Circular orbits in Einstein-Gauss-Bonnet gravity

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The stability under radial and vertical perturbations of circular orbits associated to particles orbiting a spherically symmetric center of attraction is studied in the context of the n-dimensional: the Newtonian theory of gravitation, Einstein's general relativity, and the Einstein-Gauss-Bonnet theory of gravitation. The presence of a cosmological constant is also considered. We find that this constant as well as the Gauss-Bonnet coupling constant are crucial to have stability for $n > 4$.

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I. INTRODUCTION

The physical laws for nonrelativistic phenomena are usually written in a three-dimensional space and for relativistic phenomena in a four-dimensional spacetime. In both cases these laws admit natural extensions to spaces with an arbitrary number of dimensions; these can be taken as the possible existence of an operative principle (or principles) which in addition to these laws leads us to determine the "true dimensions" of the spacetime, the arena where the physical phenomena occur.

Kant observed that the three-dimensionality of the usual space could be related with Newton's inverse square law: "The reason for the three-dimensionality of the space is yet unknown. It is likely that the three-dimensionality of space results from the law according to which the forces on the substances act on each other. The three-dimensionality seems to result from the fact that the substances of the existing work act on each other in such a way that the strength of the effect behaves like the inverse of the square of the distances" [1,2].

Since in a first approximation galaxies and planetary systems can be considered as formed by particles moving in circular orbits around an attraction center, a possible principle to determine the dimensionality of the space is the existence of stable circular orbits. The study of the stability of circular orbits moving in potential solutions of the n-dimensional Laplace equations can be found in [3]. These studies on stability culminate with Ehrenfest's work [4], where he shows that in the Newtonian-Keplerian problem there exists a stable limited orbit if and only if the dimension of the space is three (the potential must be null at infinity).

In Einstein's gravity, Tangherlini [5] studied the stability under radial perturbation of orbits associated to test particles moving around an n-dimensional Schwarzschild black hole. He found stability for $n = 4$ only, i.e., in complete concordance with the Newtonian case. The dimension of the constant time slice $n - 1$ will be denoted by d.

Generalization of Newton's theory that consists in the addition of a cosmological constant was studied by several authors; see, for instance, Refs. [6] (cosmological context) and [7] (potential theory context). Clearly, this generalization is motivated by the success of Einstein's gravity with a cosmological constant. An example of a model that makes use of a cosmological constant is the Λ cold dark matter model (ΛCDM) [8] that explains in a simple way the observed acceleration of the Universe as well as the cosmic microwave background.

The Hilbert Lagrangian (the Ricci scalar times the square root of the metric determinant) is a divergence in two dimensions. We have that the Hilbert action is proportional to the Euler characteristic of the two-dimensional manifold (Gauss-Bonnet theorem [9]). In four dimensions we have that the Bach-Lanczos Lagrangian [10,11] (a particular combination of terms quadratic in the Riemann-Christoffel curvature tensor) is a divergence, so in four dimensions this Lagrangian has no dynamical significance, like the Hilbert Lagrangian in two dimensions. The simplest generalization of the Hilbert Lagrangian for n-dimensional spaces ($n > 4$) can be obtained by adding the Bach-Lanczos Lagrangian to the Hilbert one. This new theory of gravitation, the Einstein-Gauss-Bonnet (EGB) theory, has some remarkable mathematical properties; see, for instance, [12].

Another outstanding feature of the Einstein-Gauss-Bonnet theory is that the Bach-Lanczos action appears as the leading quadratic correction term of the Hilbert action in the expansion of the supersymmetric string theory [13]. To be more precise, the Bach-Lanczos action appears exactly in the expansion of the $E_8 \times E_8$ heterotic string model [14].

The aim of this paper is to study the stability under radial perturbations as well as vertical perturbations of circular orbits related to test bodies orbiting a spherically symmet-

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ric center of attraction in the above-mentioned theories of gravity in an arbitrary number of dimensions.

