Nontrivial Interactions of Gravitational and Electromagnetic Waves with Cosmic Strings

Patricio S. Letelier

Departamento de Matematica Aplicada, Instituto de Matemática, Estatística e Ciência da Computação, Universidade Estadual de Campinas, 13081 Campinas, São Paulo, Brazil

(Received 20 July 1990)

The singular structure of the spacetime associated to a cosmic string interacting with either a plane-fronted gravitational wave or a pencil of electromagnetic radiation is analyzed. We find that depending on the value of the string constant the interaction can produce either a directional or an essential curvature singularity along the cosmic string.

PACS numbers: 04.20.Jb, 04.30.+x, 11.17.+y, 98.80.Cq

In numerical simulations of the evolution of a network of cosmic strings,[1] as well as in the study of the evolution of a single string,[2] the spacetime is fixed *a priori.* Usually, either the Minkowski or the Friedman-Robertson-Walker metric is taken as the background spacetime. In general, no considerations are made about the backreaction of the cosmic string on the metric. For nonstatic strings the production of gravitational radiation can be significant and in a more accurate scenario cannot be disregarded.[3] Also, the radiation reaction on the motion of the string should be considered. The exact equation of motion of strings in which the gravitational backreaction modifies the string evolution is not known.[4] In other words, almost all the information that we have about the evolution of cosmic strings does not take into account the string's own gravitational field.

Because of the intrinsic nonlinearities of the Einstein equations, as well as the string evolution equation, the interaction of the gravitational field with the cosmic string may not be trivial in the sense that the interaction may change the nature of the spacetime singularities that describe the cosmic string. For instance, the description of a cosmic string by the same singularities that describe a usual rod will dramatically change some of the string properties that produce gravitational-lens effects and are used to seed galaxies.[5]

Solutions of the Einstein equations have been studied that describe a cosmic string of infinite length in interaction with (a) black holes located along the string,[6] (b) cylindrical gravitational waves with one[7] or two degrees of freedom[8] that have as a symmetry axis the string, (c) other infinitely long strings forming a bundle of parallel strings,[9] and (d) one[10] or several cosmic walls[11] perpendicular to the string. In all these cases there is a rather trivial interaction; at most the interaction changes the value of the string constant. The singular structure of the spacetime, even in the more elaborate cases, is rather simple: We have the superposition of a conic singularity and the singularities that represent the object in interaction. In other words, the string only shows up in the spacetime curvature as a single Dirac distribution with support on the infinite line occupied by the string.

In this Letter we study exact solutions of the Einstein equations that represent (a) the interaction of a plane-fronted gravitational wave with an infinite cosmic string perpendicular to the wave front, and (b) a pencil of electromagnetic radiation traveling parallel to a cosmic string. In both cases we find a very different singular structure of the Riemann-Christoffel curvature tensor than the one already mentioned. We find that the interaction of the strings with either the gravitational waves or the electromagnetic field may give rise to three different types of singularities: (1) the usual conic one found in the already studied cases, (2) a directional singularity, and (3) an essential singularity. The appearance of the different singularities is related to the value of the string constant.

Let us consider the line element

$$ds^2 = H\,du^2 + 2\,du\,dv + 2A\,du\,dx + 2B\,du\,dy - Fe^{-4V}(dx^2+dy^2)\,, \tag{1}$$

where H, A, and B are functions of u, x, and y; V is a function of x and y; and F is a function of u only. The metric (1) is a special case of the general metric that admits a null vector with zero covariant derivative.[12] In particular, we shall be interested in the case where the functions F, A, and B are restricted by

$$F=(au-u_0)^2,\quad A_{,y}-B_{,x}=0,\quad A_{,x}+B_{,y}=0\,, \tag{2}$$

where a and u_0 are arbitrary constants.

