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The Sigma invariants of Thompson's group F

Robert Bieri, Ross Geoghegan and Dessislava H. Kochloukova*

Abstract. Thompson's group F is the group of all increasing dyadic PL homeomorphisms of the closed unit interval. We compute $\Sigma^m(F)$ and $\Sigma^m(F;\mathbb{Z})$, the homotopical and homological Bieri–Neumann–Strebel–Renz invariants of F, and show that $\Sigma^m(F) = \Sigma^m(F;\mathbb{Z})$. As an application, we show that, for every m, F has subgroups of type F_{m-1} which are not of type FP_m (thus certainly not of type F_m).

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1. Introduction

1.1. The group F. Let F denote the group of all increasing piecewise linear (PL) homeomorphisms[1]

$$x\colon [0,1] \to [0,1]$$

whose points of non-differentiability $\in [0,1]$ are dyadic rational numbers, and whose derivatives are integer powers of 2. This is known as Thompson's Group F; it first appeared in [22].

The group F has an infinite presentation

$$\langle x_0, x_1, x_2, \cdots \mid x_i^{-1} x_n x_i = x_{n+1} \text{ for } 0 \le i < n \rangle. \tag{1.1}$$

Let $F(i)$ denote the subgroup $\langle x_i, x_{i+1}, \dots \rangle$. The presentation (1.1) displays F as an HNN extension with base group $F(1)$, associated subgroups $F(1)$ and $F(2)$, and stable letter x_0; see [17], Proposition 9.2.5, or [13] for a proof. Thus F is an ascending[2] HNN-extension whose base and associated subgroups are isomorphic to F.

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[1] Here, PL homeomorphisms are understood to act on $[0,1]$ on the left as in [15] rather than on the right as in [13].

[2] See Subsection 2.1 for the definition.

The correspondence between the generators x_i in the presentation (1.1) and PL homeomorphisms is as in [15]. For example, the generator x_0 corresponds to the PL homeomorphism with slope $\frac{1}{2}$ on $[0, \frac{1}{2}]$, slope 1 on $[\frac{1}{2}, \frac{3}{4}]$, and slope 2 on $[\frac{3}{4}, 1]$.

The group F has type F_∞, i.e., there is a $K(F, 1)$-complex with a finite number of cells in each dimension [13]. Therefore F is finitely presented and has type FP_∞. Furthermore, F has infinite cohomological dimension [13], $H^*(F, \mathbb{Z}F)$ is trivial [14], F does not contain a free subgroup of rank 2 [10], and the commutator subgroup F' is simple [12], [15]. It is known that F has quadratic Dehn function [18]. The group of automorphisms of F was calculated in [9].

1.2. The Sigma invariants of a group. By a (*real*) *character* on G we mean a homomorphism $\chi\colon G \to \mathbb{R}$ to the additive group of real numbers. For a finitely generated group G the *character sphere* $S(G)$ of G is the set of equivalence classes of non-zero characters modulo positive multiplication. This is best thought of as the "sphere at infinity" of the real vector space $\mathrm{Hom}(G, \mathbb{R})$. The dimension d of that vector space is the torsion-free rank of G/G', and the sphere at infinity has dimension $d - 1$. We denote by $[\chi]$ the point of $S(G)$ corresponding to χ.

We recall the Bieri–Neumann–Strebel–Renz (or Sigma) invariants of a group G. Let R denote a commutative ring[3] with $1 \neq 0$, and let $m \geq 0$ be an integer. When G is of type F_m (resp. $\mathrm{FP}_m(R)$) the homotopical invariant $\Sigma^m(G)$ (resp. the homological invariant $\Sigma^m(G; R)$), is a subset of $S(G)$. In both cases we have $\Sigma^{m+1} \subseteq \Sigma^m$. We refer the reader to [7] for the precise definition, confining ourselves here to a brief recollection:

1.2.1. $m = 0$. All groups have type F_0 and type $\mathrm{FP}_0(R)$. By definition $\Sigma^0(G) = \Sigma^0(G; R) = S(G)$. This will only be of interest when we consider subgroups of F in Section 3.