In Sec. II we study the stability of circular orbits in an n-dimensional Newtonian context with a cosmological constant. In Sec. III we consider the stability of circular orbits in the EGB theory. We divide the study of stability in two. First, in Sec. IV, we consider the case without a cosmological constant. In the next section (Sec. V) we analyze the case where this constant is not null. We end by summarizing our results in Sec. VI.

II. d-DIMENSIONAL NEWTONIAN CENTER OF ATTRACTION WITH COSMOLOGICAL CONSTANT

The Laplace equation with a cosmological constant is

$$\nabla^2 \Phi + \Lambda = 0, \tag{1}$$

where ∇^2 is the Laplacian operator in d dimensions and Φ and Λ are the Newtonian potential and the cosmological constant, respectively. For spherical symmetry, Eq. (1) reduces to

$$\frac{1}{r^{n-2}} \frac{\partial}{\partial r}\left(r^{n-2}\frac{\partial}{\partial r}\Phi\right) + \Lambda = 0 \tag{2}$$

that has as a solution for $n \geq 4$

$$\Phi = -\frac{C}{n-3}\frac{1}{r^{n-3}} - \frac{\Lambda}{2(n-1)}r^2, \tag{3}$$

where C is an integration constant that for $n = 4$ ($d = 3$) is proportional to the mass of the center of attraction ($C = GM$, with G Newton's constant of gravitation).

The Newton equations for an axially symmetric potential can be written as

$$\ddot{R} - R\dot{\varphi}^2 = -\frac{\partial \Phi}{\partial R}, \tag{4}$$

$$\frac{d}{dt}(R^2\dot{\varphi}) = 0, \tag{5}$$

$$\ddot{z}^i = -\frac{\partial \Phi}{\partial z^i}, \tag{6}$$

where the overdots denote derivation with respect to t, $(R, \varphi, z^1, \ldots, z^{d-2})$ are d-dimensional cylindrical coordinates, and $\Phi = \Phi(R, z^1, \ldots, z^{d-2})$. Defining the effective potential $\Phi_{\text{eff}} = \Phi + h^2/(2R^2)$, with $h = R^2\dot{\varphi}$ ($=$ constant), we find

$$\ddot{R} = -\frac{\partial \Phi_{\text{eff}}}{\partial R}, \qquad \ddot{z}^i = -\frac{\partial \Phi_{\text{eff}}}{\partial z^i}. \tag{7}$$

Now let us study the stability of the circular orbit $R = R_0$, $z^i = 0$. We shall assume that the potential has reflection symmetry on the planes $z^i = 0$, i.e., $\Phi = \Phi[R, (z^1)^2, \ldots, (z^{d-2})^2]$. Expanding the effective potential around this circular orbit, we find

$$\Phi_{\text{eff}} = \Phi_{\text{eff}}(R_0, 0, \ldots, 0) + \frac{1}{2}\frac{\partial^2 \Phi_{\text{eff}}(R_0, 0, \ldots, 0)}{\partial R^2}(R - R_0)^2$$

$$+ \frac{1}{2}\sum_{i=1}^{d-2}\frac{\partial^2 \Phi_{\text{eff}}(R_0, 0, \ldots, 0)}{\partial (z^i)^2}(z^i)^2 + \cdots. \tag{8}$$

From Newton's equations we find the motion for the perturbations $\rho = R - R_0$ and $\zeta^i = z^i - 0$:

$$\ddot{\rho} + \kappa^2\rho = 0, \qquad \ddot{\zeta}^i + \nu_i^2\zeta^i = 0, \tag{9}$$

where

$$k^2 = \frac{\partial^2 \Phi_{\text{eff}}(R_0, 0, \ldots, 0)}{\partial R^2}$$

$$= \frac{\partial^2 \Phi(R_0, 0, \ldots, 0)}{\partial R^2} + \frac{3}{R}\frac{\partial \Phi(R_0, 0, \ldots, 0)}{\partial R}, \tag{10}$$

$$\nu_i^2 = \frac{\partial^2 \Phi_{\text{eff}}(R_0, 0, \ldots, 0)}{\partial (z^i)^2} = \frac{\partial^2 \Phi(R_0, 0, \ldots, 0)}{\partial (z^i)^2}. \tag{11}$$