The Einstein equations for the metric (1) with the restriction (2) give the energy-momentum tensor (EMT) as

$$T_{\mu\nu}=wl_\mu l_\nu+\rho(l_\mu k_\nu+l_\nu k_\mu)\,, \tag{3}$$

where

$$w=\frac{e^{4V}}{16\pi F}(H_{,xx}+H_{,yy})\,,\quad \rho=\frac{e^{4V}}{4\pi F}(V_{,xx}+V_{,yy})\,, \tag{4}$$

$$k_\mu=(H/2)\delta^u_\mu+\delta^v_\mu+A\delta^x_\mu+B\delta^y_\mu\,, \tag{5a}$$

$$l_\mu=\delta^u_\mu,\quad m_\mu=-\sqrt{F}e^{-2V}\delta^x_\mu\,,\quad n_\mu=-\sqrt{F}e^{-2V}\delta^y_\mu\,. \tag{5b}$$

We have that

$$k^\mu k_\mu=l_\mu l^\mu=l_\mu m^\mu=l_\mu n^\mu=k_\mu n^\mu=k_\mu m^\mu=m_\mu n^\mu=0\,,$$
$$k^\mu l_\mu=-m_\mu m^\mu=-n_\mu n^\mu=1\,. \tag{6}$$

As usual, the expression (3) is obtained assuming that the components of the Riemann-Christoffel curvature tensor are sufficiently regular to construct a well-defined Ricci tensor and Ricci scalar in order to have meaningful Einstein equations. In the case that the curvature tensor is singular, say along a line, it may be that (3) is no longer valid on that line. The determination of the EMT associated to a curvature singularity is not an easy task. For instance, for singular points and lines there does not exist a general theory of distributions in curved spaces. Even more, the existence of such a theory has been questioned.[13] There are some particular occasions in which the EMT associated to singular lines has been determined.[9,14-16] On the other hand, the EMT associated to singular surfaces can always be determined.[16,17]

The metric (1) admits several important particular cases; three of the most studied are the following.

(i) Cosmic strings.— When

$$H=A=B=0\,, \tag{7}$$

$$V=2\lambda \ln r\,, \tag{8}$$

where $r=(x^2+y^2)^{1/2}$, the metric (1) represents the usual cosmic string of constant linear mass density λ.[9] This interpretation is based on the fact that the EMT (3) with the restrictions (6) and (7) reduces to the EMT of an infinitely long cosmic string,

$$T_s^{\mu\nu}=\rho_s(t^\mu t^\nu - z^\mu z^\nu)\,. \tag{9}$$

The density ρ_s is the distribution with support in the string,

$$\rho_s=\lambda\delta(x)\delta(y)/\sqrt{g_2},\quad \sqrt{g_2}=(au-u_0)^2 r^{8\lambda}\,, \tag{10}$$

and t^μ and z^μ are two orthonormal vectors defined by $\sqrt{2}t^\mu=k^\mu+l^\mu$ and $\sqrt{2}z^\mu=k^\mu-l^\mu$ ($t^\mu t_\mu=-z^\mu z_\mu=1$ and $t^\mu z_\mu=0$). The deficit angle $\Delta\alpha$ is related to λ by the relation $\Delta\alpha=8\pi\lambda$. For cosmic strings originating in phase transitions in the very early Universe, λ is estimated to be between 10^{-5} and 10^{-6}.

(ii) Gravitational waves.— When

$$V=0\,, \tag{11}$$

$$H_{,xx}+H_{,yy}=0\,, \tag{12}$$

the metric (1) represents a plane-fronted gravitational wave propagating in vacuum, and the EMT (3) is null in this case. This solution is a generalization of a metric studied by Kundt.[18] An example of this wave is provided by

$$H=U_1(u)\mathrm{Re}f(\zeta)+U_2(u)\mathrm{Im}f(\zeta)\,, \tag{13}$$

$$\begin{aligned} A&=\mathrm{Re}[\phi_1(u)h(\zeta)+\phi_2(u)g(\zeta)]\,,\\ B&=\mathrm{Im}[\phi_1(u)h(\zeta)+\phi_2(u)g(\zeta)]\,, \end{aligned} \tag{14}$$

where U_1, U_2, ϕ_1, ϕ_2, f, h, and g are functions of the indicated arguments and $\zeta=x+iy$. The functions U_1 and U_2 are associated with the different polarizations of the gravitational field.

