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ERROR-BLOCK CODES AND POSET METRICS

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Abstract. Let $P = (\{1, 2, \ldots, n\}, \leq)$ be a poset, let $V_1, V_2, \ldots, V_n$ be a family of finite-dimensional spaces over a finite field $\mathbb{F}_q$ and let

$$V = V_1 \oplus V_2 \oplus \ldots \oplus V_n.$$

In this paper we endow V with a poset metric such that the P-weight is constant on the non-null vectors of a component V_i, extending both the poset metric introduced by Brualdi et al. and the metric for linear error-block codes introduced by Feng et al.. We classify all poset block structures which admit the extended binary Hamming code $[8; 4; 4]$ to be a one-perfect poset block code, and present poset block structures that turn other extended Hamming codes and the extended Golay code $[24; 12; 8]$ into perfect codes. We also give a complete description of the groups of linear isometries of these metric spaces in terms of a semi-direct product, which turns out to be similar to the case of poset metric spaces. In particular, we obtain the group of linear isometries of the error-block metric spaces.

1. Introduction

Classically, coding theory takes place in finite-dimensional linear spaces $\mathbb{F}_q^n$ over a finite field $\mathbb{F}_q$ that are equipped with a metric, the most common ones being the Hamming and Lee metrics. One of the main problems of the theory is to find a k-dimensional subspace in $\mathbb{F}_q^n$, the space of n-tuples over the finite field $\mathbb{F}_q$, with the largest possible minimum distance.

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In Hamming spaces this problem has a matricial version, which was generalized by Niederreiter in 1987 (see [13]). Inspired in this work, Brualdi, Graves and Lawrence (see [3]) provided in 1995 a wider setting for the same problem: using partially ordered sets and defining the concept of poset-codes, they introduced the concept of codes with a poset-metric. This has been a fruitful approach, since many new perfect codes have been found with such poset metrics (see [1], [3], [8], [7] and [11]). The existence of new perfect codes is related to the fact that the packing radius with respect to a poset metric is greater than the packing radius with respect to the Hamming metric.

A particular and important instance of poset-codes and poset metric spaces are the spaces introduced by Rosenbloom and Tsfasman in 1997 (see [18]), where the posets taken into consideration have a finite number of disjoint chains of equal size. These metrics are useful in the case of interference in several consecutive channels, starting from the last, which are occupied by a priority user. This poset space has been investigated by several authors, such as Skriganov [19], Quistorff [17], Ozen and Siap [14], Lee [10], Dougherty and Skriganov [5] and Panek, Firer and Alves [15].

Another generalization of the classic Hamming distance was recently proposed by Feng, Xu and Hickernell, the so-called π-distance (or π-metric) (see [6]). As opposed to what happens with a poset metric, the packing radius of a given code with respect to a π-distance is smaller than its Hamming packing radius.

In this work we show how the problem with the packing radius can be ameliorated when a π-metric is weighted by a partial order P, just as it was done in [3] with the Hamming metric. We combine the usual poset metric on a vector space, proposed by Brualdi et al. in [3] and studied by several authors in the following, with the recently introduced error-block metric by Feng et al. in [6]. In section two we describe how it can be used to turn classical codes (extended binary Hamming code $[8;4;4]$ and extended binary Golay code $[24;12;8]$) into perfect codes. In section three we determine and describe the group of linear isometries of a poset block space and finally, in the last section, we work out the cases when the block and poset structures are considered separately.

2. Poset block metric spaces

Let $[n] := \{1, 2, \ldots, n\}$ be a finite set with n elements and let $\leq$ be a partial order on $[n]$. We call the pair $P := ([n], \leq)$ a *poset*. We say that k is *smaller than* j if $k \leq j$ and $k \neq j$. An *ideal* in $([n], \leq)$ is a subset $I \subseteq [n]$ that contains every element that is smaller than or equal to some of its elements, i.e., if $j \in I$ and $k \leq j$ then $k \in I$. Given a subset $X \subset [n]$, we denote by $\langle X \rangle$ the smallest ideal containing X, called the *ideal generated by* X; if $X = \{i\}$ then we write $\langle i \rangle$ instead of $\langle X \rangle$ or $\langle \{i\} \rangle$. We denote by $\langle i \rangle^*$ the *difference* $\langle i \rangle - \{i\} = \{j \in [n] : j < i\}$.

Two posets P and Q are *isomorphic* if there exists an order-preserving bijection $\phi : P \to Q$, called an *isomorphism*, whose inverse is order preserving; that is, $x \leq y$ in P if and only if $\phi(x) \leq \phi(y)$ in Q. An isomorphism $\phi : P \to P$ is called an *automorphism*.

Now let

$$\pi : [n] \to \mathbb{N}$$

be a map such that $\pi(i) > 0$ for all $i \in [n]$. We will call this map π a *label*[*] over $[n]$. If $k_i = \pi(i)$, we take V_i as the $\mathbb{F}_q$-vector space $V_i = \mathbb{F}_q^{k_i}$ for every $1 \le i \le n$, and we define the vector space V as the direct sum

$$V := V_1 \oplus V_2 \oplus \ldots \oplus V_n$$

which is isomorphic to $\mathbb{F}_q^N$, where $N = k_1 + k_2 + \ldots + k_n$. Each vector in V can be written in a unique way as

$$v = v_1 + v_2 + \ldots + v_n,$$

$v_i \in V_i$ for $1 \le i \le n$. Denoting by $B_i = \{e_{i1}, e_{i2}, \ldots, e_{ik_i}\}$ the canonical basis of V_i, $1 \le i \le n$, each vector v_i in V_i can be written uniquely as

$$v_i = a_{i1}e_{i1} + a_{i2}e_{i2} + \ldots + a_{ik_i}e_{ik_i},$$

$a_{ij} \in \mathbb{F}_q$, $1 \le j \le k_i$.

Given a poset $P = ([n], \le)$ and $v = v_1 + v_2 + \ldots + v_n \in V$, the *$\pi$-support of v* is the set

$$\operatorname{supp}(v) := \{i \in [n] : v_i \neq 0\},$$

and we define the *(P,π)-weight* of v to be the cardinality of the smallest ideal containing $\operatorname{supp}(v)$:

$$w_{(P,\pi)}(v) = |\langle \operatorname{supp}(v) \rangle|.$$

If u and v are two vectors in $\mathbb{F}_q^N$, then their *(P,π)-distance* is defined by

$$d_{(P,\pi)}(x,y) = w_{(P,\pi)}(x-y).$$

If

$$\Theta_j(i) = \{I \subseteq P : I \text{ ideal}, \ |I| = i, \ |\operatorname{Max}(I)| = j\}$$

where $\operatorname{Max}(I)$ is the set of maximal elements in the ideal $I \subseteq P$ and

$$B_{(P,\pi)}(u;r) = \left\{v \in V : d_{(P,\pi)}(u,v) \le r\right\}$$

is the *ball with center u and radius r*, then the number of vectors in a ball of radius r equals

$$\left|B_{(P,\pi)}(u;r)\right| = 1 + \sum_{i=1}^{r} \sum_{j=1}^{i} \sum_{I \in \Theta_j(i)} \prod_{m \in \operatorname{Max}(I)} \left(q^{k_m} - 1\right) \prod_{l < m; m \in \operatorname{Max}(I)} q^{k_l}.$$

The number of vectors in a ball of radius r does not depend on its center.

An $\left[N;k;\delta_{(P,\pi)}\right]$ *linear poset block code* is a k-dimensional subspace $C \subseteq \mathbb{F}_q^N$, where $\mathbb{F}_q^N$ is endowed with a poset block metric $d_{(P,\pi)}$ and

$$\delta_{(P,\pi)}(C) = \min\left\{w_{(P,\pi)}(c) : 0 \neq c \in C\right\}$$

is the *(P,π)-minimum distance* of the code C.

The (P,π)-distance is a metric[†] on V which combines and extends both the usual poset metric on a vector space, proposed by Brualdi et al. in [3] and studied by

[*]The pair (P,π) can be identified with a quoset (quasi-ordered set); see [2], for instance.