1.2.2. $m = 1$. Let X be a finite set of generators of G and let Γ^1 be the corresponding Cayley graph, with G acting freely on Γ^1 on the left. The vertices of Γ^1 are the elements of G and there is an edge joining the vertex g to the vertex gx for each $x \in X$.

For any non-zero character $\chi\colon G \to \mathbb{R}$, and for any real number i define $\Gamma^1_{\chi \geq i}$ to be the subgraph of Γ spanned by the vertices

$$G_{\chi \geq i} = \{g \in G \mid \chi(g) \geq i\}.$$

By definition, $[\chi] \in \Sigma^1(G)$ if and only if $\Gamma^1_{\chi \geq 0}$ is connected. For a detailed treatment of Σ^1 from a topological point of view, see [17], Section 18.3.

1.2.3. $m = 2$. Let $\langle X \mid T \rangle$ be a finite presentation of G. Choose a G-invariant orientation for each edge of Γ^1 and then form the corresponding Cayley complex

[3] Only the rings $\mathbb{Z}$ and $\mathbb{Q}$ will play a role in this paper.

Γ^2 by attaching 2-cells equivariantly to Γ^1 using attaching maps indicated by the relations in T. Define $\Gamma^2_{\chi \geq i}$ to be the subcomplex of Γ^2 consisting of $\Gamma^1_{\chi \geq i}$ together with all the 2-cells which are attached to it.

By definition, $[\chi] \in \Sigma^2(G)$ if and only if $[\chi] \in \Sigma^1(G)$ and there is a nonpositive d such that the map

$$\pi_1(\Gamma^2_{\chi \geq 0}) \to \pi_1(\Gamma^2_{\chi \geq d}), \tag{1.2}$$

induced by the inclusion of spaces $\Gamma^2_{\chi \geq 0} \subseteq \Gamma^2_{\chi \geq d}$ is zero (and $\Gamma^1_{\chi \geq 0}$ is connected). See, for example, [28]. Note that Γ^2 is the 2-skeleton of the universal cover of a $K(G, 1)$-complex which has finite 2-skeleton.

1.2.4. $m > 2$. The higher $\Sigma^m(G)$ are defined similarly, for groups of type F_m, using the m-skeleton, Γ^m, of the universal cover of a $K(G, 1)$-complex having finite m-skeleton. See [7].

1.2.5. The homological case. For a commutative ring R, the homological Sigma invariants $\Sigma^m(G; R)$ are defined similarly when the group G is of type $\mathrm{FP}_m(R)$, using a free resolution of the trivial (left) RG-module R which is finitely generated in dimensions $\leq m$; see [7] for details. Among the basic facts to be used below, which hold for all rings R, are: $\Sigma^1(G) = \Sigma^1(G; R)$; and $\Sigma^m(G) \subseteq \Sigma^m(G; R)$ when both are defined (i.e., when G has type F_m). If G is finitely presented then "type F_m" and "type $\mathrm{FP}_m(\mathbb{Z})$" coincide. In that case, $\Sigma^m(G; \mathbb{Z})$ can also be understood from the above topological definition of $\Sigma^m(G)$, replacing statements about homotopy groups by the analogous statements about reduced $\mathbb{Z}$-homology groups; more precisely, one requires

$$\tilde{H}_{k-1}(\Gamma^k_{\chi \geq 0}) \to \tilde{H}_{k-1}(\Gamma^k_{\chi \geq d}), \tag{1.3}$$

to be trivial for all $k \leq m$.

Remark. The definition of Σ^1 given here agrees with the now-established conventions followed, for example, in [7] and in [2]. It differs by a sign from the Σ^1-invariant defined in [6]. This arises from our convention that RG-modules are left modules, while in [6] they are right modules.

1.3. Some facts about Sigma invariants. It is convenient to write "$[\chi] \in \Sigma^\infty$" as an abbreviation for "$[\chi] \in \Sigma^m$ for all m".

