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CAUCHY COMPLETENESS IN ELEMENTARY LOGIC

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Abstract. The inverse of the distance between two structures $\mathscr{A} \not\equiv \mathscr{B}$ of finite type τ is naturally measured by the smallest integer q such that a sentence of quantifier rank $q-1$ is satisfied by $\mathscr{A}$ but not by $\mathscr{B}$. In this way the space Str^τ of structures of type τ is equipped with a pseudometric. The induced topology coincides with the elementary topology of Str^τ. Using the rudiments of the theory of uniform spaces, in this elementary note we prove the convergence of every Cauchy net of structures, for any type τ.

§1. Introduction. For all topological notions used in this paper we refer to [7]. We let $L_{\omega\omega}$ be elementary logic and $L^\tau_{\omega\omega}$ be the set of all first-order sentences of type τ. Following Tarski, the *elementary topology* of Str^τ is given by the following closure operator C: for any $K \subseteq \mathrm{Str}^\tau$,

$$C(K) = \bigcap\{\,\mathrm{Mod}(\varphi) \mid \varphi \in L^\tau_{\omega\omega} \text{ and } K \subseteq \mathrm{Mod}(\varphi)\,\}.$$

Two structures $\mathscr{A}$ and $\mathscr{B}$ of type τ are elementarily equivalent, in symbols, $\mathscr{A} \equiv \mathscr{B}$, if and only if $C(\{\mathscr{A}\}) = C(\{\mathscr{B}\})$. As shown by Tarski [10], the family of equivalence classes $\mathrm{Str}^\tau/\equiv$ is a totally disconnected compact Hausdorff space: a base is given by the collection $\mathscr{E}^\tau = \{\,\mathrm{Mod}(\varphi) \mid \varphi \in L^\tau_{\omega\omega}\,\}$ of elementary classes.

While members of $\mathrm{Str}^\tau/\equiv$ are proper classes, $\mathrm{Str}^\tau/\equiv$ itself can be indexed by the *set* of complete theories in $L^\tau_{\omega\omega}$, and $\mathscr{E}^\tau$ can be indexed by the set of sentences of $L^\tau_{\omega\omega}$. Classes indexed by sets are known as *small* classes, and are frequently used in topological abstract model theory (see, e.g., [1], [5], [9] and [2]). From the foundational viewpoint, large topological spaces endowed with small topologies are no more problematic than sets. One can naturally speak of interior, closure, compactness, convergence and Cauchy completeness (of spaces with small uniformity bases) without burdening notation and terminology.

As is well known, the compactness of $L_{\omega\omega}$ is the following property: Whenever τ is a type and Σ is a subset of $L^\tau_{\omega\omega}$ such that for every finite subset $\Delta \subseteq \Sigma$,

$$\mathrm{Mod}(\Delta) = \bigcap_{\varphi\in\Delta} \mathrm{Mod}(\varphi) \neq \emptyset,$$

then

$$\mathrm{Mod}(\Sigma) = \bigcap_{\varphi\in\Sigma} \mathrm{Mod}(\varphi) \neq \emptyset.$$

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For a proof one can use Łos's theorem, to the effect that for every set $\{\mathscr{A}_i\}_{i\in I} \in \mathrm{Str}^\tau$, every sentence $\varphi \in L^\tau_{\omega\omega}$, and every ultrafilter U over I, the ultraproduct $\prod_U \mathscr{A}_i$ satisfies the condition

$$(*) \qquad \prod_U \mathscr{A}_i \in \mathrm{Mod}(\varphi) \iff \{\, i \in I \mid \mathscr{A}_i \in \mathrm{Mod}(\varphi)\,\} \in U.$$

Following [5, p. 216], for each $\Delta \in \mathscr{P}_\omega(\Sigma)$ = set of finite subsets of Σ, one now chooses a model $\mathscr{A}_\Delta \in \mathrm{Mod}(\Delta)$, and shows that $\prod_U \mathscr{A}_\Delta \in \mathrm{Mod}(\Sigma)$, for a certain ultrafilter U over $\mathscr{P}_\omega(\Sigma)$.