[†]It is clear that the (P,π)-distance is symmetric and positive defined. We now claim that the (P,π)-distance satisfies the triangle inequality. In fact, if $u, v \in \mathbb{F}_q^N$ then

$$\begin{aligned} d_P(u,v) &= |\langle \operatorname{supp}(u-v) \rangle| = |\langle \operatorname{supp}(u+z-z-v) \rangle| \\ &\le |\langle \operatorname{supp}(u-z) \rangle \cup \langle \operatorname{supp}(z-v) \rangle| \\ &\le |\langle \operatorname{supp}(u-z) \rangle| + |\langle \operatorname{supp}(z-v) \rangle| \\ &= d_P(u,z) + d_P(z,v) \end{aligned}$$

for all $z \in \mathbb{F}_q^N$.

several authors, and the recently introduced error-block metric by Feng et al. in [6]; we will call $(V, d_{(P,\pi)})$ a *poset block metric space*. When the label π satisfies $\pi(i) = 1$ for all $i \in [n]$ the (P,π)-distance is the poset metric d_P proposed by Brualdi et al. and when P is the antichain order of n elements, i.e., $i \leq j$ in P if and only if $i = j$, the (P,π)-distance is the error-block metric d_π proposed by Feng et al.. In case both conditions occur ($\pi(i) = 1$ for all $i \in [n]$ and P is the antichain order), the poset-block-metric reduces to the usual Hamming metric d_H of classical coding theory. In this case, whenever needed to stress that we refer to the Hamming space, we use the index H to denote the Hamming metric d_H, the parameters of a linear code, $[N; k; \delta_H]_H$, and the support $\mathrm{supp}_H(u) = \{i : u_i \neq 0\}$ of a vector $u = (u_1, u_2, \ldots, u_N) \in \mathbb{F}_q^N$.

3. Perfect poset block codes

Let d be a metric on V and let C be a subset of V. The *packing radius* $R_d(C)$ of C is the greatest positive real number r such that any two balls of radius r centered at (distinct) elements of C are disjoint. We say a code C is $R_d(C)$*-perfect* if the union of the balls of radius $R_d(C)$ centered at the elements of C covers all V.

The number of vectors in (V, d_H), (V, d_P) and (V, d_π) whose distance to a fixed vector $u \in V$ is at most equal to r, respectively, is given by

$$|B_H(u;r)| = \sum_{i=0}^{r} \binom{n}{i} (q-1)^i,$$

$$|B_P(u;r)| = 1 + \sum_{i=1}^{r} \sum_{j=1}^{i} (q-1)^j q^{i-j} \Omega_j(i)$$

and

$$|B_\pi(u;r)| = 1 + \sum_{i=1}^{r} \sum_{\substack{J \subset [n] \\ |J|=i}} \prod_{m \in J} \left(q^{k_m} - 1\right),$$

where $\Omega_j(i)$ equals the number of ideals of P with cardinality i having exactly j maximal elements.

Note that

$$B_P(u;r) \subseteq B_H(u;r) \subseteq B_\pi(u;r)$$

for any $u \in V$; this implies

$$R_{d_\pi}(C) \leq R_{d_H}(C) \leq R_{d_P}(C).$$

In [7] Hyun and Kim, based on the second of the inequalities above, classified all the posets P that make the extended binary Hamming code a 2-perfect code or a 3-perfect code. For spaces with a π-metric we have a more delicate situation: the packing radius of the extended binary Hamming code is equal either to zero or one (details in the example below). In this sense, the (P,π)-metrics improve this situation. In the following, we list all poset block metrics that turn the extended binary Hamming code $[8,4,4]_H$ into a 1-perfect code and some orders that turn the extended binary Golay code $[24,12,8]_H$ into a 1-perfect or 2-perfect code. We begin by classifying all perfect codes over V when P is a chain.

Proposition 3.1. *Let π be a label over $[n]$ and $V = V_1 \oplus V_2 \oplus \ldots \oplus V_n$ a vector space such that $\dim(V_i) = \pi(i)$ for each $1 \leq i \leq n$. Consider on V the (P,π)-metric* *where P is the linear order defined by $1 < 2 < \ldots < n$. Then, a linear code $C \subseteq V$ is r-perfect iff there is a linear transformation*

$$L : V_{r+1} \oplus V_{r+2} \oplus \ldots \oplus V_n \to V_1 \oplus V_2 \oplus \ldots \oplus V_r$$

such that

$$C = \{(L(v), v) : v \in V_{r+1} \oplus V_{r+2} \oplus \ldots \oplus V_n\}.$$

Proof. Indeed, we know that $w_{(P,\pi)}(u) = \max\{i : u_i \neq 0\}$, hence

$$B_{(P,\pi)}(0; r) = V_1 \oplus V_2 \oplus \ldots \oplus V_r.$$

Given such a linear transformation L, we have that

$$C = \{(L(v), v) : v \in V_{r+1} \oplus V_{r+2} \oplus \ldots \oplus V_n\}$$

is a linear code and $(L(v), v) = (0, 0)$ iff $v = 0$, so that $\delta_{(P,\pi)}(C) = r + 1$. It follows that $B_{(P,\pi)}(0; r)$ does not contain any element of C other then its center. Moreover, it is immediate to see that, given

$$0 \neq c = (L(v), v) \in C$$

then

$$B_{(P,\pi)}(c; r) = \{(y, v) : y \in V_1 \oplus \cdots \oplus V_r\}$$

is disjoint from $B_{(P,\pi)}(0; r)$. Since

$$|C| = |V_{r+1} \oplus V_{r+2} \oplus \ldots \oplus V_n|$$

and

$$\left|B_{(P,\pi)}(0; r)\right| = |V_1 \oplus V_2 \oplus \ldots \oplus V_r|$$

it follows that C is an r-perfect code.

Assuming now that C is r-perfect. Given $(u, v), (u', v) \in C$, then $(u, v) - (u', v) = (u - u', 0) \in C$. So the weight of $(u - u', 0) \in C$ is at most r, which implies $(u - u', 0) \in B_{(P,\pi)}(0; r)$. Since C is r-perfect it follows that $u - u' = 0$. Therefore every element $v \in V_{r+1} \oplus V_{r+2} \oplus \ldots \oplus V_n$ determines a unique element $\tilde{v} \in C$ and hence determines a function $L(v)$ such that $\tilde{v} = (L(v), v)$. Since C is a linear subspace of V it follows that L is a linear transformation. □

We note that if $\pi(i) = 1$ for $i = 1, 2, \ldots, n$ then we get the poset space $\left(\mathbb{F}_q^n, d_P\right)$ over the chain P; this result shows that there are more perfect codes in this space than the ones described in [11, Corollary 3.2].

Example 3.2. Let $\pi : [n] \to \mathbb{N}$ be a label such that $\pi(1) + \pi(2) + \ldots + \pi(n) = 2^m$ and define $m_j = \pi(1) + \pi(2) + \ldots + \pi(j)$ for $j \in \{1, 2, \ldots, n\}$ and $m_0 = 0$. Let $V = V_1 \oplus V_2 \oplus \ldots \oplus V_n$ be a vector space such that $\dim(V_i) = \pi(i)$ for each $1 \leq i \leq n$. Note that $v \in V_j$ if and only if $\mathrm{supp}_H(v) \subset \{m_{j-1} + 1, \ldots, m_{j-1} + \pi(j) - 1, m_j\}$.

We denote by $\mathcal{H}(m)$ the $[2^m; 2^m - 1 - m; 4]_H$ extended binary Hamming code (see [12]). Let

$$\mathcal{B} = \{\mathrm{supp}(c) : c \in \mathcal{H}(m), w_H(c) = 4\}$$

be the set of the supports of all minimal weight codewords in $\mathcal{H}(m)$ and $\mathcal{P} := \{1, 2, \ldots, 2^m\}$. It is well known (see [12]) that the pair $(\mathcal{P}, \mathcal{B})$ is a 3-$(2^m, 4, 1)$ design, that is, given a subset $X \subset \mathcal{P}$ with three elements, there is a unique block $\mathrm{supp}(c) \in \mathcal{B}$ such that $X \subset \mathrm{supp}(c)$.