Among the principal results of Σ-theory for a group G of type F_m (resp. type $\mathrm{FP}_m(R)$) are: (1) $\Sigma^m(G)$ (resp. $\Sigma^m(G; R)$) is an open subset of the character sphere $S(G)$, and (2) $\Sigma^m(G)$ (resp. $\Sigma^m(G; R)$) classifies all normal subgroups N of G containing the commutator subgroup G' by their finiteness properties in the following sense:

Theorem 1.1 ([7], [27], [28]). *Let G be a group of type F_m (resp. type $\mathrm{FP}_m(R)$) with a normal subgroup N such that G/N is abelian. Then N is of type F_m (resp. FP_m)*

if and only if for every non-zero character χ *of* G *such that* $\chi(N) = 0$ *we have* $[\chi] \in \Sigma^m(G)$ *(resp.* $[\chi] \in \Sigma^m(G; R)$*).*

A non-zero character is *discrete* if its image in $\mathbb{R}$ is an infinite cyclic subgroup. A special case of Theorem 1.1 (the only one we will use) is:

Corollary 1.2. *If the non-zero character* χ *is discrete then its kernel has type* F_m *(resp. type* $\mathrm{FP}_m(R)$*) if and only if* $[\chi]$ *and* $[-\chi]$ *lie in* $\Sigma^m(G)$ *(resp.* $\Sigma^m(G; R)$*).*

The invariants $\Sigma^m(G)$ and $\Sigma^m(G; R)$ have been calculated for only a few families of groups G, even fewer when $m > 1$. For metabelian groups G of type F_m there is the still-open Σ^m-Conjecture: $\Sigma^m(G)^c = \Sigma^m(G; \mathbb{Z})^c = \mathrm{conv}_{\leq m} \Sigma^1(G)^c$, where[4] $\mathrm{conv}_{\leq m}$ denotes the union of the (spherical) convex hulls of all $\leq m$-tuples; this is known for $m = 2$ [19] but only for larger m under strong restrictions on G [20], [24]. A complete description of $\Sigma^m(G)$ and $\Sigma^m(G; \mathbb{Z})$ for any right angled Artin group G is given in [23]. Recently the homotopical invariant $\Sigma^m(G)$ has been generalized to an invariant of group actions on proper CAT(0) metric spaces [2]; the corresponding invariants for the natural action of $SL_n(\mathbb{R})$ on its symmetric space have been calculated: for $n = 2$ (action by Möbius transformations on the hyperbolic plane) in [3], and for $n > 2$ in [26]. A similar generalization of the homological case, $\Sigma^m(G; R)$, to the CAT(0) setting will appear in [5].

1.4. Sigma invariants of $\boldsymbol{F}$**.** In this paper we calculate the Sigma invariants $\Sigma^m(F)$ and $\Sigma^m(F; R)$ of the group F. For $x \in F$ and $i = 0$ or 1 let $\chi_i(x) := log_2 x'(i)$, i.e., the (right) derivative of the map x at 0 is $2^{\chi_0(x)}$ and the (left) derivative of x at 1 is $2^{\chi_1(x)}$. In terms of the presentation (1.1) $\chi_0(x_0) = -1$ and $\chi_0(x_i) = 0$ for $i \geq 1$, while $\chi_1(x_i) = 1$ for all $i \geq 0$. These two characters are linearly independent. Thus $[\chi_0]$ and $[\chi_1]$ are not antipodal points of the circle $S(F)$. From (1.1) we see that the real vector space $\mathrm{Hom}(F, \mathbb{R})$ has dimension 2, so these two characters span $\mathrm{Hom}(F, \mathbb{R})$. It follows that the convex sum of $[\chi_0]$ and $[\chi_1]$ is a well-defined interval in the circle $S(F)$; its members are the points $\{[a\chi_0 + b\chi_1] \mid a, b > 0\}$. We call it the "shorter interval". We call χ_0 and χ_1 the "special" characters.

There is a useful automorphism ν of F which is most easily expressed when F is regarded as a group of PL homeomorphisms as above: it is conjugation by the homeomorphism $t \mapsto (1 - t)$; if one draws the graph of the PL homeomorphism $x \in F$ in the square $[0, 1] \times [0, 1]$ then the graph of $\nu(x)$ is obtained by rotating that square through the angle π. This ν induces an automorphism of $\mathrm{Hom}(F, \mathbb{R})$ and consequently an automorphism of $S(F)$ which permutes the elements of $\Sigma^m(F)$ (resp. $\Sigma^m(F; R)$). In particular, it swaps the points $[\chi_0]$ and $[\chi_1]$. We refer to this as "ν-symmetry" of the Sigma invariants.

