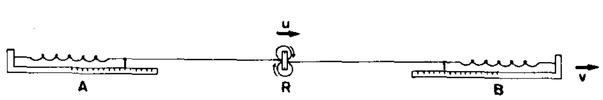

Fig. 2. Two dynamometers A and B are attached by threads to a system of reels R. The analysis of this system shows that the force produced by B upon R is equal to the force produced by A upon R, whatever the speed v of B.

The system of reels R is acted upon only by the forces produced by A and B. Its speed is constant, and therefore these forces must be numerically equal, relative to any reference system. If the proper forces produced by these springs are equal (that is, if the forces produced by the springs relative to their proper reference systems are equal), then the forces produced by springs in longitudinal motion are invariant.

Relative to a reference system attached to R, the system of threads and springs is completely symmetric. Therefore the elongations of the springs relative to R must be equal, and the proper elongations of the springs will also be equal. Therefore the proper forces produced by the dynamometers are also equal, and the force produced by a spring in longitudinal motion is invariant. This is the usually accepted relativistic transformation law for longitudinal forces.

III. TRANSVERSE MOTION

In order to compare forces produced by springs that are moving in a direction perpendicular to their elongations, it is necessary to use another imaginary device, described below.

Suppose an infinite solid rod D is transversely pushed toward opposite directions by two infinite series of parallel springs A,B [Fig. 3(a)]. Each spring is fastened at equal distances to solid rods at one extremity; these rods are constrained to maintain a constant distance between them by a device that is not shown in the figure. The other end of the springs is attached to frictionless balls that slide at D's surface. All springs are built of the same alloy, with equal dimensions. They are kept under identical proper conditions, with equal elongations. Their proper forces upon D are, therefore, equal.

The systems of springs A and B travel in opposite directions with equal speeds relative to D [Fig. 3(a)]. Relative to D, the forces produced by each spring are equal, and their mutual distances are also equal. Hence the mean resultant force acting upon each section of length L of the rod D is zero. Therefore D will have no acceleration.

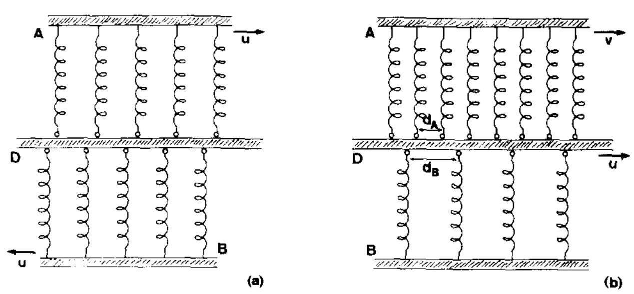

Fig. 3. Solid rod of infinite length D is pushed by two infinite sets of equal springs A and B. Relative to D the whole situation is symmetrical (a), but relative to B the distance between the springs of the system A undergoes Lorentz contraction (b). The force produced by the springs of system A must decrease in the same ratio as the length, in order that the equilibrium of D might be accounted relative to B.

Relative to B [Fig. 3(b)], the system of springs A travels at a speed v. Relative to B, the distance between adjacent springs of the system A is d_A and the distance between adjacent springs of the system B is d_B. Owing to Lorentz contraction the following relation will hold between these quantities:

$$d_A = d_B(1 - v^2/c^2)^{1/2}. \quad (3)$$

Relative to B the mean number n_B of B springs that act upon a section of length L of rod D is

$$n_B = L/d_B, \quad (4)$$

and the mean number n_A of A springs acting upon this same section of the rod must be

$$n_A = L/d_A. \quad (5)$$

The mean total forces produced by each system upon each section of D must be equal, since D is at equilibrium. Hence,

$$n_B F_B = n_A F_A, \quad (6)$$

where F_A and F_B are, respectively, the forces produced by each A spring or B spring upon D, relative to B. From (4), (5), and (6) we derive

$$F_B/d_B = F_A/d_A \quad (7)$$

and therefore, from (3),

$$F_A = F_B(1 - v^2/c^2)^{1/2}. \quad (8)$$

Therefore, relative to B, the force produced by each A spring upon D is smaller than the force produced by each B spring upon D. Transverse motion does change the force produced by the springs, in a ratio equal to the Lorentz-contraction factor. Relation (B) is the usually accepted relativistic transformation law for transverse forces, and is equivalent to (2).