Suppose there is some $i \in \{1, 2, \ldots, n\}$ such that $\pi(i) = 2$. Since supports of the codewords of minimum weight 4 in $\mathcal{H}(m)$ form a 3-$(2^m, 4, 1)$ design, there is a minimal codeword $c \in \mathcal{H}(m)$ satisfying

$$|\mathrm{supp}_H(c) \cap \{m_{i-1} + 1, m_i\}| = 2.$$

It follows that $w_\pi(c) \leq 3$ and hence

$$R_{d_\pi}(\mathcal{H}(m)) = \left\lfloor \frac{d_\pi(\mathcal{H}(m)) - 1}{2} \right\rfloor \leq 1.$$

Suppose now that $\pi(i) > 2$ for some $i \in \{1, 2, \ldots, n\}$. The design structure of the pair $(\mathcal{P}, \mathcal{B})$ implies the existence of a codeword $c \in C$ such that $w_H(c) = 4$ and such that

$$|\mathrm{supp}(c) \cap \{m_{i-1} + 1, m_{i-1} + 2, \ldots, m_i\}| \geq 3$$

which implies $w_\pi(c) \leq 2$ and hence

$$R_{d_\pi}(\mathcal{H}(m)) = \left\lfloor \frac{d_\pi(\mathcal{H}(m)) - 1}{2} \right\rfloor = 0.$$

Let $\mathcal{H}(3)$ be the $[8; 4; 4]_H$ extended binary Hamming code. Then a parity check matrix for $\mathcal{H}(3)$ is

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let $\pi : [s] \to \mathbb{N}$ be a label such that $\pi(1) + \pi(2) + \ldots + \pi(s) = 8$ and $\pi(i) = 4$ for some $i \in \{1, 2, \ldots, s\}$ (note that $1 < s \leq 5$). It follows from the last example that the packing radius of $\mathcal{H}(3)$ with respect to a block metric d_π is zero. This situation can be avoided if we endow $V = \mathbb{F}_2^8$ with a poset block metric.

Given $X \subseteq [8]$, we define V_X to be the subspace

$$V_X = \left\{ v \in \mathbb{F}_2^8 : \mathrm{supp}_H(v) \subseteq X \right\}.$$

Since the supports of the codewords of minimum weight of $\mathcal{H}(3)$ form a $3-(8, 4, 1)$ design, there is $X' \subseteq [8]$, with $|X'| = 4$, such that $|\mathrm{supp}_H(c) \cap X'| \leq 3$ for every $c \in \mathcal{H}(3)$ with $w_H(c) = 4$. We denote by

$$\Gamma^{(1)}(P) := \{j \in [s] : |\langle j \rangle| = 1\}$$

the set of minimal elements of the poset $P = ([s], \preceq)$.

Theorem 3.3. *Let X' be as above, π be a label on $[s]$ such that*

$$\pi(1) + \pi(2) + \ldots + \pi(s) = 8$$

and $V = V_1 \oplus V_2 \oplus \ldots \oplus V_s$ with V_j isomorphic to $\mathbb{F}_2^{\pi(j)}$ for all $j \in [s]$, where $V_i = V_{X'}$. Then an order $P = ([s], \preceq)$ turns the extended binary Hamming code $\mathcal{H}(3)$ into a 1-perfect code if and only if $\Gamma^{(1)}(P) = \{i\}$, where $\pi(i) = 4$, and the block V_i does not contain any codeword of minimum weight.

Proof. Let $X' \subset [8]$ be as above and assume that $\Gamma^{(1)}(P) = \{i\}$, where $\pi(i) = 4$ and that V_i does not contain any codeword of minimum weight. We claim first that the (P, π)-minimum weight of $\mathcal{H}(3)$ is at least 2. In fact, X' was chosen in a way that no non-zero vector of $\mathcal{H}(3)$ has its support contained in X'; since i is the only minimal element of P, any non-zero $c \in H(3)$ has a non-zero coordinate $j > i$ and its (P, π)-weight is at least 2.

The balls of radius 1 in $\left(V, d_{(P,\pi)}\right)$ have the right size: since $I = \langle i \rangle = \{i\}$ is the only ideal in $P = ([n], \leq)$ which has only one element and $\dim\left(V_{X'}\right) = 4$,

$$\left|B_{(P,\pi)}(u;1)\right| = 1 + \left(2^4 - 1\right) = 2^4.$$

We claim now that the balls of radius 1 centered at elements of $\mathcal{H}(3)$ are pairwise disjoint. Suppose that there are $u \in V$ and $c \in \mathcal{H}(3)$ such that $d_{(P,\pi)}(0,u) \leq 1$ and $d_{(P,\pi)}(c,u) \leq 1$. The first inequality yields $\mathrm{supp}_H(u) \subseteq X'$; hence $w_{(P,\pi)}(c-u) = d_{(P,\pi)}(c,u) \geq 2$, which is a contradiction. It follows that

$$B_{(P,\pi)}(c;1) \cap B_{(P,\pi)}(c';1) = \varnothing$$

for every $c \neq c' \in \mathcal{H}(3)$. From this and from the fact that

$$\left|B_{(P,\pi)}(u;1)\right| \cdot |\mathcal{H}(3)| = 2^4 \cdot 2^4 = 2^8,$$

we conclude that $\mathcal{H}(3)$ is a 1-perfect code.

Assume now that (P,π) is a poset block structure that turns $\mathcal{H}(3)$ into a 1-perfect code. If there is a minimal coordinate $i \in \Gamma^{(1)}(P)$ such that the corresponding block space has dimension $k_i > 4$, then

$$\left|B_{(P,\pi)}(0;1)\right| \geq 1 + \left(2^{k_i} - 1\right) = 2^{k_i} > 2^4,$$

and hence $\mathcal{H}(3)$ cannot be 1-perfect, since the code has 2^4 elements of length 8 and $2^{k_i} \cdot 2^4 > 2^8$.

Suppose now that $\left|\Gamma^{(1)}(P)\right| = r > 1$. Let $k_1, k_2, \ldots, k_r$ be the dimension of the corresponding block spaces. In this case we have that

$$\left|B_{(P,\pi)}(0;1)\right| = 1 + \sum_{i=1}^{r}\left(2^{k_i} - 1\right) = 1 - r + \sum_{i=1}^{r} 2^{k_i}.$$

Since the code is 1-perfect, we must have that $1 - r + \sum_{i=1}^{r} 2^{k_i} = 2^4$ or equivalently $\sum_{i=1}^{r} 2^{k_i} = 15 + r$. Being the sum in this last equation an even number, we can discard the cases when r is also even and so we are left with the cases $r = 3, 5$ or 7. Considering that $\sum_{i=1}^{r} k_i \leq 7$, direct computations show that the above equality cannot hold if $r = 5$ or 7 and, if $r = 3$, it holds only if $(k_1, k_2, k_3) = (3,3,1)$ (up to permutation) and if there is a unique coordinate i_0 such that $|\langle i_0 \rangle| = 2$. In this case, there is a codeword c of minimum Hamming weight such that $i_0 \notin \mathrm{supp}_H(c)$. In every binary linear code either the i-th coordinate c_i is 0 for each codeword c, or half the codewords have $c_i = 0$; since $\mathcal{H}(3)$ has 16 codewords, 14 of which are of minimum weight, there is $c \in \mathcal{H}(3)$ such that $w_H(c) = 4$ and $i_0 \notin \mathrm{supp}_H(c)$. Hence $\mathrm{supp}_H(c) = \{i_1, i_2, i_3, i_4\} \subset \Gamma^{(1)}(P)$ and $w_{(P,\pi)}(c) = 1$. Now, if $v \neq c$ is any vector with $w_{(P,\pi)}(v) = 1$, then $w_{(P,\pi)}(c - v) = 1$ and hence $B_{(P,\pi)}(0;1) \cap B_{(P,\pi)}(c;1) \neq \varnothing$ and the code is not 1-perfect. It follows that if P turns $\mathcal{H}(3)$ into a 1-perfect code then $\left|\Gamma^{(1)}(P)\right| = 1$.