The theorems of this paper can now be stated:

[4] It is customary to use the notation A^c for the complement of the set A in a character sphere; e.g. $\Sigma^m(G)^c$ or $\Sigma^m(G; R)^c$.

Theorem A. $\Sigma^1(F)$ *consists of all points of* $S(F)$ *except* $[\chi_0]$ *and* $[\chi_1]$. *The points of* $S(F)$ *lying in the open convex hull of* $[\chi_0]$ *and* $[\chi_1]$, *i.e., in the shorter interval, are in* $\Sigma^1(F)$ *but are not in* $\Sigma^2(F)$. *The other (longer) open interval between* $[\chi_0]$ *and* $[\chi_1]$ *is the set* $\Sigma^\infty(F)$. *The sets* $\Sigma^m(F;R)$ *and* $\Sigma^m(F)$ *coincide for all* m *and any ring* R.

One part of this is not new: $\Sigma^1(F)$ was computed in [6].

Theorem B. *For every* $m \geq 1$, F *contains subgroups of type* F_{m-1} *which are not of type* $\mathrm{FP}_m(\mathbb{Z})$ (*thus certainly not of type* F_m).

Theorem A is proved in Section 2, and Theorem B is proved (using [4]) in Section 3.

Acknowledgment. We thank Dan Farley who asked about the possibility of embedding powers of F in F to get non-normal subgroups of F with more interesting finiteness properties than can be found among the kernels of characters on F itself. His question led to the writing of the paper [4] and thus to our Theorem B.

2. Proof of Theorem A

2.1. Σ^0 and Σ^1. By an *ascending* HNN extension we mean a group presented by $\langle H, t \mid t^{-1}ht = \phi(h) \text{ for } h \in H\rangle$ where $\phi\colon H \to H$ is a monomorphism. Such a group is denoted by $H*_{\phi,t}$.

We begin by citing:

Theorem 2.1. *Let* G *decompose as an ascending HNN extension* $H*_{\phi,t}$. *Let* $\chi\colon G \to \mathbb{R}$ *be the character given by* $\chi(H) = 0$ *and* $\chi(t) = 1$.

(1) *If* H *is of type* F_m (*resp.* $\mathrm{FP}_m(R)$) *then* $[\chi] \in \Sigma^m(G)$ (*resp.* $[\chi] \in \Sigma^m(G;R)$).
(2) *If* H *is finitely generated and* ϕ *is not onto* H *then* $[-\chi] \in \Sigma^1(G)^c$.

Proof. The homological case of (1) for all m is [24], Proposition 4.2, and the homotopical case for $m = 2$ is a special case of [25], Theorem 4.3. The homotopical case of (1) for all m then follows.

(2) is elementary: we recall the argument. Let N be the kernel of χ. By (1) and Corollary 1.2, (2) is equivalent to claiming that the group N is not finitely generated. The hypothesis that ϕ is not onto implies $t^{-1}Ht$ is a proper subgroup of H. Thus $N = \bigcup_{n\geq 1} t^n H t^{-n}$ is a proper ascending union, so it cannot be finitely generated. □

Applying Theorem 2.1 together with "ν-symmetry" to the group F, i.e., $G = F$, $t = x_0$, $H = F(1)$, and $\chi = -\chi_0$, we get part of Theorem A:

Corollary 2.2. $\{[-\chi_0], [-\chi_1]\} \subseteq \Sigma^\infty(F)$ *and* $\{[\chi_0], [\chi_1]\} \subseteq \Sigma^1(F)^c$.

Theorem 8.1 of [6] is the assertion that the complement of the two-point set $\{[\chi_0], [\chi_1]\}$ is precisely[5] $\Sigma^1(F)$.

