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Bose-Einstein condensates relative phase measurements through atomic homodyne detection

B. R. da Cunha and M. C. de Oliveira
Instituto de Física "Gleb Wataghin," Universidade Estadual de Campinas, 13083-970, Campinas-SP, Brazil
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The dynamics of a two-mode Bose-Einstein condensate trapped in a double-well potential results approximately in an effective Rabi oscillation regime of exchange of population between both wells for sufficiently strong overlap between the modes functions and sufficiently large condensates. Facing the effective Rabi oscillation as a temporal atomic beam splitter we propose a nondestructive measurement process allowing an atomic homodyne detection, thus yielding indirect relative phase information about one of the two-mode condensates. This proposal is achieved through a secondary optical homodyne detection, in which the field is allowed to dispersively interact with one of the condensate modes inside an optical cavity.

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I. INTRODUCTION

Since the first experimental achievements of Bose-Einstein condensation with a diluted atomic gas [1–3], the investigation related to the detection of condensate relative phases and more specifically to the actual condensate quantum state determination has generated considerable literature [4–6]. A significant amount of discussion has been directed toward the detection of the relative phase of two Bose-Einstein condensates (BECs), either in the form of interference between two independent BECs [4,7–9] or due to indirect light probe of independent condensates [10]. Essentially, acquiring information about the BEC relative phase enables one to observe many interesting dynamical phenomena such as Josephson effect [9,11–21] and the transition from superfluid to Mott insulator [22]. However, the knowledge of such a phase could also be employed for partial or total inference on the condensate quantum state, through tomographic reconstruction as proposed by many authors in the past [23–25].

Indeed, an atomic interferometric device implemented with high controllable parameters can be envisaged as an atomic beam splitter [4,26,27], which would certainly be important for schemes of quantum state reconstruction of BECs [23–25]. It was previously noticed in Ref. [28] that a two-mode BEC trapped in a double-well potential could be envisaged as the atomic version of the balanced optical homodyne detection (BOHD), where the coherent tunneling of atoms would play the role of a temporal atomic beam splitter. As is well known, BOHD consists of mixing the signal field with a coherent local oscillator (LO) on a 50:50 beam splitter to yield the necessary phase sensitivity for signal field quadratures detection [29,30]. In the same sense, the phase of a signal atomic BEC mode could thus be determined by counting the difference of atoms in the wells of the trapping potential [28]. However, in Ref. [28] only an approximate description of a two-mode BEC was given by neglecting cross collision between atoms in different wells. As such, the approximate calculations were valid only for a small number of atoms (small condensate) and short time, determined by the ratio between tunneling and collision frequencies. If the cross collisions between atoms trapped in a double well potential are considered, a significant increase in the atom tunneling rate, for a large number of condensed atoms, can occur leading to an effective linear Rabi regime of population oscillation between the trap wells [21]. This regime of oscillation is optimal for atomic homodyne detection of a signal BEC phase if the number of atoms in each well can be inferred from available experimental techniques since the Josephson coupling between distinct modes of a BEC trapped in a double well potential can thus be regarded as a reliable system for realizing a temporal atomic beam splitter [27].

In this paper we develop the formal procedures for detecting a condensate relative phase through atomic homodyne detection. For the proper atomic homodyne detection process, a secondary detection process able to determine the difference of atoms in the trap wells is needed. The approach which we believe is the most promising in measuring the phase of a condensate is based on an extension of the homodyne measurements on a BEC proposed by Corney and Milburn in Ref. [14]. One of the wells of the double-well system is placed inside an optical cavity, which is far off resonance with respect to any dipole transition in the atomic sample, allowing a dispersive interaction between the light field and the atomic gas. Hence the effect of the atoms is to shift the phase of the cavity field by a given amount dependent on the balance of bosons in both wells, which may be measured by homodyne interferometry reflecting the internal dynamics of the condensate. Consequently, we simulate the homodyne current and its relation to the quadrature phase, possibly foreseeing experimental measured quantities and showing how the presence of cross collisions enables a dynamical regime ideal for such homodyne interferometry scheme. This paper is organized as follows. In Sec. II we derive the atomic system Hamiltonian and discuss the evolution of the condensate phase quadratures. In Sec. III we develop a detection model based on optical homodyne detection. In Sec. IV approximate solutions for the equations discussed in Sec. III are given, allowing the determination of the condensate relative phase through BOHD. In Sec. V the backaction on the condensate phase due to the continuous measurement process is analyzed and finally in Sec. VI a conclusion closes the paper.