Let $\Gamma^{(1)}(P) = \{i\}$. We already know that $k_i \leq 4$. Since $|B(0;1)| = 2^{k_i}$, if $k_i < 4$ it follows that the poset block structure (P,π) does not turn $\mathcal{H}(3)$ into a 1-perfect code. Assuming $k_i = 4$, we find that the block space V_i cannot contain any codeword of minimum weight $c \in \mathcal{H}(3)$, since this would imply $w_{(P,\pi)}(c) = 1$. □

We remark that if P, in the theorem 3.3, is a chain, the extended Hamming code $\mathcal{H}(3)$ is one of the codes described in Proposition 3.1 (as it should be). Reordering the blocks if necessary, we may take $i = 1$ $(V_{X'} = V_1)$; denoting the remaining component $V_2 \oplus V_3 \oplus \ldots \oplus V_s$ by W we can write $V = V_1 \oplus W$. Consider the canonical projection

$$T : V \to W$$

defined as $T(c_1, c_2, \ldots, c_8) = (c_5, c_6, c_7, c_8)$. Since $\ker(T) = V_1$ and no non-zero codeword $v \in \mathcal{H}(3)$ is contained in V_1, $\ker(T) \cap \mathcal{H}(3) = 0$ and therefore $S := T|_{V_1}$ is a linear isomorphism from V_1 onto W. It follows that

$$\mathcal{H}(3) = \left\{ (S^{-1}(v), v) : v \in W \right\}.$$

Example 3.4. Let $\pi : [n] \to \mathbb{N}$ be a label such that $\pi(1) + \pi(2) + \ldots + \pi(n) = 24$ and $\pi(i) = 2$ for some $i \in \{1, 2, \ldots, n\}$, and define $m_j = \pi(1) + \pi(2) + \ldots + \pi(j)$ for $j \in \{1, 2, \ldots, n\}$. Let $V = V_1 \oplus V_2 \oplus \ldots \oplus V_n$ be a vector space such that $\dim(V_i) = \pi(i)$ for each $1 \le i \le n$. Note that $v \in V_j$ if and only if $\mathrm{supp}_H(v) \subset \{m_{j-1}+1, \ldots, m_{j-1}+\pi(j)-1, m_j\}$.

Let $\mathcal{G}_{24}$ be the $[24; 12; 8]_H$ extended binary Golay code (see [12]) and $c \in \mathcal{G}_{24}$ be such that $w_H(c) = 8$. As the supports of the codewords of weight 8 in $\mathcal{G}_{24}$ form a 5-$(24, 8, 1)$ design (see [12]), we can choose c in such a way that

$$|\mathrm{supp}_H(c) \cap \{m_{i-1}+1, m_i\}| = 2.$$

Under these conditions we have that $w_\pi(c) \le 7$ and therefore

$$R_{d_\pi}(\mathcal{G}_{24}) = \left\lfloor \frac{d_\pi(\mathcal{G}_{24}) - 1}{2} \right\rfloor \le 3.$$

We remark that if $\pi(i) > 2$ then $R_{d_\pi}(\mathcal{G}_{24}) < 3$: since the supports of the codewords of minimum weight of $\mathcal{G}_{24}$ form a 5-$(24, 8, 1)$ design, there is $c \in \mathcal{G}_{24}$ such that

$$|\mathrm{supp}_H(c) \cap \{m_{i-1}+1, m_{i-1}+2, \ldots, m_i\}| \ge 3$$

and therefore $w_\pi(c) < 7$.

However, there are non-trivial poset-block structures in $[24]$ that turn $\mathcal{G}_{24}$ into a 1-perfect code. We just need a subset $Y \subset [24]$ with $|Y| = 12$ that does not contain the support of any codeword of minimum weight of $\mathcal{G}_{24}$. There is at least one such subset; otherwise, every subset of 12 elements contains the support of a codeword of minimum weight. For each 12-subset, pick one vector whose support is contained in it; since each 8-subset is contained in $\binom{16}{4}$ of the 12-subsets, there should be at least $\binom{24}{12}/\binom{16}{4}$ codewords of minimum weight in $\mathcal{G}_{24}$; since $\binom{24}{12}/\binom{16}{4} > 749$, the number of codewords of minimum weight in $\mathcal{G}_{24}$, there is at least one subset Y with 12 elements containing no codewords of minimum weight of the Golay code.

Let Y be as above and consider a label $\pi : [s] \to \mathbb{N}$ such that $\pi(1) + \pi(2) + \ldots + \pi(s) = 24$ and $\pi(i) = 12$ for some $i \in [s]$ with $V_i = V_Y$. If $\Gamma^{(1)}(P) = \{i\}$, then we have that

$$B_{(P,\pi)}(0; 1) = \left\{ v \in \mathbb{F}_2^{24} : \mathrm{supp}(v) \subset Y \right\} \tag{1}$$

and hence $\left| B_{(P,\pi)}(0; 1) \right| = 2^{|Y|} = 2^{12}$. If c is a codeword of minimum weight then $w_{(P,\pi)}(c) \ge 2$, because $\mathrm{supp}(c)$ must have an element not contained in Y and Y is the only block of height 1. On the other hand, if $v \in B_{(P,\pi)}(0, 1)$, it follows from (1) that $\mathrm{supp}(c - v) \nsubseteq Y$, and hence that $w_{(P,\pi)}(c - v) \ge 2$. We conclude that $B_{(P,\pi)}(0, 1) \cap B_{(P,\pi)}(c, 1) = \varnothing$ for every $c \in \mathcal{G}_{24}$. Since

$$\left| B_{(P,\pi)}(0, 1) \right| \cdot |\mathcal{G}_{24}| = 2^{24}$$

we find that $\mathcal{G}_{24}$ is 1-perfect for any poset block structure satisfying the following condition: it has a unique block of weight 1 with 12 elements that does not contain any codeword of minimum weight of the Golay code.

A similar reasoning shows that $\mathcal{G}_{24}$ is a 2-perfect code with a poset block structure such that $\Gamma^{(1)}(P)$ and $\Gamma^{(2)}(P)$ have each a unique block V_i and V_j respectively, such that $\dim(V_i)+\dim(V_j)=12$ and $V_i \oplus V_j$ does not contain (non-zero) codewords of $\mathcal{G}_{24}$.

4. Groups of linear isometries

Let $\left(V, d_{(P,\pi)}\right)$ be a poset block space. A *linear isometry* T of the metric space $\left(V, d_{(P,\pi)}\right)$ is a linear transformation $T: V \to V$ that preserves (P,π)-distance:

$$d_{(P,\pi)}\left(T(u), T(v)\right) = d_{(P,\pi)}(u,v)$$

for every $u, v \in V$. Equivalently, a linear transformation T is an isometry if $w_{(P,\pi)}(T(u)) = w_{(P,\pi)}(u)$ for every $u \in V$. A linear isometry of $\left(V, d_{(P,\pi)}\right)$ is said to be a (P,π)-*isometry*. Since an isometry must be injective, a linear isometry is an invertible map and it is easy to see that its inverse is also a linear isometry. It follows that the set of all linear isometries of the poset block space $\left(V, d_{(P,\pi)}\right)$ is a group. We denote it by $GL_{(P,\pi)}(V)$ and call it *the group of linear isometries of* $\left(V, d_{(P,\pi)}\right)$.

Linear isometries are used to classify linear codes in equivalence classes, since they take linear code onto linear code and preserve length, dimension, minimum distance and other parameters. So it is just natural to call two linear codes *equivalent* if one is the image of the other under a linear isometry.

In [4], [18], [10] and [16] the groups of linear isometries (with label $\pi(i) = 1$ for all $i \in P$) were determined for the Rosenbloom-Tsfasman space, generalized Rosenbloom-Tsfasman space, crown space and arbitrary poset-space respectively. In [15] we describe the full symmetry group (which includes non-linear isometries) of a poset block space (with constant label equal to 1) that is a product of Rosenbloom-Tsfasman spaces. In this work, we give a complete description of the groups of linear isometries, for any given label π and poset P.

We remark that the initial idea is the same as in [16]: to associate to each isometry T an automorphism ϕ_T of the underlying poset P (Theorem 4.10). The main differences are that we follow a more coordinate-free approach and that the dimensions of the blocks pose a new restraint. We first study two subgroups of isometries, one of isometries induced by automorphisms of P that preserve labels and the other of isometries that induce the identity map on P. Next we prove some results on linear isometries analogous to those of [16], plus a result on preservation of block dimensions, and conclude that $GL_{(P,\pi)}(V)$ is the semi-direct product of those subgroups.

4.1. Two subgroups of linear isometries.

In this section we present two subgroups of linear isometries of $(V, d_{(P,\pi)})$. Afterwards it will be shown that $GL_{(P,\pi)}(V)$ is the semi-direct product of these groups.

Let $B = \{e_{i,j} : 1 \leq i \leq n, 1 \leq j \leq k_i\}$ be a basis for V where for each i, $B_i = \{e_{i,j} : 1 \leq j \leq k_i\}$ is the canonical basis of $V_i = \mathbb{F}_q^{k_i}$.

Given a poset $P = ([n], \leq)$ we denote by $\operatorname{Aut}(P)$ the group of order automorphisms of P.

Definition 4.1. Let $\pi : [n] \to \mathbb{N}$ be a label and $P = ([n], \leq)$ be a poset. The subgroup of automorphisms $\phi \in \mathrm{Aut}\,(P)$ such that

$$k_{\phi(i)} = \pi(\phi\,(i)) = \pi(i) = k_i$$

for all $i \in [n]$ is denoted by $\mathrm{Aut}\,(P, \pi)$ and is called the *group of automorphisms of* (P, π).