2.2. The "longer" interval. The following is proved by combining two theorems of H. Meinert, namely [24], Proposition 4.1, and [25], Theorem B:

Theorem 2.3. *Let G decompose as an ascending HNN extension $H*_{\phi,t}$. Let $\chi\colon G \to \mathbb{R}$ be a character such that $\chi|H \neq 0$. If H is of type F_∞ and if $[\chi|H] \in \Sigma^\infty(H)$ then $[\chi] \in \Sigma^\infty(G)$.*

We use this to show that whenever $\chi\colon F \to \mathbb{R}$ is such that $\chi(x_1) < 0$ we always have $[\chi] \in \Sigma^\infty(F)$. Recall that F is an HNN extension with base group $F(1) = \langle x_1, x_2, \dots\rangle$, associated subgroups $F(1)$ and $F(2)$ and with stable letter x_0, where $F(i) = \langle x_i, x_{i+1}, \dots\rangle$. As $\{x_i\}_{i\geq 1}$ are conjugate in F we see that $\chi(x_1) = \chi(x_i) < 0$ for all $i \geq 1$. Let $\tilde{\chi}$ be the restriction of χ to $F(1)$. If we identify $F(1)$ with F via the isomorphism that sends x_i to x_{i-1} for $i \geq 1$, then $\tilde{\chi}$ gets identified with $-\chi_1$ and, by Corollary 2.2, $[-\chi_1] \in \Sigma^\infty(F)$. Thus we have:

Corollary 2.4.
$$\{[\chi] \in S(F) \mid \chi(x_1) < 0\} \subseteq \Sigma^\infty(F). \tag{2.1}$$

This shows that the open interval in the circle $S(F)$ from $[\chi_0]$ to $[-\chi_0]$ which contains $[-\chi_1]$ lies in $\Sigma^\infty(F)$. By ν-symmetry its image under ν has the same property, and this enlarges the interval in question to cover the whole "long" open interval between $[\chi_0]$ and $[\chi_1]$. In summary:

Proposition 2.5. *All of $S(F)$ except possibly the closed convex sum of the points $[\chi_0]$ and $[\chi_1]$ lies in $\Sigma^\infty(F)$.*

2.3. The "shorter" interval. For the homotopical version of Theorem A we could simply apply the following:

Theorem 2.6. [21] Let G be a finitely presented group which has no free non-abelian subgroup. Then[6] $\mathrm{conv}_{\leq 2}\, \Sigma^1(G)^c \subseteq \Sigma^2(G)^c$.

However, the homological version of Theorem 2.6 is only known under restrictive conditions, so we proceed in a manner which handles the homotopical and homological versions at the same time. We begin by citing:

Theorem 2.7. *Let G have no non-abelian free subgroups and have type* $\mathrm{FP}_2(R)$. *Let $\tilde{\chi}\colon G \to \mathbb{R}$ be a non-zero discrete character. Then G decomposes as an ascending HNN extension $H*_{\phi,t}$ where H is a finitely generated subgroup of* $\ker(\tilde{\chi})$, *and $\tilde{\chi}(t)$ generates the image of $\tilde{\chi}$.*

[5] But note the change of conventions explained in the remark at the end of Section 1.2.

[6] See Section 1.3 for the definition of $\mathrm{conv}_{\leq 2}$.

This is an immediate consequence of [8], Theorem A. That theorem yields an HNN extension, and the hypothesis about free subgroups ensures it is an ascending HNN extension.[7]

We apply Theorem 2.7 to understand $\Sigma^2(F;R)$. Consider the non-zero character $a\chi_0 + b\chi_1$ where $a, b \in \mathbb{Q}$. Let $G := \ker(a\chi_0 + b\chi_1)$. Since F/F' is a free abelian group of rank 2, it is not hard to see that $G = \langle F', t\rangle$ for some $t \in F$. For the same reason, there is a non-zero discrete character $\tilde{\chi}\colon G \to \mathbb{R}$ whose kernel is F' such that $\tilde{\chi}(t)$ generates $\mathrm{im}(\tilde{\chi})$. We assume that G has type $\mathrm{FP}_2(R)$ and we consider what this implies. By Theorem 2.7 the existence of $\tilde{\chi}$ implies that G decomposes as $H*_{\phi,t}$ where H is a finitely generated subgroup of F'. The group F' consists of all PL homeomorphisms whose left and right slopes are 1. Since H is finitely generated, there must exist $\epsilon > 0$ such that all elements of H are supported in the interval $[\epsilon, 1-\epsilon]$. We may assume ϵ is so small that the PL homeomorphism t is linear on $[0,\epsilon]$ and on $[1-\epsilon, 1]$.