Thus when $\kappa^2 > 0$ we have that the radial perturbation remains bounded; i.e., we have linear stability. These perturbations are usually named epicyclic perturbations [15]. In a similar way, when $\nu_i^2 > 0$ we have that the "vertical" perturbation ζ^i also remains bounded. In this case, we say that we have linear stability under perturbations along the direction z^i. For the spherically symmetric potential (3), we find

$$\kappa^2 = (5 - n)CR_0^{1-n} - 4\Lambda/(n - 1), \tag{12}$$

$$\nu_i^2 = \nu^2 = CR_0^{1-n} - \Lambda/(n - 1). \tag{13}$$

To derive the previous equations we used $r = [R^2 + \sum_{i=1}^{d-2}(z^i)^2]^{1/2}$. Note that, because of the spherical symmetry of the potential, we have that all of the frequencies for all of the vertical perturbations are equal.

The relations (12) and (13) tell us that when $\Lambda = 0$ we have stability only for $n = 4$. For $\Lambda > 0$ and $n = 4$, we have stability whenever $R_0^3 < 3C/(4\Lambda)$. This last relation puts an upper limit to the size of the stable galactic structures. For $\Lambda > 0$ and $n \geq 5$, we have $\kappa^2 < 0$, and the circular orbits are not stable under epicyclic perturbations. Now for $\Lambda < 0$ we have stable circular orbits for $n = 4, 5$, and for $n \geq 6$ we have stability for R_0 limited by the lower bound $R_0^* \equiv [(n - 5)(n - 1)C/(-4\Lambda)]^{1/(n-1)}$ ($< R_0$). In other words, for certain values of n the presence of a cosmological constant can stabilize the circular orbits, sometimes with limitations in the size of the orbit.

III. EINSTEIN-GAUSS-BONNET THEORY OF GRAVITATION

The Einstein-Gauss-Bonnet action is

$$S = \int d^n x\sqrt{-g}\left[\frac{1}{\kappa_n^2}(R - 2\Lambda + \alpha L_{\text{GB}})\right] + S_{\text{matter}}, \tag{14}$$

where R is the n-dimensional Ricci scalar ($n > 4$) and κ_n is defined as $\kappa_n \equiv \sqrt{8\pi G_n}$, where G_n is the n-dimensional gravitational constant. We use units such that the speed of light is 1 ($c = 1$). The Gauss-Bonnet term L_{GB} is given by

$$L_{GB} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\delta}R^{\mu\nu\rho\delta}, \qquad (15)$$

and α is a coupling constant. S_{matter} is the part of the action that describes the matter. We use the conventions $R^a_{bcd} = \partial_c \Gamma^a_{bd} + \cdots$, $R_{\mu\nu} = R^\rho_{\mu\rho\nu}$, and $R = R^\mu_\mu$. The action (14) is the simplest one for $n > 4$ built with topological terms. For the general case, see [16].

From (14) we find

$$G_{\mu\nu} - \alpha H_{\mu\nu} - \Lambda g_{\mu\nu} = \kappa_n^2 T_{\mu\nu}, \qquad (16)$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R,$$
$$H_{\mu\nu} = 2[RR_{\mu\nu} - 2R_{\mu\alpha}R^\alpha_\nu - 2R^{\alpha\beta}R_{\mu\alpha\nu\beta} + R^{\alpha\beta\gamma}_\mu R_{\nu\alpha\beta\gamma}]$$
$$- \tfrac{1}{2}g_{\mu\nu}L_{GB}. \qquad (17)$$