Thus far we have not discussed the direction of the forces produced by the springs. In classical mechanics the force is always parallel to the elongation of the springs. In relativistic mechanics it is known that this is not the case in oblique motion.[17] In order to find the transformed direction of a force in relativistic mechanics, one may decompose it into its longitudinal and transverse components, transform these components, and add them again. This process obviously presupposes that the directions of longitudinal and transverse forces are not changed in transformation. It is not difficult to prove this, as is shown below.

In the case of longitudinal motion the system is symmetric relative to the direction of the threads. Therefore the direction of the force must be parallel to the direction of the threads. The existence of any longitudinal motion does not break this symmetry, and therefore does not change the direction of the force. If there is an oblique motion, there is no symmetry relative to the direction of the threads, and the argument can no longer be applied.[18]

In the case of transverse motion, let us analyze the problem relative to B [Fig. 3(b)]. All B springs are at rest relative to this system. The forces that these springs produce upon D must therefore have the direction of their elongations. The forces produced by the B springs are therefore perpendicular to D. Now, if the forces produced by the A springs upon D were not perpendicular to D, there would be a resultant force parallel to D, and D would be accelerated. This is not the case, as can be seen from the hypotheses. Therefore, relative to B, the forces produced by the moving A springs are also perpendicular to D, that is, they have the same direction as the proper forces produced by the A springs. Therefore this kind of transformation does not change the direction of transverse forces.

Notice that all these properties of forces have been derived from thought experiments, without the use of actual experimental data. We have derived the force transformations from relativistic kinematics (Lorentz transformations) and from the proposed method for measuring forces. Something similar to this is done in the derivation of the relativistic transformations of time and length: they are deduced from the relativistic postulates and from the chosen measurement procedure for time and length. It seems that the similarity between both procedures recommends this kind of derivation of force transformations.

IV. FINAL COMMENTS

In this paper we have presented an empirical elucidation of force that allows the derivation of the relativistic force transformations. Nothing in this elucidation is really new. All the necessary ideas were available at the time when Einstein published his first papers. It is indeed remarkable that nobody has hitherto used ideas such as these in order to develop an operational relativistic dynamics.

In the above treatment of forces, use was made of extended bodies (threads, rods) in the derivation of force transformations. This is a weak point in our approach, since the behavior of extended bodies in special relativity is far from simple.[19] If we read the details of the derivation of the transformation law for longitudinal forces again, an important gap may be found. We have supposed that a thread with small (negligible) mass was used to connect the springs to the system of reels. This made it possible to consider only the action of the springs upon the reels and overlook the forces acting upon the threads. But in special relativity, no *stressed* thread is devoid of mass. Besides its "normal" (unstressed) mass, it will have an additional mass Δm equal to[20]

$$\Delta m = FL/c^2, \quad (9)$$

where F is the stress, L the length of the thread, and c the speed of light. As the thread unwinds it acquires a linear speed and its linear momentum increases. Relative to R, as B recedes with a speed u, the rate of mass change of the thread will be

$$dm/dt = FdL/(c^2 dt) = Fu/c^2 \quad (10)$$

and the rate of momentum change of the thread will be

$$dp/dt = Fu^2/c^2. \quad (11)$$

The resultant force acting upon the thread is therefore nonzero, and the force produced by the spring upon R will be smaller than F. Its value will be

$$F' = F(1 - u^2/c^2). \quad (12)$$

Hence, if a spring is *directly* attached to the system of reels (as in the case shown in Fig. 1) and another spring is attached to the unwinding thread, experiment will show that the forces produced by the two springs are not equal, their difference being a second-order relativistic effect due to the non-negligible mass of the stressed thread. Therefore, if Bouasse's suggestion is strictly followed, experiment will "prove" that the force of a longitudinally moving spring decreases with motion. We were careful in avoiding this difficulty, using a system of two reels and two threads (Fig. 2). The above criticism does not apply to our modification of Bouasse's method.

It has sometimes been proposed[21] that the currently accepted force transformations of special relativity are wrong. The elucidation proposed in this paper is incompatible with such claims, and may be considered a new argument against those reformers. It is very difficult to see how they would describe the cases of equilibrium studied in this article.

ACKNOWLEDGMENT

This work was supported by a research grant from the Brazilian National Council for Scientific and Technological Development (C.N. Pq.).