(iii) Pencils of light.— When $V=0$, and H is taken as

$$H=8\omega \ln r_1\,, \tag{15}$$

where $r_1\equiv[(x-a_1)^2+(y-b_1)^2]^{1/2}$, with ω, a_1, and b_2 constants, the EMT (3) reduces to

$$T_r^{\mu\nu}=w_r l^\mu l^\nu,\quad w_r=\omega\delta(x-a_1)\delta(y-b_1)/\sqrt{g_2}\,. \tag{16}$$

Thus the metric (1) in the present case represents a beam of electromagnetic radiation of zero cross section located at $x=a_1$ and $x=b_1$. The constant ω is the energy per proper length of the beam of radiation. The metric (1) with the same restrictions as before and $H=\psi(\zeta)\overline{\psi}(\zeta)$, ψ an arbitrary function, represents a beam of directed electromagnetic radiation. Note that for either of the functions H mentioned above we have $H_{,xx}+H_{,yy}\neq 0$ on the line defined by $x=a_1$ and $y=b_1$. These metrics are generalizations of metrics studied some years ago by Bonnor.[19]

Recently, another particular case of the metric (1) has been considered in connection with strongly gravitating cosmic strings.[20] In this case the functions U_1 and U_2 are taken as being distributions in order to have a null curvature tensor outside the string world sheet. In this Letter we shall consider two other particular cases of (1) with the restrictions (2).

(a) Cosmic strings with gravitational waves.— We claim that the metric (1) with F, A, and B given by (2), V by (8), and H a solution of Eq. (12) represents a usual cosmic string of infinite length placed on the line $r=0$ interacting with a plane-fronted wave. This interpretation is based, as usual, on the fact that in this case the EMT (3) reduces to the EMT (9) (the EMT of a cosmic string located along the $r=0$ axis) and that when $\lambda=0$, we end up with the plane gravitational wave described in (ii). Note that in the present case ρ_s is given by (10) and it does not change with the interaction, but t^μ and z^μ that are related to l^μ and k^μ as in (i) do change. Now k^μ is a function of H, A, and B [cf. Eq. (5a)].

(b) Cosmic strings and pencils of light.— When H is given by (15) and V by (8), we have that the EMT (3) reduces to

$$T^{\mu\nu}=T_s^{\mu\nu}+T_r^{\mu\nu}\,. \tag{17}$$

Therefore the metric (1) in this case represents the superposition of a beam of electromagnetic radiation of zero cross section located at $x=a_1$ and $y=b_1$ and a cosmic string placed at $x=y=0$.

We want to remark that the interpretation of the above two particular cases relies on the hypothesis of regularity of the curvature tensor that gives rise to Eq. (3). In order to determine if the given interpretation is always correct we shall study the singular structure of the curvature tensor.

The curvature invariants

$$\begin{aligned} I_1&=R_{\mu\nu\lambda\sigma}k^\mu k^\lambda m^\nu m^\sigma\,,\quad I_2=R_{\mu\nu\lambda\sigma}k^\mu k^\lambda n^\nu n^\sigma\,,\\ I_3&=R_{\mu\nu\lambda\sigma}k^\mu k^\lambda m^\nu n^\sigma\,,\quad I_4=R_{\mu\nu\lambda\sigma}m^\mu m^\lambda n^\nu n^\sigma\,, \end{aligned} \tag{18}$$