II. EFFECTIVE RABI REGIME AND BEC QUADRATURES DETECTION

In this section we present the quantum dynamics of a BEC trapped in a double well potential in order to justify the homodyne measurements discussed in later sections. The condensate model used in the following discussion has been studied in previous papers [16,17] and so we only present an overview of it. Consider a double well potential trapping a Bose-Einstein condensate. The potential barrier is considered to be symmetric and the chemical potential is such that only two single-particle states are below the barrier separating the two wells, but in such a way that cross collisions between bosons of both wells may not be negligible. Those hypotheses enable a treatment of the many-body problem within a two-mode approximation. The well known bosonic many body Hamiltonian in the interaction picture is

$$\hat{H} = \int d^3r\, \hat{\Psi}^\dagger(\mathbf{r})\left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r})\right]\hat{\Psi}(\mathbf{r}) + \frac{U_0}{2}\int d^3r\, \hat{\Psi}^\dagger(\mathbf{r})\hat{\Psi}^\dagger(\mathbf{r})\hat{\Psi}(\mathbf{r})\hat{\Psi}(\mathbf{r}), \quad (1)$$

where m is the atomic mass, $U_0=4\pi\hbar^2 a/m$ measures the strength of the two-body interaction, a is the s-wave scattering length, and $\hat{\Psi}^\dagger$ and $\hat{\Psi}$ are the Heisenberg picture field operators. If we consider a dilute gas in order that only s-scattering interactions are not negligible we may define the tunneling (Ω), self-collision (κ), and cross-collision (η, Λ) rates, respectively, as

$$\Omega = \frac{2}{\hbar}\int d^3\mathbf{r}\, u_1^*(\mathbf{r})[V(r) - \tilde{V}^{(2)}(\mathbf{r}-\mathbf{r}_1)]u_2(\mathbf{r}), \quad (2)$$

$$\kappa = \frac{U_0}{2\hbar}\int d^3\mathbf{r}\, |u_i|^4, \quad (3)$$

$$\eta = \left(\frac{U_0}{2\hbar}\right)\int d^3\mathbf{r}\, u_i^* u_i u_j^* u_j, \quad (4)$$

$$\Lambda = \left(\frac{U_0}{2\hbar}\right)\int d^3\mathbf{r}\, u_j^* u_i u_i^* u_i, \quad (5)$$

where $V^{(2)}$ is the harmonic approximation of the trapping potential around each minimum and u_i is the ith mode function such that $\hat{a}_i(t)=\int d^3\mathbf{r}\, u_i^*(\mathbf{r})\hat{\psi}(\mathbf{r},t)$. It is then possible to write down a two-mode single-particle Hamiltonian as

$$\hat{H} = \hbar[2\Lambda(N-1)+\Omega][\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a}] + \hbar\eta[\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a}]^2 + \hbar(\kappa-\eta)[(\hat{a}^\dagger)^2(\hat{a}^2) + (\hat{b}^\dagger)^2(\hat{b}^2)], \quad (6)$$

where we have used the bosonic field operators relation to isomorphically map the many-body problem into the single-particle one. Note that for both η and Λ going to zero the Hamiltonian (6) accounts for the first order approximation considered in previous works [16–19]. So η and Λ account for the overlap of the wave functions on distinct wells, being thus a second order correction to these previous works. If we now introduce the Schwinger angular momentum representation

$$\hat{J}_x = \frac{1}{2}(b^\dagger b - a^\dagger a), \quad (7)$$

$$\hat{J}_y = \frac{i}{2}(b^\dagger a - a^\dagger b), \quad (8)$$

$$\hat{J}_z = \frac{1}{2}(a^\dagger b + b^\dagger a), \quad (9)$$

the Hamiltonian in Eq. (6) then becomes

$$\hat{H} = \hbar[\Omega + 2\Lambda(N-1)]\hat{J}_z + 4\hbar\eta\hat{J}_z^2 + 2\hbar(\kappa-\eta)\hat{J}_x^2, \quad (10)$$

where we have neglected terms proportional to N and N^2 since they correspond only to a shift in the energy scale. The Casimir invariant is

$$\hat{J}^2 = \frac{\hat{N}}{2}\left(\frac{\hat{N}}{2}+1\right), \quad (11)$$

which is analogous to an angular momentum model with total eigenvalue given by $j=N/2$. We may now use the Heisenberg picture to write the equations of motion for the angular momentum operators as follows

$$\dot{\hat{J}}_x = -\hbar[\Omega + 2\Lambda(N-1)]\hat{J}_y - 4\hbar\eta[\hat{J}_y,\hat{J}_z]_+, \quad (12)$$

$$\dot{\hat{J}}_y = \hbar[\Omega + 2\Lambda(N-1)]\hat{J}_x - 2\hbar(\kappa-3\eta)[\hat{J}_z,\hat{J}_x]_+, \quad (13)$$

$$\dot{\hat{J}}_z = 2\hbar(\kappa-\eta)[\hat{J}_y,\hat{J}_x]_+, \quad (14)$$

where $[\cdot,\cdot]_+$ are anticommutators. This system of differential equations can be solved numerically, and shows a number of interesting effects, such as self-trapping as discussed in Refs. [16–19] in the absence of cross-collision terms ($\eta=\Lambda=0$). Note that the cross collision terms η and Λ in Eqs. (12)–(14) can only be neglected if they are very small and moreover when N is not very large so that $\Omega > 2\Lambda(N-1)$. Whenever $N \approx \Omega/2\Lambda$ the cross-collision term Λ should be considered. Since η is of the same order of Λ, it must also be considered to correctly describe the system dynamics. In that form Eqs. (12) and (13) essentially show that when the cross collision between the localized modes is taken into account, the self-collisions rate occurring in each mode as given by κ decreases to an effective self-collision rate $\kappa'=\kappa-\eta$. Also the tunneling rate is increased as given by $\Omega+2\Lambda(N-1)$, and so dependent not only on the cross-collisional rate Λ but also on the number of atoms N in the trap. The larger N is, the larger will be the new tunneling rate.