[1]The simple form of force transformations described here is usually found in papers on relativistic statics: J. H. Fremlin, Cont. Phys. **10**, 179 (1969); J. W. Butler, Am. J. Phys. **38**, 360 (1970); S. Aranoff, Nuovo Cim. **10 B**, 155 (1972). For derivations, see : T. M. Helliwell, *Introduction to Special Relativity* (Allyn and Bacon, Boston, 1966), p. 180; R. Resnick, *Introduction to Special Relativity* (Wiley, New York, 1968), p. 147. A simple form of the equations in vector formalism has been presented by B. U. Stewart, Am. J. Phys. **47**, 50 (1979).

[2]L. Page, N. I. Adams, *Electrodynamics* (Van Nostrand, New York, 1940), p. 176; J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962), p. 377. Earlier forms of the derivation may be found in A. Einstein, Jahrb. Radioakt. Elektron. **4**, 411 (1907) [translation and commentaries by H. M. Schwartz, Am. J. Phys. **45**, 512, 811, 899 (1977)]; and in Max von Laue, *La Théorie de la Relativité* (Gauthier-Villars, Paris, 1924), tome I, p. 224. The converse derivation of electromagnetic field transformations from the relativistic transformation of force has been presented by E. Tsantes, Am. J. Phys. **46**, 294 (1978), in an interesting approach where force transformations are derived from the properties of radiation forces.

[3]G. N. Lewis and R. C. Tolman, Proc. Am. Acad. Arts Sci. **44**, 711 (1909); Phil. Mag. **18**, 510 (1909).

[4]Besides references given in note (1) above, see: J. C. Nickerson and R. T. McAdory, Am. J. Phys. **43**, 615 (1975); G. Cavalleri and G. Spinelli, Lett. Nuovo Cim. **26**, 261 (1979); and references found therein.

[5]J. L. Synge, *Relativity: the Special Theory* (North-Holland, Amsterdam, 1958), p. 162; V. Fock, *The Theory of Space, Time and Gravitation* (Pergamon, New York, 1959), p. 45.

[6]T. M. Helliwell, *Introduction to Special Relativity* (Allyn and Bacon, Boston, 1966), p. 73; J. H. Smith, *Introduction to Special Relativity* (Benjamin, New York, 1965), p. 122; E. F. Taylor and J. A. Wheeler, *Spacetime Physics* (Freeman, San Francisco, 1966), p. 101; W. G. V. Rosser, *An Introduction to the Theory of Relativity* (Butterworths, London, 1964), p. 174.

[7]See note (3). An analysis of this and other derivations of the mass-velocity relation in special relativity may be found in: R. de A. Martins, Rev. Bras. Fís. **10**, 315 (1980).

[8]A. Einstein, Ann. Physik **17**, 891 (1905). See the analysis of this paper by A. I. Miller, Am. J. Phys. **45**, 1040 (1977).

[9]P. W. Bridgman, *The Logic of Modern Physics* (MacMillan, New York, 1928).

[10]G. Gutting, Phil. Sci. **39**, 51 (1972). For the relation between operationalism and relativity see: R. de A. Martins, Manuscr. **5**, 103 (1981).

[11]A. Einstein, Bull. Am. Math. Soc. **41**, 223 (1935). Planck's contribution to relativistic dynamics has also a mathematical flavor, and is described by S. Goldberg, Hist. Stud. Phys. Sci. **7**, 125 (1976).

[12]In most modern treatments, force is just a shorthand for dp/dt. It is even sometimes said that the second law of Newtonian mechanics is "a definition of how force is to be measured": G. Stephenson and C. W. Kilmister, *Special Relativity for Physicists* (Longmans and Green, London, 1958), p. 50. This is a return to Kirchhoff's position relative to classical mechanics. See: C. T. O'Sullivan, Am. J. Phys. **48**, 131 (1980). Notice that the Newtonian concept cannot be used directly in special relativity since its properties become mutally incompatible: the classical concept of force loses its univocity and a new empirical elucidation is required to provide an operational meaning to this concept: P. Frank, *Foundations of Physics* (University of Chicago, Chicago, 1946), p. 455; J. Giedymin, Br. J. Phil. Sci. **24**, 270 (1973).