The character $\tilde{\chi}$ expresses G as a semidirect product of F' and $\mathbb{Z}$. Thus we have $F' = \bigcup_{n\geq 1} t^n H t^{-n}$. So for each $x \in F'$ there is some $n > 0$ such that $t^{-n}xt^n \in H$, and hence the support of $t^{-n}xt^n$ lies in $[\epsilon, 1-\epsilon]$.

This implies that the support of x lies in $[t^n(\epsilon), t^n(1-\epsilon)]$, and hence these end points have subsequences converging to 0 and 1 respectively as x varies in F'. If t has slope ≥ 1 on $[0,\epsilon]$ then $t(\epsilon) \geq \epsilon$ so $t^n(\epsilon) \geq \epsilon$ for all $n > 0$. Therefore t must have slope < 1 near 0. Similarly t must have slope < 1 near 1. Since $a\chi_0(t) + b\chi_1(t) = 0$ it follows that (still assuming G has type $\mathrm{FP}_2(R)$) $ab < 0$. Expressing the contrapositive, we have

Proposition 2.8. *If $ab > 0$ then $\ker(a\chi_0 + b\chi_1)$ does not have type $\mathrm{FP}_2(R)$.*

Now assume a and b are positive and rational. Write $\chi = a\chi_0 + b\chi_1$; thus χ is discrete. By Corollary 1.2, $\ker(\chi)$ has type $\mathrm{FP}_2(R)$ if and only if both $[\chi]$ and $[-\chi]$ lie in $\Sigma^2(F;R)$. But by Proposition 2.5 $[-\chi] \in \Sigma^2(F;R)$. So $[\chi]$ cannot lie in $\Sigma^2(F;R)$.

Proposition 2.9. *No point in the open convex sum of $[\chi_0]$ and $[\chi_1]$ (i.e., the shorter open interval) lies in $\Sigma^2(F;R)$.*

Proof. We have just shown that a dense subset of the open convex sum lies in $\Sigma^2(F;R)^c$, and since $\Sigma^2(F;R)$ is open in $S(F)$ this is enough. □

The proof of Theorem A is completed by recalling that for any ring R

(1) $\Sigma^1(F;R) = \Sigma^1(F)$, and
(2) $\Sigma^m(F) \subseteq \Sigma^m(F;R)$.

[7] The equivalence of "almost finitely presented" with respect to R, the term actually used in [8], and $\mathrm{FP}_2(R)$ is well known: see, for example, Exercise 3 of [11], VIII 5.

3. Subgroups of F with different finiteness properties

As before, we denote the complement of any subset A of a sphere by A^c. The Direct Product Formula for homological Sigma invariants (which is not always true) reads as follows:

$$\Sigma^n(G \times H; R)^c = \bigcup_{p=0}^{n} \Sigma^p(G; R)^c * \Sigma^{n-p}(H; R)^c.$$

Here, $*$ refers to "join" of subsets of the spheres $S(G)$ and $S(H)$ which are considered to be subspheres of the sphere $S(G \times H)$. In particular, when $p = 0$ or n one of these sets is empty, and then the join is treated in the usual way: e.g., $A * \emptyset = A$.

It has been known for many years that one inclusion of the Direct Product Formula is always true:

Theorem 3.1 (Meinert's inequality).

$$\Sigma^n(G \times H; R)^c \subseteq \bigcup_{p=0}^{n} \Sigma^p(G; R)^c * \Sigma^{n-p}(H; R)^c$$

and

$$\Sigma^n(G \times H)^c \subseteq \bigcup_{p=0}^{n} \Sigma^p(G)^c * \Sigma^{n-p}(H)^c.$$

Meinert did not publish this, but a proof can be found in [16], Section 9. The paper [1] also contains a proof of the homotopy version.