It is easily seen from the system of equations (12)–(14) that the presence of cross collision can inhibit self-trapping in the limit where $\kappa-\eta \ll \Omega+2\Lambda(N-1)$, especially when $\eta \to \kappa$. In such a case, Eqs. (12)–(14) approximately result in

$$\dot{\hat{J}}_x = -\hbar\hat{J}_y, \quad (15)$$

$$\dot{\hat{J}}_y = \hbar \Omega' \hat{J}_x, \tag{16}$$

$$\dot{\hat{J}}_z \approx 0, \tag{17}$$

$\hat{J}_z$ thus being approximately a constant of motion. Here we have defined $\Omega' \equiv \Omega + 2\Lambda(N-1) + 8\kappa \hat{J}_z(0)$ as the new tunneling frequency, which explicitly depends on the cross-collision rate, on N, and on the initial condition for $\hat{J}_z$. This regime could in principle be attained for shallow traps, but with the potential barrier height sufficiently larger than the condensate chemical potential, $V_0 > \mu_c$, so that the two mode approximation can still be employed. The new set of equations is thus easily solved to give

$$\hat{J}_x(t) = \hat{J}_x(0)\cos \Omega' t + \hat{J}_y(0)\sin \Omega' t. \tag{18}$$

If we suppose that initially both wells are equally populated then $\hat{J}_x(0)=0$ and the last equation reduces to

$$\hat{J}_x(t) = \sin(\Omega' t)\hat{J}_y(0). \tag{19}$$

As we shall see the dynamical regime imposed by Eq. (19) is optimal for atomic homodyne detection.

By hypothesis we suppose that one of the two modes of the condensate (let us say mode B) is prepared in a coherent state [4,28,32] in such a way that $\beta = |\beta| e^{i\theta}$. $\langle \hat{J}_y \rangle$ can be rewritten as

$$\langle \hat{J}_y \rangle = \frac{i}{2}|\beta|(\langle \hat{a}^\dagger \rangle e^{i\theta} - \langle \hat{a} \rangle e^{-i\theta}) = -|\beta| \langle \hat{X}_{\theta-\pi/2} \rangle, \tag{20}$$

where $\hat{X}_{\theta-\pi/2}$ is the quadrature operator of the mode A. It is directly seen that

$$\langle \hat{J}_x(t) \rangle = |\beta| \sin(\Omega' t) \langle \hat{X}_{\theta-\pi/2} \rangle, \tag{21}$$

which is the well-known result for balanced homodyne detection [29,30] times a coherent amplitude dependent through Ω' on the geometry of the trap, the total number of particles, and the initial condition of $\hat{J}_z$. It is also interesting to write the normalized operator $\hat{S}_i \equiv \hat{J}_i / N$ with $i \in (x,y,z)$. The result in Eq. (21) means that even for a large number of atoms, the self-trapping is totally suppressed and coherent oscillation takes place. We note that in this regime the frequency of oscillation increases with the total number of bosons in the system as $\Omega' = \Omega + 2\Lambda(N-1) + 8N\kappa \hat{S}_z(0)$, in such a way that the correspondent period decreases. This effective Rabi regime allows the double well trap to be envisaged as a realization of a temporal atomic beam splitter. Hence the sine function modulating the homodyne current is an analogue to the beam-splitter transmissivity factor. For an ideal 50:50 beam splitter the optimal situation would be such that

$$\Omega' t = (2n+1)\pi/2, \tag{22}$$

where $n \in N$. It is preferable to write Eq. (21) as

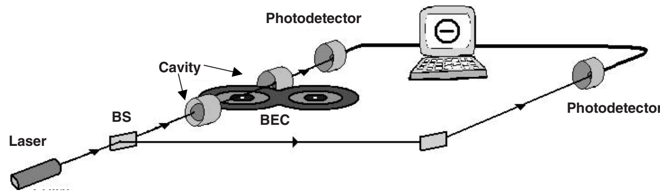

FIG. 1. Atomic homodyne detection scheme. One of the BEC modes interacts dispersively with the cavity field, after [17]. The phase shift suffered by the cavity field is dependent on the imbalance of atoms in the BEC modes. Thus a secondary BOHD on the cavity output field allows inference of the condensate relative phase.

$$\langle \hat{X}_{\theta-\pi/2} \rangle = \frac{1}{|\beta|} \langle \hat{J}_x \rangle, \tag{23}$$

since the experimentally measured quantity would be the population difference given by the right-hand side of the above equation. Note that, since the above derivation is for matter field instead of the BOHD, the quadratures are indeed given by combinations of the center of mass position and momentum operators for the mode A relative to the mode B center of mass. Thus the atomic homodyne detection would essentially correspond to BEC mode A center of mass position and momentum measurements.