for the metric (1) with the restrictions (2) reduce to

$$\begin{aligned} &I_1+I_2=-8\pi w,\quad I_4=-8\pi\rho\,,\\ &I_1=J_1+J_2+J_3,\quad I_3=J_4+J_5+J_6\,, \end{aligned} \tag{19}$$

with ρ and w defined in (4), and

$$\begin{aligned} J_1 &= (e^{4V}/F)(A_{,xu} - H_{,xx}/2)\,, \\ J_2 &= (2e^{4V}/F)(A_{,u}V_{,x} - B_{,u}V_{,y})\,, \\ J_3 &= -(e^{4V}/F)(H_{,x}V_{,x} - H_{,y}V_{,y})\,, \\ J_4 &= (e^{4V}/F)(A_{,yu} - H_{,xy}/2)\,, \\ J_5 &= (2e^{4V}/F)(A_{,u}V_{,y} + B_{,u}V_{,x})\,, \\ J_6 &= -(e^{4V}/F)(H_{,x}V_{,y} + H_{,y}V_{,x})\,. \end{aligned} \tag{20}$$

Now we shall study the curvature invariants associated to the metric (1) with V given by (8), F by (2), H by (13), A and B by (14) with the specialization

$$\begin{aligned} &h = f' = (\zeta)^{\alpha}, \quad g = (\zeta)^{\beta}, \\ &U_1 = 2\,\mathrm{Re}\,\phi_1', \quad U_2 = 2\,\mathrm{Im}\,\phi_1', \quad \mathrm{Im}\,\phi_2 = 0\,, \end{aligned} \tag{21}$$

where the prime indicates differentiation with respect to the argument, and α and β are real constants such that $\alpha \neq -1$. In this case the metric (1) represents a single cosmic string interacting with the particular gravitational wave described by (13), (14), and (21), i.e., a special case of the situation (a). The quantities J in this case are

$$J_1 = \beta r^{8\lambda+\beta-1}(\phi_2'/F)\cos[(\beta-1)\theta]\,, \tag{22}$$

$$J_2 = (4\lambda/\beta)J_1 + 2\lambda r^{8\lambda+\alpha-1}\{U_1\cos[(\alpha+1)\theta] + U_2\sin[(\alpha+1)\theta]\}/F\,, \tag{23}$$

$$J_3 = 2\lambda r^{8\lambda+\alpha-1}\{-U_1\cos[(\alpha-1)\theta] + U_2\sin[(\alpha-1)\theta]\}/F\,, \tag{24}$$

$$J_4 = -\beta r^{8\lambda+\beta-1}(\phi_2'/F)\sin[(\beta-1)\theta]\,, \tag{25}$$

$$J_5 = (4\lambda/\beta)J_1 + 2\lambda r^{8\lambda+\alpha-1}\{U_1\cos[(\alpha-1)\theta] + U_2\sin[(\alpha-1)\theta]\}/F\,, \tag{26}$$

$$J_6 = 2\lambda r^{8\lambda+\alpha-1}\{U_1\sin[(\alpha-1)\theta] - U_2\cos[(\alpha-1)\theta]\}/F\,, \tag{27}$$

where θ denotes the usual polar angle $\theta = \arctan(y/x)$.

First we shall consider the invariants for the case in which the cosmic string is not present. Setting $\lambda = 0$ in (22)-(27) we have that only J_1 and J_4 are different from zero; also $w = \rho = 0$ in this case. Furthermore, if we take $\beta \geq 1$, the functions J_1 and J_4 are regular on $r=0$. In consequence, for this particular gravitational wave we have that the curvature invariants $I_1 = -I_2 = J_1$, $I_3 = J_4$, and $I_4 = 0$ are regular on $r=0$. When $\beta < 1$ and $\lambda = 0$ the invariants $I_1 = -I_2 = J_1$, $I_3 = J_4$ have an essential singularity on $r=0$ and in consequence there is a rod located on $r=0$. The actual computation of the EMT associated with the rod is not an easy task as we pointed out before, but its presence can be noticed either by the behavior of the motion of test particles near $r=0$ or by the evolution of the geodesic deviation. The equation that describes the geodesic deviation of test particles is particularly useful in this case due to its explicit dependence on the curvature tensor. Thus, when the string is not present, to have a pure gravitational wave we need $\beta \geq 1$.