In the above formula $(*)$, the ultraproduct of the $\mathscr{A}_i$'s can be replaced by more general constructions:

DEFINITION 1. Let $\{\mathscr{A}_i\}_{i\in I}$ be a set of structures of the same type τ. Let U be an ultrafilter over I. Then $\lim_U \mathscr{A}_i \subseteq \mathrm{Str}^\tau$ is defined by

$$\lim_U \mathscr{A}_i = \bigcap \{\, \mathrm{Mod}(\varphi) \mid \{\, i \in I \mid \mathscr{A}_i \in \mathrm{Mod}(\varphi)\,\} \in U\,\}.$$

To the best of our knowledge, $\lim_U$ was first introduced in [5, p. 223 ff] for Boolean spaces, and in [3, p. 11 ff] for more general spaces.

Consider the following abstract form of Łos's Theorem:

AŁT For every type τ, every set $\{\mathscr{A}_i\}_{i\in I}$ in Str^τ and every ultrafilter U over I, $\lim_U \mathscr{A}_i \neq \emptyset$.

Trivially, AŁT is equivalent to compactness in every *small* logic, i.e., in any model-theoretic logic L such that the collection of sentences in L of any type τ is a set.

To conclude this section, let us mention that in [8, p. 255], one can find another proof of the compactness theorem related to (though without explicit mention of) Łos's theorem. In fact, it is shown that for every family of structures $K = \{\mathscr{A}_i\}_{i\in I}$, the cluster points of K coincide with the elements of

$$\Big\{ \prod_U \mathscr{A}_i \mid U \text{ is an ultrafilter over } I \Big\}.$$

§2. The uniformity of Str^τ**.** Fraïssé [4, p. 129] first observed that for any finite type τ the elementary topology of Str^τ is uniformizable; using partial isomorphisms, he explicitly introduced a uniformity base.

Following Karp [6], we shall instead use n-equivalence, $\equiv_n$. Recall that two structures are called *n-equivalent* if and only if they satisfy the same sentences of quantifier rank $\leq n$.

Omitting unnecessary τ-superscripts, for every finite type τ, we define the *uniformity base* $\mathscr{B}_I$ for Str^τ by $\mathscr{B}_I = \{\mathscr{U}_n\}_{n\in\omega}$, where for each $n \in \omega$,

$$\mathscr{U}_n = \{\, (\mathscr{A}, \mathscr{B}) \in \mathrm{Str}^\tau \times \mathrm{Str}^\tau \mid \mathscr{A} \equiv_n \mathscr{B}\,\}.$$

Given a (possibly infinite) type τ and a finite set of sentences $\Phi \subseteq L^\tau_{\omega\omega}$, let us write $\mathscr{A} \equiv_\Phi \mathscr{B}$ if and only if the structures $\mathscr{A}$ and $\mathscr{B}$ satisfy the same sentences of Φ. Letting now

$$\mathscr{U}_\Phi = \{\, (\mathscr{A}, \mathscr{B}) \in \mathrm{Str}^\tau \times \mathrm{Str}^\tau \mid \mathscr{A} \equiv_\Phi \mathscr{B}\,\},$$

we define the uniformity base $\mathscr{B}_{II}$ by

$$\mathscr{B}_{II} = \{ \mathscr{U}_\Phi \mid \Phi \text{ a finite set of sentences of type } \tau \}.$$

Let τ be a finite type. Since for each n there exist only finitely many pairwise inequivalent sentences of quantifier rank $\leq n$, the bases $\mathscr{B}_I$ and $\mathscr{B}_{II}$ are *uniformly equivalent*, in the sense that for each $n \in \omega$ there exists a finite set of sentences $\Phi \subseteq L^\tau_{\omega\omega}$ such that $\mathscr{U}_\Phi \subseteq \mathscr{U}_n$, and vice versa.

As an immediate consequence of the definition of $\mathscr{B}_{II}$, for every type τ and sentence $\varphi \in L^\tau_{\omega\omega}$ we have

$$\mathscr{U}_{\{\varphi\}}[\mathscr{A}] = \{ \mathscr{B} \mid (\mathscr{A},\mathscr{B}) \in \mathscr{U}_{\{\varphi\}} \} = \begin{cases} \mathrm{Mod}(\varphi), & \text{if } \mathscr{A} \models \varphi \\ \mathrm{Mod}(\neg\varphi), & \text{if } \mathscr{A} \not\models \varphi. \end{cases}$$

Thus $\mathscr{B}_{II}$ generates the elementary topology of Str^τ, and the resulting uniform space is *totally bounded.*