III. ATOMIC HOMODYNE DETECTION

For the complete implementation of the homodyne atomic detection a measuring process sensitive to the difference of atoms in the two modes is needed. Here we propose a possible implementation by letting one of the condensate modes interact with a far off resonance cavity light field. The cavity output field is recombined at a 50:50 beam splitter in a second stage BOHD as depicted in Fig. 1. The scheme is very similar to that proposed in Ref. [17]. One of the wells of the double-well system is placed in one arm of an optical cavity. The cavity is driven by a coherent field at the cavity frequency. A dispersive interaction between the light field and the atomic gas is supposed in such a way that the field is far off resonance with respect to any dipole transition of the atomic species. Hence the effect of the atoms is to shift the phase of the cavity field by a determined amount dependent on the balance of bosons in both wells. If the atom number in the cavity oscillates, so will the phase shift. Any tunneling of the condensate will be manifested in a modulated phase shift of the optical field exiting the cavity. In order to detect the light phase shift, it is considered a common BOHD scheme. The light leaving the cavity is thus recombined with the reference beam in a 50:50 beam splitter and allowed to fall on the photodetectors.

Throughout the following calculations we assume a bad-cavity limit, where any related atomic spontaneous emission rate is much smaller than the cavity field relaxation rate thus being neglected. In that regime the cavity field is undepleted and if the cavity light field is assumed to be far-detuned from any atomic resonance, the interaction Hamiltonian is effectively given [17] by

$$\hat{H}=\int d^3r\,\hat{\Psi}^\dagger(\mathbf{r})[\hat{H}_{\text{c.m.}}-\hbar\mu g(\mathbf{r})\hat{c}^\dagger\hat{c}]\hat{\Psi}(\mathbf{r}), \tag{24}$$

where $\hat{c}$ and $\hat{c}^\dagger$ are the cavity field operators, $g(\mathbf{r})$ is the intensity mode function, and $\mu=\Omega_R^2/4\Delta$, with Rabi frequency Ω_R, optical detuning Δ, and $\hat{H}_{\text{c.m.}}$ describing the center of mass motion. We may then write the above Hamiltonian in a single-particle formalism introducing the condensate operators $\hat{a}$ and $\hat{a}^\dagger$ and averaging over the optical mode function, resulting in the following interaction Hamiltonian:

$$\hat{H}_I=-\hbar\xi\hat{c}^\dagger\hat{c}\hat{a}^\dagger\hat{a}=-\hbar\frac{N}{2}\xi\hat{c}^\dagger\hat{c}-\hbar\xi\hat{c}^\dagger\hat{c}\hat{J}_x, \tag{25}$$

where ξ is the interaction strength. Since the cavity field is undepleted the total number of photons inside the cavity is a constant of motion. However, the cavity field phase evolves with time. The phase time evolution can then be found by considering the Heisenberg equation for the photon annihilation operator if the undepleted cavity field is assumed to be in a coherent state. Thus we find that

$$\dot{\phi}=-\xi\left(\frac{N}{2}+\langle\hat{J}_x\rangle\right), \tag{26}$$

showing the direct dependence of the cavity field phase with the condensate imbalance operator $\langle\hat{J}_x\rangle$. If we now suppose the BEC is being monitored in a balanced homodyne way as in Fig. 1, then a well known result is that for balanced homodyne detection schemes, the difference between both fields arriving at the photodetectors is proportional to the phase in such a way that

$$\langle\hat{\mathcal{J}}_{xf}\rangle=-|d|\langle\hat{\mathcal{X}}_{\phi-\pi/2}\rangle, \tag{27}$$

where $\langle\hat{\mathcal{J}}_{xf}\rangle$ stands for the photon counting difference at the photodetectors, $|d|$ is the eigenvalue of the reference beam annihilation operator, and $\mathcal{X}$ is the cavity field quadrature operator. In this last equation the light field phase ϕ varies with time depending on the condensate dynamics following Eq. (26). Hence the measured photon difference gives us indirect information about the internal structure of the condensate since it relates itself directly to the relative phase of the condensate in both wells of the trapping potential. A schematic circuit involving both the atomic and the optical homodyne detection process is depicted in Fig. 2. It is clear that ϕ is a phase shift conditioned on the number of atoms in the BEC mode inside the cavity.

We can develop this proposal further by writing

$$\langle\hat{\mathcal{X}}_{\phi-\pi/2}(t)\rangle=\langle\hat{\mathcal{X}}_{\phi(t)-\pi/2}\rangle=-\frac{i}{2}(\langle\hat{c}^\dagger\rangle e^{i\phi(t)}-\langle\hat{c}\rangle e^{-i\phi(t)}) \tag{28}$$

since the field that goes through the other branch of the beam splitter does not have *a priori* time dependence, being $\phi(t)$ as given by Eq. (26). If we assume both light beams being detected in a coherent state, then

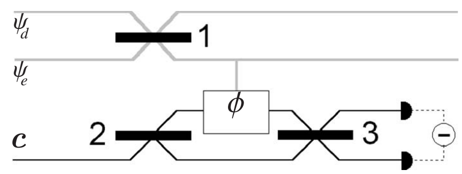

FIG. 2. Circuit representing the atomic homodyne detection process. Gray lines are for the condensate modes and dark lines for the optical fields. Beam-splitter 1 is the double-well trapping potential, while 2 and 3 are optical 50:50 beam splitters. Atoms in the mode inside the cavity interact dispersively with its field yielding a phase ϕ conditioned to the number of atoms in the mode. The light phase is detected through BOHD, thus giving information about the relative condensate phase.

$$\langle\hat{\mathcal{J}}_{xf}\rangle=-|c||d|\sin[\phi(t)]. \tag{29}$$

We now have an explicit relation between the experimentally measured quantity ($\langle\hat{\mathcal{J}}_{xf}\rangle$) and the condensate imbalance operator. In order now to access the relative condensate phase we need to obtain a relationship between the quadrature phase operator and the imbalance one. In the following we find such relationships and provide some real insight on possible experiments measuring such quantities.