Now we shall study the invariants for the string interacting with the particular gravitational wave described by (21) with $\beta \geq 1$. We can distinguish three different cases.

(1) When $8\lambda + \alpha > 1$ and $\beta \geq 1$, all the functions J as well as the curvature invariants I_1, I_2, and I_3 are regular on $r=0$ and I_4 has only the conic singularity characterized by the delta function that appears in ρ_s. Thus, we have that the interaction of the string with the gravitational wave does not change the singular behavior of the curvature on $r=0$.

(2) When $8\lambda + \alpha = 1$ and $\beta \geq 1$, except for J_1 and J_4 all the J's have a directional singularity. Hence the value of curvature invariants I_1, I_2, and I_3 on the line $r=0$ depends on the angle at which we approach this line. I_4 has a conic singularity as in the preceding case. Hence, we have that the interaction of the string with the gravitational wave produces along the axis $r=0$ the superposition of a directional and a conic singularity. Note that when the string is not present, $\lambda = 0$, the conic singularity disappears. In this particular case the interaction changes the nature of the curvature singularity. For some authors,[21] the appearance of directional singularities is an indication that the coordinate patch used in the neighborhood of the singularity is inappropriate.

(3) When $8\lambda + \alpha < 1$ and $\beta \geq 1$, the quantities J_2, J_3, J_5, and J_6 present an essential singularity. So, I_1, I_2, and I_3 have an essential singularity on $r=0$ and I_4 a conic one. Therefore the interaction of the string with the gravitational wave gives rise to a naked singularity all along the string. Also, as in the preceding case, when the string is not present ($\lambda = 0$) this essential singularity disappears. We can think of this type of singularity as produced by a condensation of the gravitational field along the cosmic string. The EMT associated to the singularity on $r=0$ is no longer (9) and (10). Because of the presence of the essential singularity the EMT splits into two parts: The first one is the EMT (9) and (10) and the second represents the distribution of tensions along the essential singularity; in particular we can have "hoop tensions."[16] In summary, the structure of the EMT will look more like a usual rod than the one of a cosmic string. Again, the correctness of this statement can be seen from the study of the motion of test particles near $r=0$. In general, the effect of the essential singularity will dominate over the conic one. The appearance of naked singularities in metrics that represent infinitely long lines is a known property of this class of solutions to

the Einstein equations.[22]

For the complementary case, $\beta < 1$, the most interesting situations are obtained when $8\lambda+\alpha > 1$ and either $8\lambda+\beta=1$ or $8\lambda+\beta > 1$. In both cases we have the interaction of a gravitational wave, that has an essential curvature singularity on $r=0$, with a cosmic string. The interaction changes the nature of the singularity; in the first case the interaction changes the essential singularity into a directional one and in the second the essential singularity is regularized. In both cases the conic singularity that describes the string is always present.

The singular behavior of the interaction of a cosmic string with the gravitational wave represented by (21) is not only a property of this last set of functions. There are a variety of functions f, h, etc., that describe gravitational waves interacting with a single string that have the same singular behavior of the curvature tensor already studied. As an example, we can mention that any set of functions that behave near $r=0$ as (21) will produce the same singular structure. Moreover, by choosing the functions V and H given by (8) and (13), respectively, and A and B given by (21) with $\phi_2=\mathrm{Im}\phi_1=0$ and $h=\zeta^a$ we have a metric that represents a pencil of light that interacts gravitationally with a cosmic string in a way that reproduces the singular structure of the curvature tensor already studied. Also, for a beam of electromagnetic radiation characterized by the function ψ it is not difficult to find functions h, g, etc., that give rise to a similar singular structure of the curvature invariants.