It is proved in [4] that the Direct Product Formula holds when R is a field. On the other hand, an example in [29] shows that the formula does not always hold when $R = \mathbb{Z}$. However, it is shown in [4] that when $\Sigma^n(G; \mathbb{Z}) = \Sigma^n(G; \mathbb{Q})$ for all n then the Direct Product Formula does hold when $R = \mathbb{Z}$. Writing F^r for the r-fold direct product of copies of F, one concludes (by induction on r) that the formula holds for F^r when $R = \mathbb{Z}$. More precisely, we have:

Theorem 3.2. *Let $r \geq 2$. Then, for all n,*

$$\Sigma^n(F^r; \mathbb{Z})^c = \bigcup_{p=0}^{n} \Sigma^p(F; \mathbb{Z})^c * \Sigma^{n-p}(F^{r-1}; \mathbb{Z})^c$$

and $\Sigma^n(F^r) = \Sigma^n(F^r; \mathbb{Z})$.

Proof. Only the last sentence requires some explanation. It follows from Meinert's Inequality (Theorem 3.1) together with the fact that for any group G we have $\Sigma^m(G) \subseteq \Sigma^m(G; R)$. □

Theorem A implies that $\Sigma^m(F)^c$ is a (spherical) 1-simplex if $m \geq 2$, is the 0-skeleton of that 1-simplex when $m = 1$, and is empty (i.e., the (-1)-skeleton of the 1-simplex) when $m = 0$. And that 1-simplex has the property that it is disjoint from its negative. It follows from Theorem 3.2 that $\Sigma^m(F^r)^c$ is the $(m-1)$-skeleton of a spherical $(2r-1)$-simplex in the $(2r-1)$-sphere $S(F^r)$, a simplex which is disjoint from its negative.

We now prove Theorem B. Consider $[\chi]$ in $S(F^r)$ which lies in the $(m-1)$-skeleton but not in the $(m-2)$-skeleton of the $(2r-1)$-simplex. Since the discrete characters are dense we can always choose χ discrete. Then $[\chi]$ lies in $\Sigma^m(F^r)^c \cap \Sigma^{m-1}(F^r)$ while $[-\chi]$ lies in $\Sigma^m(F^r)$. Thus, by Corollary 1.2, the kernel of χ has type F_{m-1} but not type $\mathrm{FP}_m(\mathbb{Z})$ when $m < 2r-1$. Now, F contains copies of F^r for all r; for example, let $0 < t_1 < \cdots < t_{r-1} < 1$ be a subdivison of $[0,1]$ into r segments where the subdivision points are dyadic rationals. The subgroup of F which fixes all the points t_i is a copy of F^r. Thus Theorem B is proved.

Example. Here is an explicitly described subgroup $G_r \leq F$ which has type F_{2r-1} but does not have type $\mathrm{FP}_{2r}(\mathbb{Z})$. Fix a dyadic subdivision of $[0,1]$ into r subintervals as above. Let G_r denote the subgroup of F consisting of all elements x for which the product of the numbers in the following set D_r equals 1. The members of D_r are: the left and right derivatives of x at the $(r-1)$ subdivision points t_i, the right derivative of x at 0, and the left derivative of x at 1. This subgroup of F (we consider F^r embedded in F as above) corresponds to the barycenter of the $(2r-1)$-simplex, and thus has the claimed properties.

Remark 3.3. This example is "structurally stable" in the following sense: The interior of the $(2r-1)$-simplex is open in the sphere $S(F^r)$. Thus all the points in that open set which correspond to discrete characters on F^r (they are dense) give rise to groups $\widetilde{G}_r$ with exactly the finiteness properties possessed by G_r. These groups $\widetilde{G}_r$ should be thought of as all the normal subgroups of F^r "near" G_r which have infinite cyclic quotients.

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R. Bieri, FB 12 - Institut für Mathematik, Johann Wolfgang Goethe-Universität Frankfurt, Robert-Mayer-Str. 6–8, 60325 Frankfurt am Main, Germany

E-mail: bieri@math.uni-frankfurt.de

R. Geoghegan, Department of Mathematical Sciences, Binghamton University (SUNY), Binghamton, NY 13902-6000, U.S.A.

E-mail: ross@math.binghamton.edu

D. H. Kochloukova, Department of Mathematics, State University of Campinas (UNICAMP), Cx. P. 6065, 13083-970 Campinas, SP, Brazil

E-mail: desi@ime.unicamp.br