IV. APPROXIMATE SOLUTIONS

Assuming again the cavity is driven by a strong coherent field and strongly damped, the cavity field is undepleted and in the $\kappa-\eta\ll\Omega'$ limit the Heisenberg equations for the BEC operators are given by

$$\dot{\hat{J}}_x=-\Omega'\hat{J}_y-4\kappa[\hat{J}_y,\hat{J}_z]_+, \tag{30}$$

$$\dot{\hat{J}}_y=\Omega'\hat{J}_x+4\kappa[\hat{J}_x,\hat{J}_z]_++\xi\hat{c}^\dagger\hat{c}\hat{J}_z, \tag{31}$$

$$\dot{\hat{J}}_z=-\xi\hat{c}^\dagger\hat{c}\hat{J}_y. \tag{32}$$

In the following we suppose that the BEC in the cavity is strongly embedded in the photon field in such a way that

$$\epsilon\equiv\frac{\kappa}{\xi N_f}\ll 1, \tag{33}$$

where $N_f\equiv\langle\hat{c}^\dagger\hat{c}\rangle$. In other words, we say that the density of the BEC is extremely small compared to that of the photons in the cavity. It is an experimental fact that such an approximation is quite correct since BEC densities range from 10^{12} to 10^{13} atoms/cm^3.

In the following, we write down a solution for the set of differential equations given by Eqs. (21)–(23) up to first order in ϵ by expanding the Schwinger operators as $\hat{J}_i=\sum_k\epsilon^k\hat{J}_i^{(k)}$. The zeroth order solution follows directly by integration and it reads

$$\langle \hat{J}_x^{(0)}(t)\rangle = \frac{\Omega'}{\omega}|\beta|\langle \hat{X}^{(0)}_{\theta-\pi/2}(t-\pi/2\omega)\rangle, \tag{34}$$

where $\omega^2 \equiv \Omega'^2 + \xi^2 N_f^2$. The first order solution may be found by considering the homogeneous solution (zeroth order) and applying the variation of parameters

$$\langle \hat{J}_x^{(1)}(t)\rangle = \frac{\Omega'|\beta|}{[1+\cos^2(2\omega t)]}\left[\frac{3t}{2} + \frac{1}{4\omega}\cos(2\omega t)\sin(2\omega t) - i\frac{3\omega}{\Omega'^2}\sin^2(\omega t)\right]\langle \hat{X}^{(1)}_{\theta-\pi/2}(t)\rangle. \tag{35}$$

Thus, up to first order in ϵ, the full solution reads

$$\langle \hat{J}_x\rangle \simeq \langle \hat{J}_x^{(0)}\rangle + \epsilon\langle \hat{J}_x^{(1)}\rangle. \tag{36}$$

This is a complicated function of time but, as was already mentioned, it shows that by measuring the imbalance of population in the wells we acquire information about the relative phase between both BEC modes as is expressed in the quadrature phase operator $\hat{X}_{\theta-\pi/2}(t)$.

We may now plug those results into Eq. (26) and integrate it to obtain in zeroth order in ϵ

$$\phi^{(0)}(t) = \xi\left(\frac{\Omega'|\beta|}{\omega^2}\langle \hat{X}^{(0)}_{\theta-\pi/2}(t)\rangle - \frac{Nt}{2}\right), \tag{37}$$

and in first order in ϵ

$$\phi^{(1)}(t) = -\xi\left\{\frac{N}{2}t + \frac{\Omega'|\beta|}{[1+\cos^2(2\omega t)]}\left[\frac{3t^2}{4} + \frac{1}{16\omega^2}\sin^2(2\omega t) - i\frac{3\omega}{2\Omega'^2}\left(t - \frac{1}{\omega}\sin(\omega t)\cos(\omega t)\right)\right]\langle \hat{X}^{(1)}_{\theta-\pi/2}(t)\rangle\right\}. \tag{38}$$

Hence, up to first order, the full solution reads

$$\phi \simeq \phi^{(0)} + \epsilon\phi^{(1)}. \tag{39}$$

As a matter of fact, it is possible to acquire information about the atomic gas quadrature from the quadrature of the light field by an atomic temporal homodyne scheme followed by a balanced light field homodyne detection. Though this solution is quite accurate, for actual experimental data it may suffice to write down the solution up to zeroth order in ϵ. Consider for instance $\kappa/\Omega = 0.02$, $\xi = 10^{-3}$ Hz, $\Omega = 10^3$ Hz, $N_f = 10^{10}$, $\Omega' = 9$ Hz, and $N = 1000$. For such a situation $\epsilon \sim 10^{-9}$, corroborating a zeroth order approximation of the problem. Henceforth the phase ϕ reads

$$\phi(t) \simeq \phi^{(0)}(t) = \xi\left(\frac{\Omega'|\beta|}{\omega^2}\langle \hat{X}^{(0)}_{\theta-\pi/2}(t)\rangle - \frac{Nt}{2}\right). \tag{40}$$