Therefore, either the gravitational or electromagnetic waves can interact with cosmic strings in such a way that they cause an essential singularity of the Riemann-Christoffel curvature tensor all along the position of the cosmic string.

We want to stress that the substitution of a conic singularity by an essential singularity to represent the string will produce quite different effects on the motion of massive particles as well as photons near the string. For instance, the geodesic deviation equation will have a completely different set of solutions. Also, the dynamics related to the intersection of strings for nonconic singularities is quite different; in the present case the strings interact like the usual rods.

In conclusion, we think that because of the peculiarities of the Einstein equations, as well as the cosmic string evolution equation, the interactions of the strings with the gravitational field produced by different fields such as the electromagnetic field or by the presence of massive bodies need to be better understood in order to construct consistent models of interacting strings. We recall that these fields will always be present in a cosmologic or cosmogonic scenario.

Most of the algebra of this work was performed using the program of algebraic manipulation μ-TENSOR.[23]

[1] D. P. Bennett and F. R. Bouchet, Phys. Rev. Lett. **63**, 2776 (1989); A. Albrecht and N. Turok, Phys. Rev. D **40**, 973 (1989); M. Sakellariadou and A. Vilenkin, Phys. Rev. D **37**, 885 (1988).

[2] P. S. Letelier, Class. Quantum Grav. **7**, 597 (1990).

[3] V. P. Frolov, W. Israel, and W. G. Unruh, Phys. Rev. D **39**, 1084 (1989).

[4] P. S. Letelier, Phys. Lett. A **143**, 103 (1990).

[5] See, for instance, A. Vilenkin, Phys. Rep. **121**, 263 (1985).

[6] B. Linet, Class. Quantum Grav. **4**, L33 (1987); S. Chandrasekhar and B. C. Xanthopoulos, Proc. Roy. Soc. London **423**, 387 (1989).

[7] J. Garriga and E. Verdaguer, Phys. Rev. D **36**, 2250 (1987).

[8] A. Economou and D. Tsoubelis, Phys. Rev. Lett. **61**, 2045 (1988); B. D. Papadopoulos and B. C. Xanthoupoulos, Phys. Rev. D **41**, 2512 (1990).

[9] P. S. Letelier, Class. Quantum Grav. **4**, L75 (1987).

[10] P. S. Letelier, Class. Quantum Grav. **6**, L207 (1989).

[11] P. S. Letelier, Class. Quantum Grav. **7**, L203 (1990).

[12] See, for instance, P. S. Letelier, Gen. Rel. Grav. **11**, 367 (1979), and references therein.

[13] R. Geroch and J. Traschen, Phys. Rev. D **36**, 1017 (1987).

[14] W. Israel, Phys. Rev. D **15**, 935 (1977).

[15] D. D. Sokolov and A. A. Starobinsky, Dokl. Akad. Nauk SSSR **234**, 1043 (1977) [Sov. Phys. Dokl. **22**, 312 (1977)].

[16] A. H. Taub, J. Math. Phys. **21**, 1423 (1980).

[17] W. Israel, Nuovo Cimento **44B**, 1 (1966); **48B**, 463 (1966).

[18] W. Kundt, Z. Phys. **163**, 77 (1961).

[19] W. B. Bonnor, Commun. Math. Phys. **13**, 163 (1969).

[20] D. Garfinkle, Phys. Rev. D **41**, 1112 (1990).

[21] D. Papadopoulos, B. Stewart, and L. Witten, Phys. Rev. D **24**, 320 (1981).

[22] See, for instance, D. Kramer *et al.*, *Exact Solutions of Einstein's Field Equations* (Cambridge Univ. Press, London, 1980).

[23] J. F. Harper and C. C. Dyer, *The μ-TENSOR Reference Manual* (University of Toronto, Toronto, 1986).