To each $\phi \in \mathrm{Aut}\,(P, \pi)$ we associate the linear mapping $T_\phi : V \to V$ defined by

$$T_\phi(e_{i,j}) = e_{\phi(i),j}.$$

Note that the definition of T_ϕ only makes sense if $\dim\,(V_i) = \dim\left(V_{\phi(i)}\right)$, i.e, if $k_{\phi(i)} = k_i$, and this is why we only use automorphisms that preserve labels.

Proposition 4.2. *Let ϕ be an automorphism of (P, π). The linear mapping T_ϕ associated to ϕ is a linear isometry of $\left(V, d_{(P,\pi)}\right)$, and the map $\Phi : \mathrm{Aut}(P, \pi) \to GL_{(P,\pi)}(V)$ defined by $\phi \mapsto T_\phi$ is an injective group homomorphism.*

Proof. Let $v = \sum_{i,j} a_{ij} e_{i,j} \in V$. Then

$$\begin{aligned} \mathrm{supp}\,(T_\phi(v)) &= \mathrm{supp}\left(\sum_{i,j} a_{ij} e_{\phi(i),j}\right) \\ &= \{\phi(i) \in P : a_{ij} \neq 0 \text{ for some } j\} \\ &= \{\phi(i) \in P : i \in \mathrm{supp}(v)\}\,. \end{aligned}$$

Since ϕ is an automorphism of P, if $I = \langle \mathrm{supp}(v) \rangle$, then $|I| = |\phi(I)|$ and

$$\phi(I) = \langle \{\phi(i) : i \in \mathrm{supp}(v)\} \rangle = \langle \mathrm{supp}\,(T_\phi(v)) \rangle.$$

Hence T_ϕ preserves (P, π)-weights. The map $\phi \mapsto T_\phi$ is trivially a homomorphism, for

$$T_{\phi\sigma}(e_{i,j}) = e_{(\phi\sigma)(i),j} = T_\phi(e_{\sigma(i),j}) = T_\phi T_\sigma(e_{i,j})$$

and injectivity is also straightforward from the definition of Φ. □

From the last result we conclude also that the image of Φ is a subgroup of $GL_{(P,\pi)}(V)$, isomorphic to $\mathrm{Aut}(P, \pi)$, which will be called $\mathcal{A}$ from here on. Note also that $T_\phi(V_i) = V_{\phi(i)}$.

Given $X \subseteq P$, we define V_X to be the subspace

$$V_X = \{v \in V : \mathrm{supp}(v) \subseteq X\}.$$

Proposition 4.3. *Let $T : V \to V$ be a linear isomorphism that satisfies the following condition: for each non-zero vector $v_i \in V_i$ there are a non-zero $v_i' \in V_i$ and a vector $u_i \in V_{\langle i \rangle^*}$ such that $T(v_i) = v_i' + u_i$. Then T is a linear isometry of $\left(V, d_{(P,\pi)}\right)$.*

Proof. Note that $T(V_i) \subseteq V_{\langle i \rangle}$. Let $v = v_1 + \ldots + v_n$. We have

$$T(v) = (v_1' + u_1) + \ldots + (v_n' + u_n)$$

and $T(v_j) = v_j' + u_j$ with $v_j' \neq 0$ for all j such that $v_j \neq 0$.

Let

$$u_l = u_l^1 + \ldots + u_l^n$$

be the the canonical decomposition of u_l in V, where $u_l^j \in V_j$. Note that if $u_l^i \neq 0$ then $i < l$, because $u_l \in V_{\langle l\rangle^*}$. Then the decomposition of $T(v)$ is

$$T(v) = \sum_i \left(v_i' + (u_1^i + \cdots + u_n^i)\right).$$

Let M be the set of maximal elements of $\langle \mathrm{supp}(v)\rangle$. Clearly, $M \subseteq \mathrm{supp}(v)$. Note that if $i \in M$ then all u_k^i are zero for each k, because if $u_k^i \neq 0$ then $k \in \mathrm{supp}(v)$ and $i < k$, but i is maximal in $\mathrm{supp}(v)$.

Suppose that there is $i \in M$ such that $i \notin \mathrm{supp}(T(v))$. The decomposition of $T(v)$ yields

$$v_i' + u_1^i + \ldots + u_n^i = 0.$$

But each $u_k^i = 0$, and we conclude that $v_i' = 0$, a contradiction. Hence $M \subset \mathrm{supp}(T(v))$,

$$\langle \mathrm{supp}(v)\rangle = \langle M\rangle \subseteq \langle \mathrm{supp}(T(v))\rangle$$

and $w_{(P,\pi)}(T(v)) \geq w_{(P,\pi)}(v)$.

Now let j be maximal in $\mathrm{supp}(T(v))$. The j-th component of $T(v)$ is

$$v_j' + (u_1^j + \cdots + u_n^j).$$

If $u_l^j \neq 0$ then $l \in \mathrm{supp}(v)$ and $j < l < i$ for some $i \in M$, which implies j is not maximal, contradiction. Hence all u_k^j are zero, $v_j' \neq 0$, and $j \in M$. Therefore $w_{(P,\pi)}(T(v)) = w_{(P,\pi)}(v)$. $\square$

Let $\mathcal{T}$ be the set of mappings defined in the previous proposition. We will prove in Theorem 4.10 that $\mathcal{T}$ is a subgroup of $GL_{(P,\pi)}(V)$. We can also obtain a matricial version of this group.

Now let $B = (B_{i_1}, B_{i_2}, \ldots, B_{i_n})$ be a total ordering of the basis of V such that B_{i_s} appears before B_{i_r} whenever $|\langle i_s\rangle| < |\langle i_r\rangle|$ for all $i_r, i_s = 1, 2, \ldots, n$. Renaming the elements of $P = ([n], \leq)$ if necessary, we can suppose that $i_r = r$ for all $r = 1, 2, \ldots, n$. In this manner, $B = (B_1, B_2, \ldots, B_n)$ and if $|\langle s\rangle| < |\langle r\rangle|$ then all elements of B_s come before the elements of B_r.

Theorem 4.4. *Let $B_i = \{e_{i,j} : 1 \leq j \leq k_i\}$ be a basis of V_i, $B = (B_1, \ldots, B_n)$ be an ordered basis of V where $|\langle r\rangle| \leq |\langle s\rangle|$ implies $r \leq_{\mathbb{N}} s$. If $T \in \mathcal{T}$ then*

$$T(e_{i,j}) = \sum_{s \leq i} \sum_{t=1}^{k_s} a_{st}^{ij} e_{s,t}$$

where each block $\left(a_{ri}^{rj}\right)_{1 \leq i \leq k_r}^{1 \leq j \leq k_r}$, $r = 1, 2, \ldots, n$, is an invertible matrix. Every element of $\mathcal{T}$ is represented as an upper-triangular matrix with respect to B.

Proof. Since $T \in \mathcal{T}$ we have that $T(V_i) \subseteq V_{\langle i\rangle}$. So

$$\begin{aligned} T(e_{1,1}) &= a_{11}^{11} e_{1,1} + \ldots + a_{1k_1}^{11} e_{1,k_1} \\ T(e_{1,2}) &= a_{11}^{12} e_{1,1} + \ldots + a_{1k_1}^{12} e_{1,k_1} \\ &\vdots \\ T(e_{1,k_1}) &= a_{11}^{1k_1} e_{1,1} + \ldots + a_{1k_1}^{1k_1} e_{1,k_1} \end{aligned}$$

$$\begin{aligned}
T\left(e_{2,1}\right) &= \left(a_{11}^{21}e_{1,1}+\ldots+a_{1k_1}^{21}e_{1,k_1}\right)+\left(a_{21}^{21}e_{2,1}+\ldots+a_{2k_2}^{21}e_{2,k_2}\right)\\
T\left(e_{2,2}\right) &= \left(a_{11}^{22}e_{1,1}+\ldots+a_{1k_1}^{22}e_{1,k_1}\right)+\left(a_{21}^{22}e_{2,1}+\ldots+a_{2k_2}^{22}e_{2,k_2}\right)\\
&\ \vdots\\
T\left(e_{2,k_2}\right) &= \left(a_{11}^{2k_2}e_{1,1}+\ldots+a_{1k_1}^{2k_2}e_{1,k_1}\right)+\left(a_{21}^{2k_2}e_{2,1}+\ldots+a_{2k_2}^{2k_2}e_{2,k_2}\right)
\end{aligned}$$