It is interesting however to write down the imbalance in terms of the light phase since that phase could possibly be measured by experimentalists giving indirect information about the atomic gas internal structure. In this sense it is easily seen that

$$\langle \hat{X}^{(0)}_{\theta-\pi/2}(t)\rangle \simeq \frac{\omega^2}{\Omega'|\beta|}\left(\frac{1}{\xi}\phi + \frac{Nt}{2}\right). \tag{41}$$

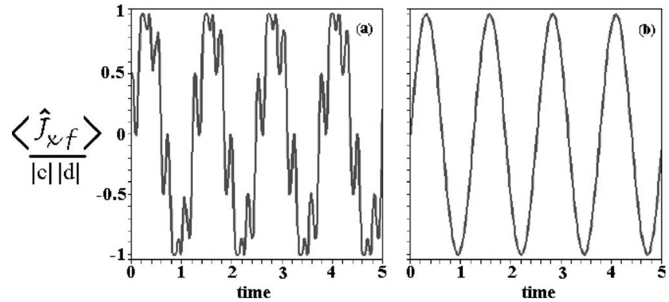

FIG. 3. Time evolution of the homodyne current as shown by Eq. (43). For both graphics $\xi = 0.01$, $\Omega' = 25$ Hz, $\omega = 30$ Hz, and $N = 10\,000$ atoms. In (a) we suppose a very large initial momentum (quadrature phase) of $\langle \hat{J}_y^{(0)}(0)\rangle = 1667$ and in (b) a small initial momentum of $\langle \hat{J}_y^{(0)}(0)\rangle = 0.001$. Time is normalized in units of Ω'.

It is possible now to write down the full zeroth order solution to the Schwinger operators mean values in order to write down an equation for the light field phase depending explicitly on time:

$$\frac{\phi}{\xi} = -\left(\frac{\Omega'}{\omega^2}\cos(\omega t)\langle \hat{J}_y^{(0)}(0)\rangle + \frac{Nt}{2}\right), \tag{42}$$

meaning that for sufficiently large initial momentum $\langle \hat{J}_y^{(0)}(0)\rangle$ the harmonic behavior should dominate and for smaller values the regime should be linear in time.

For typical data of the system described here the zeroth order terms in ϵ dominate and only coherent oscillations are observed for the imbalance of the atomic population given by the mean value of the Schwinger operator $\hat{J}_x$. In this approximation, the relationship between both phases is linear as we choose specific intervals of time when the potential barrier in the condensate acts as an ideal 50:50 temporal atomic beam splitter. These results make the model presented here an ideal system for optimal detection of the condensed phase via two homodyne detections: One temporal atomic one and a second one on the cavity output light field as depicted in Figs. 1 and 2.

It is now possible to write down an expression which shows us how the light counting difference at the detectors is directly related to the internal structure of the condensate. Combining Eqs. (41) and (42) it is readily seen that

$$\langle \hat{\mathcal{J}}_{xf}\rangle = |c||d|\sin\left[\xi\left(\frac{\Omega'}{\omega^2}\cos(\omega t)\langle \hat{J}_y^{(0)}(0)\rangle + \frac{Nt}{2}\right)\right]. \tag{43}$$

The above expression shows clearly the relationship between the measured homodyne current and the condensate quadrature phase. In this sense, it is possible to measure the homodyne current $\langle \hat{\mathcal{J}}_{xf}\rangle$ and then acquire the necessary phase info about the condensate via Eq. (42). It is easy then to see that the evolution of the homodyne current with the condensate initial momentum (quadrature phase) is simply sinusoidal and its time evolution is quite similar but with the envelope function being modulated as shown in Fig. 3 for two different choices of the initial momentum. The condensate relative phase is thus more evident for larger initial momenta. Such a result is evidently optimal for homodyne measurements and reflects the effective Rabi dynamics discussed previously.

V. MEASUREMENT BACKACTION

We have considered a measurement process which allows the inference of a condensate relative phase through optical phase detection. However, as is well known, the continuous detection process induces a backaction into the condensate, altering thus the condensate phase during the measurement process. We now analyze those effects by assuming the more realistic situation in that the cavity is driven by a strong coherent field of strength ς and is strongly damped at the rate γ. The procedure follows closely that by Corney and Milburn [17]. Hence the unconditioned evolution of the system (light field+ BEC) is governed by the following master equation (taking $\hbar=1$):

$$\dot{\hat{\rho}}_{\rm tot} = -i[\hat{H}_I,\hat{\rho}_{\rm tot}] + i\xi[\hat{c}^\dagger\hat{c}\hat{J}_x,\hat{\rho}_{\rm tot}] - i\left(\delta - \frac{N\xi}{2}\right)[\hat{c}^\dagger\hat{c},\hat{\rho}_{\rm tot}] - i\epsilon[\hat{c}^\dagger+\hat{c},\hat{\rho}_{\rm tot}] + \frac{\gamma}{2}(2\hat{c}\hat{\rho}_{\rm tot}\hat{c}^\dagger - \hat{c}^\dagger\hat{c}\hat{\rho}_{\rm tot} - \hat{\rho}\hat{c}^\dagger\hat{c}), \quad (44)$$

where the initial detuning $\delta=N\xi/2$ was chosen in order to remove the N linear dependent dispersion.