[13]G. Holton and D. Roller, *Foundation of Modern Physical Science* (Addison-Wesley, Reading, MA, 1958), p. 63; D. Halliday and R. Resnick, *Physics for Students of Science and Engineering* (Wiley, New York, 1965), p. 68; Physical Science Study Committee, *Physics* (Heath, Boston, 1960), Part III. See also: J. H. Keenan, Sci. Monthly **73**, 406 (1948). In this kind of approach, forces are measured independently of other mechanical variables, and the second law of Newtonian mechanics becomes testable. For historical references, see: M. Jammer, *Concepts of Force* (Harvard University, Cambridge, 1957); M. Hesse, *Forces and Fields* (Nelson, London, 1961); R. Dugas, *Histoire de la Mécanique* (Dunod, Paris, 1950); and other references cited in: M. Hesse, Am. J. Phys. **32**, 905 (1964). A clear exposition of this approach may be found in J. C. Maxwell, *Matter and Motion* (Dover, New York, no date).

[14]A very interesting and complete discussion of the concept of force and of force measurements in relativity is presented by H. Arzeliès, *La Dynamique Relativiste et ses Applications* (Gauthier-Villars, Paris, 1957), Vol. I, p. 143. Arzeliès at first states that forces are the basic dynamical parameters and that they are to be measured by elastic deformations, that is, by dynamometers. But later he shows that his concept leads to great difficulties. If forces are to be measured by dynamometers, no change of the value of a force could arise when they are observed from different reference systems: forces should be invariant. Hence, Arzeliès limits this operationalization to the proper force acting upon a body.

[15]"In what I have sent to you on the subject of the fall of a stone, I suppose not only the (existence of the) void, but also that the force that moves this stone acts always equally, and this is overtly repugnant to the laws of nature. Because all natural powers act plus or less, according as the subject is more or less disposed to receive their action. And it is unmistakable that a stone is not equally disposed to receive a new motion or an increase in speed when it already moves very fast as when it moves very slowly." (Descartes, in a lettter to Mersenne, 1632): R. Descartes, *Oeuvres*, edited by C. Adams and P. Tannery (Cerf, Paris, 1897), Vol. 6, p. 216. Cited by J. Wilbois, Rev. Mét. Mor. **7**, 579 (1899). For a study of Descartes' mechanics, see: R. Dugas, *La Mécanique au XVII^{e} Siècle* (Dunod, Paris, 1954), pp. 117–202.

[16]H. Bouasse, *Introduction a l'Étude des Théories de la Mécanique* (Carré, Paris, 1895), p. 103.

[17]K. A. Johns, Lett. Nuovo Cim. **4**, 351 (1970); G. Cavalleri, G. Spavieri, and G. Spinelli, Nuovo Cim. **25 B**, 348 (1975).

[18]Symmetry arguments have been used in this derivation of force transformations. Such arguments are grounded on Pierre Curie's law: The symmetry of causes survives in their effects; or, as he enunciates it: "When some causes produce certain effects, the elements of symmetry of the causes must be found in the produced effects; when certain effects show a given asymmetry, this asymmetry must be found in the causes that have produced it." P. Curie, J. Phys. Théor. Appl. **3**, 393 (1894). An interesting use of this principle in classical mechanics has been made by P. Painlevé, Bull. Soc. Franç. Phil. **5**, 27 (1905); P. Painlevé, *Les Axiomes de la Mécanique* (Gauthier-Villars, Paris, 1909). A discussion of symmetry principles in elementary physics, and relevant references, may be found in: J. Rosen. Am. J. Phys. **42**, 68 (1974).

[19]In the references cited in notes (1), (4), and (17) above, several strange properties of extended bodies in special relativity are described. No universal agreement has been reached concerning the best formulation of relativistic statics, relativistic fluid dynamics, and relativistic thermodynamics. See: A. Guessous, *Thermodynamique Relativiste* (Gauthier-Villars, Paris, 1970); H. Arzeliès, *Fluides Relativistes* (Masson, Paris, 1971); O. Gron, Am. J. Phys. **46**, 249 (1978); O. Gron, Nuovo Cim. **17 B**, 141 (1973); R. G. Newburgh, Nuovo Cim. **52 B**, 219 (1979); G. Cavalleri, G. Spavieri, and G. Spinelli, Nuovo Cim. **53 B**, 385 (1979).

[20]See Arzeliès (note 19), especially p. 83. This is a result derived in the classical (von Laue's) formulation of relativistic dynamics of extended bodies. In the new formulations this effect does not exist. See: H. Arzeliès, Nuovo Cim. **35**, 792 (1965); R. Penney, Nuovo Cim. **43 A**, 911 (1966).

[21]L. Karlov, Lett. Nuovo Cim. **3**, 37 (1970); O. Gron, Lett. Nuovo Cim. **13**, 441 (1975).