The same conclusion holds for the base $\mathscr{B}_I$, provided the type τ is *finite*. Moreover, in this case Str^τ is pseudo-metrizable. As a matter of fact, letting

$$d\colon \mathrm{Str}^\tau \times \mathrm{Str}^\tau \to \mathbb{R}$$

be defined by

$$d(\mathscr{A},\mathscr{B}) = \inf\left\{ \frac{1}{n+1} \;\middle|\; (\mathscr{A},\mathscr{B}) \in \mathscr{U}_n \right\},$$

we easily see that d is a *nonarchimedean pseudo-metrics,* in the sense that

$$d(\mathscr{A},\mathscr{C}) \leq \max\bigl(d(\mathscr{A},\mathscr{B}), d(\mathscr{B},\mathscr{C})\bigr).$$

§3. Cauchy completeness. In every uniform space we have the following well known characterization [7, p. 198]:

$$\text{Compactness} = \text{Cauchy Completeness} + \text{Total Boundedness}.$$

From our previous discussion it follows that the compactness of Str^τ is equivalent to the Cauchy completeness of its underlying uniformity.

For any finite type τ, Fraïssé [4, pp. 127–128] proved that Str^τ is Cauchy complete, whence, Str^τ is sequentially compact. To construct limits of Cauchy sequences, Fraïssé used inductive limits of certain systems of structures directed by partial isomorphisms. We present a simpler proof of the Cauchy completeness of Str^τ for arbitrary τ, only using condition AŁT. To this purpose, we first generalize Definition 1:

Definition 2. Let $\{\mathscr{U}_\alpha\}_{\alpha\in\Xi}$ be an arbitrary base of a uniformity for the elementary topology of Str^τ. Let $\{\mathscr{A}_i\}_{i\in I}$ be a set of structures of type τ, and U an ultrafilter over I. We denote by $\lim_U \mathscr{A}_i$ the collection of structures $\mathscr{A} \in \mathrm{Str}^\tau$ such that for every $\alpha \in \Xi$, the set $\{ i \in I \mid (\mathscr{A},\mathscr{A}_i) \in \mathscr{U}_\alpha \}$ is a member of U.

The dependence of $\lim_U \mathscr{A}_i$ on $\{\mathscr{U}_\alpha\}_{\alpha\in\Xi}$ is tacitly understood. In case $\{\mathscr{U}_\alpha\}_{\alpha\in\Xi}$ coincides with $\mathscr{B}_{II}$, Definition 2 coincides with Definition 1, and the following lemma becomes an equivalent reformulation of condition AŁT:

LEMMA 1. *Let $\{\mathscr{U}_\alpha\}_{\alpha\in\Xi}$ be a uniformity base for the elementary topology of* Str^τ. *Let I be a set, and U an ultrafilter over I. Then for every set $\{\mathscr{A}_i\}_{i\in I}$ of structures of type τ,* $\lim_U \mathscr{A}_i \neq \emptyset$.

PROOF. Let $\mathscr{A} = \prod_U \mathscr{A}_i$. It is sufficient to prove $\mathscr{A} \in \lim_U \mathscr{A}_i$. By way of contradiction, suppose that there exists $\alpha \in \Xi$ such that

$$\{\, i \in I \mid (\mathscr{A}, \mathscr{A}_i) \notin \mathscr{U}_\alpha \,\} = \{\, i \in I \mid \mathscr{A}_i \notin \mathscr{U}_\alpha[\mathscr{A}] \,\} \in U.$$

Since $\mathscr{U}_\alpha[\mathscr{A}]$ is open in the elementary topology and $\mathscr{A} \in \mathscr{U}_\alpha[\mathscr{A}]$, there exists a sentence $\psi_{\mathscr{A}}$ such that $\mathscr{A} \in \mathrm{Mod}(\psi_{\mathscr{A}}) \subseteq \mathscr{U}_\alpha[\mathscr{A}]$. It follows that

$$\{\, i \in I \mid \mathscr{A}_i \notin \mathscr{U}_\alpha[\mathscr{A}] \,\} \subseteq \{\, i \in I \mid \mathscr{A}_i \notin \mathrm{Mod}(\psi_{\mathscr{A}})\},$$

whence $\{\, i \in I \mid \mathscr{A}_i \models \neg\psi_{\mathscr{A}} \,\} \in U$ and, by Łos' Theorem, $\mathscr{A} \models \neg\psi_{\mathscr{A}}$, a contradiction. ⊣

DEFINITION 3. Let $\{\mathscr{U}_\alpha\}_{\alpha\in\Xi}$ be a uniformity base for the elementary topology of Str^τ, and let $(D, \leq)$ be a directed set.