It is possible to eliminate adiabatically the optical field from the master equation as in Refs. [17,33], under the hypotheses that the driving and damping terms dominate over the coupling one. This process leads to the master equation in terms of the atomic variables alone

$$\dot{\hat{\rho}} = \frac{-i}{\hbar}[\hat{H}_i,\hat{\rho}] - \frac{\Gamma}{2}[\hat{J}_x,[\hat{J}_x,\hat{\rho}]] + \mathcal{O}(\epsilon_0^3), \quad (45)$$

where the measurement strength is $\Gamma=16\xi^2\varsigma^2/\gamma^2$ and $|\xi|c_0|\langle\hat{J}_x\rangle/\gamma|=\epsilon_0\ll 1$ (large damping). We may then observe the ensemble-averaged effect of the measurement in the operator moment equations for $\eta\to\kappa$ and $\kappa-\eta\ll\Omega'$, up to first order in ϵ:

$$\langle\dot{\hat{J}}_x\rangle \simeq -\Omega'\langle\hat{J}_y\rangle, \quad (46)$$

$$\langle\dot{\hat{J}}_y\rangle \simeq \Omega'\langle\hat{J}_x\rangle + \xi|c_0|^2\langle\hat{J}_z\rangle - \frac{\Gamma}{2}\langle\hat{J}_y\rangle, \quad (47)$$

$$\langle\dot{\hat{J}}_z\rangle \simeq -\xi|c_0|^2\langle\hat{J}_y\rangle - \frac{\Gamma}{2}\langle\hat{J}_z\rangle. \quad (48)$$

Those equations are the same for the case $\kappa=0$ (coherent oscillation) considered in Ref. [17] but now the oscillation regime is attained in the presence of self- and cross collisions, for $\eta\to\kappa$ and $\kappa-\eta\ll\Omega'$. We may now solve numerically this set of equations and with the aid of Eqs. (26) and (29) find numerically the dependence of the homodyne current with the condensate phase quadrature. In Fig. 4 we depict the preselected homodyne current as a function of time for $\Gamma/\Omega'=0.0001$ and $\eta/\Omega'=0.04$. It is observed in general that even with the light field typical damping of the preselected state, the phase information of the field (relative to the condensate quadrature) is still present and shows similar behavior to that in Fig. 3(a). The chosen initial momentum for the condensate is such that the oscillatory behavior shown in Fig. 4 is due both to the linear term $N\xi t/2$ as well as the term proportional to the initial momentum in the sine argument of Eq. (43).

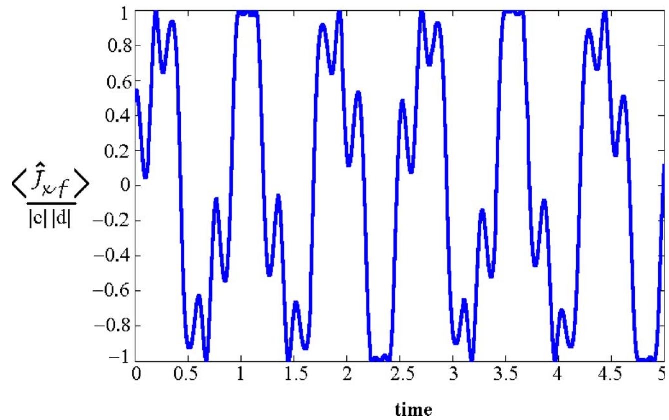

FIG. 4. (Color online) Numerical calculation for the unconditional evolution of the homodyne current when $\Gamma/\Omega'=0.0065$ and $\xi|c_0|^2/\Omega'=0.04$. Time is normalized in units of Ω'.

We now consider the postselected dynamics of the cavity field plus condensate system. It is usual then to numerically simulate stochastic realizations of quantum trajectories as already pointed out by several authors [34–38]. The resultant stochastic process is a diffusive evolution rather than the jump processes, which occur in the direct detection of atoms or individual photons since we have a condensate system continuously monitored by the optical homodyne detection scheme. In the presence of cross collisions the conditional master equation (which corresponds to an average over many runs of the experiment and many homodyne current records) for the optical field is given by

$$\left(\frac{d\hat{\rho}_c}{dt}\right)_{\rm field} = \gamma\mathcal{D}[c]\hat{\rho}_c + \sqrt{\gamma}\frac{dW(t)}{dt}\mathcal{H}[c]\hat{\rho}_c, \quad (49)$$

where $dW(t)$ is the infinitesimal Wiener increment [38], $\hat{\rho}_c$ is the density matrix that is conditioned on a particular realization of the homodyne current up to time t, and $\mathcal{D}$ and $\mathcal{H}$ are the Wiseman's superoperators [37]. The conditional stochastic Schrödinger equation then follows

$$d|\tilde{\Psi}_c(t)\rangle = dt\left[-i\hat{H} - \frac{\Gamma}{2}\hat{J}_x^2 + I(t)\hat{J}_x\right]|\tilde{\Psi}_c(t)\rangle, \quad (50)$$

where $\hat{H}$ is given by Eq. (10) plus Eq. (25) and $\tilde{\Psi}_c(t)$ describes the conditional state of the system. The measured photocurrent is

$$I(t) = 2\Gamma\langle\hat{J}_x\rangle_c + \sqrt{\Gamma}\mathcal{A}(t), \quad (51)$$

where the stochastic term $\mathcal{A}(t)$ has the correlations $\langle\mathcal{A}(t)\rangle=0$ and $\langle\mathcal{A}(t),\mathcal{A}(t')\rangle=\delta(t-t')$.