$$\vdots$$

$$\begin{aligned}
T\left(e_{n,1}\right) &= \left(a_{11}^{n1}e_{1,1}+\ldots+a_{1k_1}^{n1}e_{1,k_1}\right)+\ldots+\left(a_{n1}^{n1}e_{n,1}+\ldots+a_{nk_n}^{n1}e_{n,k_n}\right)\\
T\left(e_{n,2}\right) &= \left(a_{11}^{n2}e_{1,1}+\ldots+a_{1k_1}^{n2}e_{1,k_1}\right)+\ldots+\left(a_{n1}^{n2}e_{n,1}+\ldots+a_{nk_n}^{n2}e_{n,k_2}\right)\\
&\ \vdots\\
T\left(e_{n,k_n}\right) &= \left(a_{11}^{nk_n}e_{1,1}+\ldots+a_{1k_1}^{nk_n}e_{1,k_1}\right)+\ldots+\left(a_{n1}^{nk_n}e_{n,1}+\ldots+a_{nk_n}^{nk_n}e_{n,k_n}\right)
\end{aligned}$$

where $\left(a_{s1}^{ij},a_{s2}^{ij},\ldots,a_{sk_s}^{ij}\right)=0$ if $s\not\leq i$ and $\left(a_{i1}^{ij},a_{i2}^{ij},\ldots,a_{ik_i}^{ij}\right)\neq 0$ for all $i\in\{1,2,\ldots,n\}$. Therefore, if $[T]_{B_r}^{s}=\left(a_{si}^{rj}\right)_{1\leq i\leq k_s}^{1\leq j\leq k_r}$, $r,s\in\{1,2,\ldots,n\}$, then the matrix $[T]_B$ of T relative to the base B has the form

$$[T]_B=\begin{pmatrix}
[T]_{B_1}^1 & [T]_{B_2}^1 & [T]_{B_3}^1 & \cdots & [T]_{B_3}^1\\
0 & [T]_{B_2}^2 & [T]_{B_3}^2 & \cdots & [T]_{B_3}^2\\
0 & 0 & [T]_{B_3}^3 & \cdots & [T]_{B_3}^3\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & 0 & \cdots & [T]_{B_n}^n
\end{pmatrix}$$

where $[T]_{B_r}^s=0$ if $s\not\leq r$ and $[T]_{B_r}^r\neq 0$ for all $r\in\{1,2,\ldots,n\}$. To see that each $[T]_{B_i}^i$ is invertible, we notice that $[T]_B$ is assumed to be invertible, so that $0\neq\det\left([T]_B\right)$. But $\det\left([T]_B\right)=\prod_i\det\left([T]_{B_i}^i\right)$ and it follows that each $[T]_{B_i}^i$ is an invertible matrix. □

4.2. Group of linear isometries of $(V,d_{(P,\pi)})$.

Lemma 4.5. *Let $T\in GL_{(P,\pi)}(V)$ and $0\neq v_i\in V_i$. If $j\in\operatorname{supp}\left(T\left(v_i\right)\right)$ then $|\langle j\rangle|\leq|\langle i\rangle|$.*

Proof. By assumption $\langle j\rangle\subseteq\langle\operatorname{supp}\left(T\left(v_i\right)\right)\rangle$. It follows from this and (P,π)-weight preservation that $|\langle j\rangle|\leq|\langle\operatorname{supp}\left(T\left(v_i\right)\right)\rangle|=|\langle\operatorname{supp}\left(v_i\right)\rangle|=|\langle i\rangle|$. □

An ideal I of a poset P is said to be a *prime ideal* if it contains a unique maximal element.

Lemma 4.6. *If $T\in GL_{(P,\pi)}(V)$ and $0\neq v_i\in V_i$ then $\langle\operatorname{supp}\left(T\left(v_i\right)\right)\rangle$ is a prime ideal.*

Proof. We will first show that there is an element $j\in\operatorname{supp}\left(T\left(v_i\right)\right)$ satisfying $|\langle j\rangle|=|\langle i\rangle|$. Assume the contrary, namely that $|\langle j\rangle|<|\langle i\rangle|$ for every $j\in\operatorname{supp}\left(T\left(v_i\right)\right)$. If $\operatorname{supp}\left(T\left(v_i\right)\right)=\{i_1,i_2,\ldots,i_s\}$ then

$$T\left(v_i\right)=v_{i_1}+\ldots+v_{i_s}$$

with $0 \neq v_{i_k} \in V_{i_k}$, $k \in \{1, 2, \ldots, s\}$ and, by assumption, $|\langle i_k \rangle| < |\langle i \rangle|$ for $k \in \{1, 2, \ldots, s\}$. It follows from the linearity of T^{-1} that we have $\{i\} = \operatorname{supp}(v_i) \subseteq \cup_{k=1}^{s} \operatorname{supp}\left(T^{-1}(v_{i_k})\right)$, which implies $i \in \operatorname{supp}\left(T^{-1}(v_{i_l})\right)$ for some $l \in \{1, 2, \ldots, s\}$. Using this and Lemma 4.5, we obtain $|\langle i \rangle| \leq |\langle i_l \rangle| < |\langle i \rangle|$, which is a contradiction. Hence, there is $j \in \operatorname{supp}(T(v_i))$ such that $|\langle i \rangle| = |\langle j \rangle|$. By the (P, π)-weight preservation, such an element j is unique and the result follows. □

Lemma 4.7. *If $T \in GL_{(P,\pi)}(V)$, $i \leq j$ and $0 \neq v_t \in V_t$ for $t = i, j$, then*

$$\langle \operatorname{supp}(T(v_i)) \rangle \subseteq \langle \operatorname{supp}(T(v_j)) \rangle .$$

Proof. Lemma 4.6 states that $\langle \operatorname{supp}(T(v_i)) \rangle$ and $\langle \operatorname{supp}(T(v_j)) \rangle$ are prime ideals, so there are elements k and l such that $\langle k \rangle = \langle \operatorname{supp}(T(v_i)) \rangle$ and $\langle l \rangle = \langle \operatorname{supp}(T(v_j)) \rangle$. If $k = l$, then we are done, so assume that $k \neq l$. This means that either

$$k \in \operatorname{supp}(T(v_i) - T(v_j)) \text{ or } l \in \operatorname{supp}(T(v_i) - T(v_j)) .$$

We have three cases to consider.

(1) $k \notin \operatorname{supp}(T(v_i) - T(v_j))$. In this case, $k \in \operatorname{supp}(T(v_j))$. It follows that $\langle \operatorname{supp}(T(v_i)) \rangle = \langle k \rangle \subseteq \langle \operatorname{supp}(T(v_j)) \rangle$.

(2) $l \notin \operatorname{supp}(T(v_i) - T(v_j))$. In this case, $l \in \operatorname{supp}(T(v_i))$, so $l < k$. Hence, $\langle \operatorname{supp}(T(v_j)) \rangle = \langle l \rangle \subsetneq \langle k \rangle = \langle \operatorname{supp}(T(v_i)) \rangle$, so

$$w_{(P,\pi)}(v_j) = w_{(P,\pi)}(T(v_j)) < w_{(P,\pi)}(T(v_i)) = w_{(P,\pi)}(v_i) .$$

However, the hypothesis $i \leq j$ implies $w_{(P,\pi)}(v_i) \leq w_{(P,\pi)}(v_j)$, a contradiction.