Given a net $\{\mathscr{A}_i\}_{i\in D}$ of structures of type τ, we define $\lim_i \mathscr{A}_i$ to be the collection of structures $\mathscr{A} \in \mathrm{Str}^\tau$ such that for every $\alpha \in \Xi$ there exists $k \in D$ such that $(\mathscr{A}, \mathscr{A}_i) \in \mathscr{U}_\alpha$ for every $i \geq k$. Any such structure $\mathscr{A}$ is a limit element, in the sense of Cauchy, for the given net. Since Str^τ is not a Hausdorff space, $\mathscr{A}$ is not uniquely determined.

A net $\{\mathscr{B}_i\}_{i\in D}$ of structures of type τ is said to be a *Cauchy net* if and only if for every $\alpha \in \Xi$ there exists $k \in D$ such that for every $i, j \geq k$, $(\mathscr{B}_i, \mathscr{B}_j) \in \mathscr{U}_\alpha$.

An ultrafilter U over D is said to be *free* if and only if for each $k \in D$, the set $Y_k = \{\, i \in D \mid i \geq k \,\}$ is a member of U. Since the set $\{Y_k\}_{k\in D}$ has the finite intersection property, free ultrafilters (over directed sets without a maximum element) are a generalization of nonprincipal ultrafilters over ω.

LEMMA 2. *Let, as above, $\{\mathscr{U}_\alpha\}_{\alpha\in\Xi}$ be a uniformity base for the elementary topology of* Str^τ. *Let $(D, \leq)$ be a directed set and $\{\mathscr{B}_i\}_{i\in D}$ a Cauchy net. Then* $\lim_i \mathscr{B}_i = \lim_U \mathscr{B}_i$, *for every free ultrafilter U over D.*

PROOF. The inclusion $\lim_i \mathscr{B}_i \subseteq \lim_U \mathscr{B}_i$ immediately follows by definition.

Conversely, let $\mathscr{B} \in \lim_U \mathscr{B}_i$; then for every $\alpha \in \Xi$ there exists $X_\alpha \in U$ such that for every $i \in X_\alpha$, $(\mathscr{B}, \mathscr{B}_i) \in \mathscr{U}_\alpha$. Since $\{\mathscr{B}_i\}_{i\in D}$ is a Cauchy net, there exists $k_\alpha \in D$ such that for every $i, j \geq k_\alpha$, $(\mathscr{B}_i, \mathscr{B}_j) \in \mathscr{U}_\alpha$. For any $\alpha \in \Xi$ let $\beta \in \Xi$ be such that $\mathscr{U}_\beta \circ \mathscr{U}_\beta \subseteq \mathscr{U}_\alpha$, where the symbol $\circ$ denotes composition. Let X_β and k_β be as above. Since U is free, $X_\beta \cap Y_{k_\beta} \in U$. Pick an arbitrary $k \in X_\beta \cap Y_{k_\beta}$; then we claim that for every $i \geq k$, $(\mathscr{B}, \mathscr{B}_i) \in \mathscr{U}_\alpha$. As a matter of fact, from $k \in X_\beta$ we get $(\mathscr{B}, \mathscr{B}_k) \in \mathscr{U}_\beta$. From $i \geq k \geq k_\beta$ we now get $(\mathscr{B}_k, \mathscr{B}_i) \in \mathscr{U}_\beta$, whence $(\mathscr{B}, \mathscr{B}_i) \in \mathscr{U}_\beta \circ \mathscr{U}_\beta \subseteq \mathscr{U}_\alpha$. This settles our claim, and concludes the proof. ⊣

REMARK. The above lemma holds, *mutatis mutandis*, for any small logic.

THEOREM. *Elementary logic is Cauchy complete: for any type τ and uniformity base $\{\mathscr{U}_\alpha\}_{\alpha\in\Xi}$ for the elementary topology of* Str^τ, *(e.g., the uniformity base $\mathscr{B}_{II}$), every Cauchy net of structures of type τ converges.*

PROOF. Let $\{\mathscr{B}_i\}_{i\in D}$ be a Cauchy net of structures of type τ, indexed by some directed set D. Choose an arbitrary free ultrafilter U over D. By Lemma 1 there exists $\mathscr{B} \in \lim_U \mathscr{B}_i$. By Lemma 2, $\mathscr{B}$ is a limit of the given net. ⊣

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ADDED IN PROOF. Xavier Caicedo (in his paper *Continuous Operations on Spaces of Structures*) independently proved the Cauchy completeness of the spaces Str^τ.

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