From Eq. (50) we see that the presence of cross collisions in the $\eta\to\kappa$ $(\kappa-\eta\ll\Omega')$ limit introduces only a harmonic

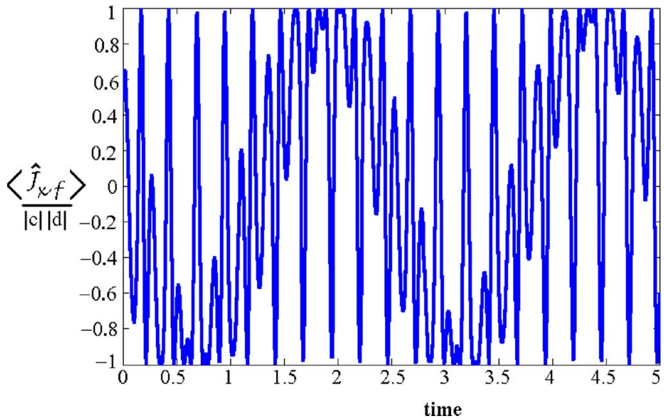

FIG. 5. (Color online) Numerical calculation for the conditional evolution of the homodyne current when $\Gamma/\Omega'=0.0001$ and $\eta/\Omega'=0.04$. Time is normalized in units of Ω'.

correction due to the $\eta\hat{J}_z^2$ term in the Hamiltonian. Then again, the cross collisions reproduce the homodyne interference pattern expected when there are no collisions at all ($\kappa=0$). Numerical simulations then show similar results to that found by Corney and Milburn [17] for only coherent oscillation dynamics ($\kappa=0$) corroborating the fact that such a system in the effective Rabi regime is optimal for homodyne detection in the sense that it attains a purely coherent oscillation of population dynamics without collapse and revival. The numerical simulations depicted in Fig. 5 show as well how the experimental measured quantity given by Eq. (29) evolves in time under the conditioned detection for $\Gamma/\Omega'=0.0001$ and $\eta/\Omega'=0.04$. We then observe that the homodyne current changes considerably in relation to the oscillatory unconditioned evolution. This is expected since the measurement backaction alters considerably the whole system state. An efficient detection process with a larger initial condensate momentum would allow the evidence of the condensate phase. It may then be possible to experimentally confirm such results by measuring the homodyne current and then inverting Eq. (29) in order to access the imbalance of population between both wells and then to acquire the desired information about the relative quadrature phase of the Bose-Einstein condensates.

VI. CONCLUDING REMARKS AND GENERAL DISCUSSION

We have shown that in the effective Rabi regime of a double-well atomic BEC [31] an optimal condition for atomic homodyne detection scheme is found which gives indirect measurement of the condensate relative phase. The double-well potential barrier acts as a temporal atomic beam splitter with the transmissivity factor varying with time and depending directly on the total number of bosons and cross-collisions strength by the corrected frequency Ω' [27]. Up to first order in ϵ, the Heisenberg equations of motion for the mean values of the Schwinger operators are exactly soluble even when the interaction with the light field is considered strong for sufficiently strong light intensities.

Typical experimental data show that it may suffice to consider only zeroth order terms in the calculations which result in a linear relationship between the light phase and the condensate quadrature. In this sense it is supposed that the light phase may be detected with the aid of the scheme proposed in Figs. 1 and 2. It consists of a two stage homodyne detection: One optical and the other on the state of one of the two-mode condensates. Hence we believe that such a system in this dynamical regime (effective Rabi) might be the appropriate choice to indirectly detect the relative phase between the two modes of a BEC in a double well potential in the form of Josephson-like tunneling in a regime of purely coherent exchange of population between both wells due to the strong presence of cross collisions. Such a conclusion is strongly supported by the calculations (analytical and numerical) discussed in this paper.

Recently an outstanding experiment was realized based on stimulated light scattering to continuously sample the relative phase of two spatially separated atomic BECs that never interact [10]. Our proposal, on the other hand, imposes that the two atomic BEC modes must be overlapping in order that the effective (stable) Rabi regime ($\kappa-\eta\ll\Omega'$) be attained. In light of Eqs. (26) and (29), we expect experimentalists to be able to measure the relative phase with present technologies on trapping potentials. Experiments should then be able to detect the relative condensate phase, as given by the model presented here, possibly opening new frontiers in quantum phase engineering. We expect that these results may be useful in further experimental and theoretical studies on the state of a Bose-Einstein condensate as well as to applications on atom optics when such systems are extrapolated to an array of BECs. As a last comment, our results are certainly relevant for reconstruction and measurement of atomic quantum states [23–25] and may be useful for future implementations on quantum communication protocols [28,39].

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