(3) $k, l \in \operatorname{supp}(T(v_i) - T(v_j))$. In this case,

$$\begin{aligned} |\langle k, l \rangle| &\leq |\langle \operatorname{supp}(T(v_i) - T(v_j)) \rangle| \\ &= |\langle \operatorname{supp}(T(v_i - v_j)) \rangle| \\ &= |\langle \operatorname{supp}(v_i - v_j) \rangle| = |\langle i, j \rangle| . \end{aligned}$$

By hypothesis, $i \leq j$, so $|\langle k, l \rangle| \leq |\langle j \rangle| = |\langle \operatorname{supp}(v_j) \rangle| = |\langle \operatorname{supp}(T(v_j)) \rangle| = |\langle l \rangle|$. We conclude that $\langle k \rangle \subseteq \langle l \rangle$, that is, $\langle \operatorname{supp}(T(v_i)) \rangle \subseteq \langle \operatorname{supp}(T(v_j)) \rangle$. □

Proposition 4.8. *If $T \in GL_{(P,\pi)}(V)$ then, for each $i \in [n]$, there is a unique j in $[n]$ such that $|\langle i \rangle| = |\langle j \rangle|$ and*

(*i*) *For each non-zero $v \in V_i$, $T(v) = v' + u'$, where v' is a non-zero vector in V_j and $u' \in V_{\langle j \rangle^*}$.*

(*ii*) *$T(V_{\langle i \rangle}) \subseteq V_{\langle j \rangle}$.*

Proof. Let $0 \neq v \in V_i$; Lemma 4.6 provides $j \in [n]$ such that $T(v) \in V_{\langle j \rangle}$ and $|\langle i \rangle| = |\langle j \rangle|$. We will show that j depends only on i. If $u \in V_i, u \neq 0, u \neq v$, then there is $k \in [n]$ such that $T(u) \in V_{\langle k \rangle}$ and $|\langle i \rangle| = |\langle k \rangle|$. Since

$$|\langle i \rangle| = w_{(P,\pi)}(u - v) = w_{(P,\pi)}(T(u) - T(v)) \geq |\langle j, k \rangle| \geq |\langle j \rangle| = |\langle i \rangle|$$

we conclude that $|\langle j, k \rangle| = |\langle j \rangle|$ and therefore $k = j$. Hence, $T(V_i) \subset V_{\langle j \rangle}$, with $|\langle i \rangle| = |\langle j \rangle|$. Since T preserves weights, if $v \neq 0$ then $T(v) = v' + u'$, where $0 \neq v' \in V_j$ and $u' \in V_{\langle j \rangle^*}$.

Suppose now that $v \in V_{\langle i \rangle^*}$; then $v = v_{i_1} + \cdots + v_{i_k}$, where $i_l < i$ for each l. It follows from Lemma 4.7 that

$$\langle \operatorname{supp}(T(v_{i_l})) \rangle \subseteq \langle \operatorname{supp}(T(v) \rangle = \langle j \rangle.$$

Hence

$$\langle \mathrm{supp}(T(v))\rangle = \bigcup_{l=1}^{k} \langle \mathrm{supp}(T(v_{i_l}))\rangle \subseteq \langle j\rangle$$

and therefore $T(V_{\langle i\rangle}) \subseteq V_{\langle j\rangle}$. ☐

Theorem 4.9. *Let $T : V \to V$ be an automorphism of (V, d_P), let $i \in P$ and let j be the unique element of P determined by $T(V_i) \subseteq V_{\langle j\rangle}$ and $|\langle i\rangle| = |\langle j\rangle|$. Then $\dim(V_i) = \dim(V_j)$.*

Proof. Let T, i and j be as above.

Since $T(V_i) \subseteq V_{\langle j\rangle}$, we may consider T as map from V_i into $V_{\langle j\rangle}$. Being $V_{\langle j\rangle^*}$ a subspace of $V_{\langle j\rangle}$, we can form the quotient space $V_{\langle j\rangle}/V_{\langle j\rangle^*}$. Since every element of $V_{\langle j\rangle}$ is expressed in a unique manner as $v_j + u_j$, where $v_j \in V_j$ and $u_j \in V_{\langle j\rangle^*}$, that quotient space is isomorphic to V_j via the map $v_j + u_j + V_j \mapsto v_j$. Therefore we have a sequence of linear maps

$$V_i \to V_{\langle j\rangle} \to \frac{V_{\langle j\rangle}}{V_{\langle j\rangle^*}} \to V_j$$

where the first map is T, the second is the canonical projection and the last one is the isomorphism above. The composite map is injective because if $0 \neq v \in V_i$ then $T(v) \notin V_{\langle j\rangle^*}$. Hence $\dim(V_i) \leq \dim(V_j)$.

On the other hand, $T^{-1}(V_j) \subseteq V_{\langle i\rangle}$. In fact, if $v \in V_i$, $v \neq 0$, and $T(v) = v' + u'$, then $T^{-1}(v') = v - T^{-1}(u')$. Hence $i \in \mathrm{supp}(T^{-1}(v'))$ and, since

$$w_{(P,\pi)}(T^{-1}(v')) = w_{(P,\pi)}(v') = |\langle j\rangle| = |\langle i\rangle|,$$

it follows that $\langle \mathrm{supp}(T^{-1}(v'))\rangle = \langle i\rangle$. We conclude from Proposition 4.8 that $T^{-1}(V_j) \subseteq V_{\langle i\rangle}$; switching the roles of V_i and V_j we get an injective map from V_j into V_i, and this proves that the dimensions are equal. ☐

Theorem 4.10. *Let $T \in GL_{(P,\pi)}(V)$, and consider the map $\phi_T : P \to P$ given by $\phi_T(i) := \max \langle \mathrm{supp}(T(v_i))\rangle$, where v_i is an arbitrary non-zero vector in V_i. Then*

- *(i) ϕ_T is an automorphism of the labelled poset (P, π).*
- *(ii) The map $\Phi : GL_{(P,\pi)}(V) \to \mathrm{Aut}(P)$ given by $T \mapsto \phi_T$ is a group homomorphism from $GL_{(P,\pi)}(V)$ onto $\mathrm{Aut}(P, \pi)$ with kernel equal to $\mathcal{T}$. In particular, $\mathcal{T}$ is a normal subgroup of $GL_{(P,\pi)}(V)$.*
- *(iii) The map $\iota : \mathrm{Aut}(P, \pi) \to GL_{(P,\pi)}(V)$ given by $\iota(\phi) = T_\phi$ satisfies $\Phi \circ \iota(\phi) = \phi$ for all $\phi \in \mathrm{Aut}(P, \pi)$ (i.e., ι is a section of Φ).*

Proof. The map ϕ_T is well-defined by Proposition 4.8 and Lemma 4.7 assures that ϕ_T is an order preserving map.

We claim that ϕ_T is one-to-one. In fact, let us suppose that $\phi_T(i) = \phi_T(j)$ with $i \neq j$. Since $\phi_T(i) = \max \langle \mathrm{supp}(T(v_i))\rangle$ and $\phi_T(j) = \max \langle \mathrm{supp}(T(v_j))\rangle$, $0 \neq v_i \in V_i$ and $0 \neq v_j \in V_j$, it follows that

$$\langle \mathrm{supp}(T(v_i))\rangle = \langle \mathrm{supp}(T(v_j))\rangle .$$

By the (P, π)-weight preservation and the linearity of T,

$$|\langle i, j\rangle| = |\langle \mathrm{supp}(T(v_i + v_j))\rangle| = |\langle \mathrm{supp}(T(v_i) + T(v_j))\rangle| . \tag{2}$$

But

$$\langle \mathrm{supp}(T(v_i) + T(v_j))\rangle \subseteq \langle \mathrm{supp}(T(v_i))\rangle \cup \langle \mathrm{supp}(T(v_j))\rangle$$

and since both ideals in the right hand are assumed to be equal and T is an isometry, it follows that

$$|\langle \text{supp}\,(T\,(v_i) + T\,(v_j))\rangle| = |\langle \text{supp}\,(T\,(v_i))\rangle| = |\langle i\rangle| \tag{3}$$

and

$$|\langle \text{supp}\,(T\,(v_i) + T\,(v_j))\rangle| = |\langle \text{supp}\,(T\,(v_j))\rangle| = |\langle j\rangle| \,. \tag{4}$$

From equations (2), (3) and (4) it follows that

$$|\langle i, j\rangle| = |\langle i\rangle| = |\langle j\rangle| \,.$$

On the other hand, if $i \neq j$ and $|\langle i\rangle| = |\langle j\rangle|$, then $|\langle i, j\rangle| > |\langle i\rangle|$. This contradiction proves that $i = j$ and we conclude that ϕ_T is one-to-one.

Since P is finite, it follows that ϕ_T is a bijection that preserves order, that is, an order automorphism. Theorem 4.9 shows that ϕ_T lies in $\text{Aut}(P, \pi)$, and this takes care of the first item.

Consider now $T, S \in GL_{(P,\pi)}(V)$, $i \in P$ and $v \in V_i$ a non-zero vector as usual. We write $\phi_T(i) = j$ and $\phi_S(j) = k$. This means that $T(v) = v_j + u_j$, with $v_j \in V_j$, $v_j \neq 0$, and $u_j \in V_{\langle j\rangle^*}$, and $S(v_j) = v_k + u_k$, where v_k and u_k satisfy analogous conditions. Now

$$ST(v) = S(v_j + u_j) = v_k + S(u_j)$$

and, since $w_{(P,\pi)}(u_j) < w_{(P,\pi)}(v_j) = w_{(P,\pi)}(v_k)$, it follows that $w_{(P,\pi)}(S(u_j)) < w_{(P,\pi)}(v_k)$. Since $S(V_j) \subseteq V_{\langle k\rangle^*}$ and $w_{(P,\pi)}(S(u_j)) < w_{(P,\pi)}(v_k) = |\langle k\rangle|$ it follows that $S(u_k) \in V_{\langle k\rangle^*}$ and $ST(v) = v_k + u'_k$, with $v_k \in V_k$, $v_k \neq 0$, and $u'_k = S(u_k) \in V_{\langle k\rangle^*}$. Hence $\phi_{ST}(i) = \phi_S\phi_T(i)$ and Φ is a group homomorphism.

Given $\phi \in \text{Aut}(P, \pi)$, $\Phi(T_\phi) = \phi$. This proves that Φ is surjective and that $\Phi \circ \iota(\phi) = \phi$ for all $\phi \in \text{Aut}(P, \pi)$, i.e., ι is a section of Φ.

Finally, $\mathcal{T} \subseteq ker(\Phi)$ because by definition $T(V_i) \subseteq V_{\langle i\rangle}$ for each $T \in \mathcal{T}$. Conversely, if $T \in ker(\Phi)$ then $T(V_i) \subseteq V_{\langle i\rangle}$ for all i and, since $w_{(P,\pi)}(T(v)) = w_{(P,\pi)}(v)$ for all $v \in V$, if $v \in V_i$ is a non-zero vector then $T(v) = v' + u'$, with $v' \in V_i$, $v' \neq 0$ (and $u' \in V_{\langle i\rangle^*}$); hence $ker(\Phi) = \mathcal{T}$. This shows also that $\mathcal{T}$ is a normal subgroup of $GL_{(P,\pi)}(V)$. □

Let $M_{r\times t}(\mathbb{F}_q)$ be the set of all $r \times t$ matrices over $\mathbb{F}_q$ and

$$\mathcal{U}(P, \pi) = \left\{ (A_{ij}) \in M_{N\times N}(\mathbb{F}_q) : \begin{array}{l} A_{ij} \in M_{k_i\times k_j}(\mathbb{F}_q) \\ A_{ij} = 0 \text{ if } i \not\leq j \\ A_{ii} \text{ is invertible} \end{array} \right\}.$$

As a consequence of the last result we have a structure theorem for $GL_{(P,\pi)}(V)$. We recall that $\mathcal{T}$ is the group of the isometries that satisfy the hypotheses of Proposition 4.3, and that $\mathcal{A}$ is the group of isometries of the form T_ϕ, $\phi \in \phi \in \text{Aut}(P, \pi)$.

Theorem 4.11. *Every linear isometry S can be written in a unique way as a product $S = F \circ T_\phi$, where $F \in \mathcal{T}$ and $T_\phi \in \mathcal{A}$. Furthermore,*

$$GL_{(P,\pi)}(V) \cong \mathcal{T} \rtimes \mathcal{A} \cong \mathcal{U}(P, \pi) \rtimes \text{Aut}(P, \pi)$$

where $\mathcal{T} \rtimes \mathcal{A}$ is the semi-direct product of $\mathcal{T}$ by $\mathcal{A}$ induced by the action of $\mathcal{A}$ on $\mathcal{T}$ by conjugation and $\cong$ denotes group isomorphism.

Proof. Given $S \in GL_{(P,\pi)}(V)$, if $\phi = \phi_S$, then $F = S \circ (T_\phi)^{-1} = S \circ (T_{\phi^{-1}})$ is in $\mathcal{T}$ and

$$S = (S \circ (T_{\phi^{-1}})) \circ T_\phi .$$

This expression shows that $GL_{(P,\pi)}(V) = \mathcal{T}\mathcal{A}$. We have seen that $\Phi \circ \iota(\phi) = \phi$ for every $\phi \in \mathrm{Aut}(P,\pi)$ and that $\Phi(T) = Id$, the identity, for all $T \in \mathcal{T}$. Since $\mathcal{A} = \iota(\mathrm{Aut}(P,\pi))$, it follows that $\mathcal{A} \cap \mathcal{T} = \{Id\}$; from this and from the fact that $\mathcal{T}$ is a normal subgroup of $GL_{(P,\pi)}(V)$ we have the first isomorphism. The second one follows from the isomorphisms $\mathcal{A} \cong \mathrm{Aut}(P,\pi)$ and $\mathcal{T} \cong \mathcal{U}(P,\pi)$. □

5. P-isometries and π-isometries

The cases when $\pi(i) = 1$ for all $i \in [n]$ and when $P = ([n], \leq)$ is an antichain give rise to spaces endowed with a P-metric and a π-metric respectively. Determination and description of the group of linear isometries of those spaces can be done as particular instance of Theorem 4.11.

In the case that the label π is such that $\pi(i) = 1$ for all $i \in [n]$, each V_i reduces to a copy of $\mathbb{F}_q$ and the poset-block-space reduces to the poset space introduced in [3]. Immediate substitution gives that‡

$$\mathcal{U}(P,\pi) = \{(a_{ij}) \in M_{n\times n}(\mathbb{F}_q) : a_{ij} = 0 \text{ if } i \nleq j \text{ and } a_{ii} \neq 0\}$$

and $\mathrm{Aut}(P,\pi) = \mathrm{Aut}(P)$. Then, the characterization of $GL_P(\mathbb{F}_q^n)$ given in [16, Corollary 1.3] follows from Theorem 4.11 as a particular case:

$$GL_{(P,\pi)}(\mathbb{F}_q^n) \cong \mathcal{U}(P,\pi) \rtimes \mathrm{Aut}(P).$$

We consider now the case when P is an antichain. Let N be a positive integer and π be a partition $(k_1, k_2, \ldots, k_n)$ of N where

$$k_1 = \ldots = k_{m_1} = l_1$$
$$\vdots$$
$$k_{m_1+\ldots+m_{l-1}+1} = \ldots = k_{m_1+\ldots+m_l} = l_r$$

with $l_1 > l_2 > \ldots > l_r > 0$. We denote such a partition π by $[l_1]^{m_1}[l_2]^{m_2}\ldots[l_r]^{m_l}$. The π-weight of $v = v_1 + v_2 + \ldots + v_n \in V$ is defined to be

$$w_\pi(v) = |\{i : v_i \neq 0\}|.$$

In our approach, this corresponds to taking $V_i = \mathbb{F}_q^{k_i}$, $V = \bigoplus_{i=1}^n V_i$, and $P = ([n], \leq)$ as the antichain of n elements, i.e., $i \leq j$ in P if and only if $i = j$. In this case $w_{(P,\pi)}(v) = w_\pi(v)$ for all $v \in V$.

Since $\langle i \rangle = \{i\}$ for each $i \in [n]$, the upper-triangular maps T take V_i isomorphically onto itself. Hence,

$$\mathcal{T} \cong GL(k_1, \mathbb{F}_q) \times GL(k_2, \mathbb{F}_q) \times \ldots \times GL(k_n, \mathbb{F}_q).$$

On the other hand, $\mathrm{Aut}(P) \cong S_n$ and $\mathrm{Aut}(P,\pi)$ can be identified with a subgroup of S_n. If $\pi = [l_1]^{m_1}[l_2]^{m_2}\ldots[l_r]^{m_l}$, then $\mathrm{Aut}(P,\pi)$ only permutes those vertices with same labels and therefore

$$\mathrm{Aut}(P,\pi) \cong S_{m_1} \times S_{m_2} \times \ldots \times S_{m_l}.$$

‡In this case, $\mathcal{U}(P,\pi)$ is the group of units of the incidence algebra $I(P,\mathbb{F}_q)$; if π is another label, one can identify the labelled poset (P,π) with a quoset (quasi-ordered set) Q and then $\mathcal{U}(P,\pi)$ is the group of units of the structural matrix algebra of Q; see for instance [2, 20].

From Theorem 4.11 it follows that

$$GL_{(P,\pi)}(V) \cong \left(\prod_{i=1}^{n} GL(k_i, \mathbb{F}_q)\right) \rtimes \left(\prod_{i=1}^{l} S_{m_i}\right).$$

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