For vacuum ($T_{\mu\nu} = 0$) spherically symmetric solutions to the EGB equations, there is no loss of generality in choosing the metric

$$ds^2 = f(r)dt^2 - f(r)^{-1}dr^2 - r^2 d\Omega^2_{n-2}, \qquad (18)$$

where $d\Omega^2_{n-2} = d\theta_2^2 + \Sigma_{j=3}^n (\Pi_{i=2}^{n-1}\sin^2\theta_i)d\theta_j^2$ is the metric of a $(n-2)$ sphere. The case in which this sphere is changed by a space of negative or null curvature has also been considered [17].

From the fact that the Einstein tensor $G^{\mu\nu}$ as well as the Gauss-Bonnet tensor $H^{\mu\nu}$ are divergence-free [16], we have that Eq. (16) tells us that $T^{\mu\nu}$ is also divergence-free. From this last fact we conclude that test particles move along geodesics [18]. We shall analyze the stability of the circular geodesics γ whose parametric equation is

$$t = t(s), \qquad r = r_0, \qquad \theta_j = \frac{\pi}{2},$$
$$j = 2, \ldots, n-1, \qquad \theta_n = \phi \in [0, 2\pi]. \qquad (19)$$

The evolution of a small perturbation ξ^μ of γ is given by the geodesic deviation equation

$$\frac{D^2\xi^\mu}{ds^2} + R^\mu_{\nu\alpha\beta}u^\nu\xi^\alpha u^\beta = 0, \qquad (20)$$

where $D/ds = u^\mu\nabla_\mu$ and $u^\mu = dx^\mu/ds$. Equation (20) can be cast in a simpler but not manifestly covariant way as

$$\ddot{\xi}^\mu + 2\Gamma^\mu_{\alpha\beta}u^\alpha\dot{\xi}^\beta + \Gamma^\mu_{\alpha\beta,\nu}u^\alpha u^\beta\xi^\nu = 0, \qquad (21)$$

where the comma denotes ordinary differentiation. For the curve (19) the system (21) reduces to

$$\ddot{\xi}^0 + \frac{f'(r_0)}{f(r_0)}\dot{t}\dot{\xi}^1 = 0, \qquad (22)$$

$$\ddot{\xi}^1 + \kappa^2\xi^1 = 0, \qquad (23)$$

$$\ddot{\xi}^j + \nu^2\xi^j = 0, \qquad (24)$$

$$\ddot{\xi}^n + \frac{2}{r_0}\dot{\phi}\dot{\xi}^1 = 0, \qquad (25)$$

where

$$\kappa^2 = \frac{f(r_0)(r_0 f''(r_0) + 3f'(r_0)) - 2r_0 f'(r_0)^2}{r_0(2f(r_0) - r_0 f'(r_0))},$$
$$\nu^2 = \frac{f'(r_0)}{r_0(2f(r_0) - r_0 f'(r_0))}. \qquad (26)$$

We shall do a study of cases depending on the values of α and Λ. The analysis starts with the stability of circular orbits for the pure Einsteinian case (no Gauss-Bonnet term). Then we study the stability of these orbits when a coupling term and/or the cosmological constant are considered.

For $\Lambda = 0$ and $\alpha = 0$, the model of spacetime reduces to the n-dimensional Schwarzschild solution. The equation for the metric function is

$$rf' - (n-3)(1-f) = 0, \qquad (27)$$

which has the general solution $f(r) = 1 - C/r^{n-3}$. C is an integration constant that will be taken as positive in order to have the Schwarzschild solution when $n = 4$ ($C = 2GM$). In this case the constants ν^2 and κ^2 can be cast as

$$\nu^2 = \frac{C(n-3)r_0^{n-3}}{r_0^{n-1}(2r_0^{n-3} - C(n-1))},$$
$$\kappa^2 = \frac{C(n-3)((5-n)r_0^{n-3} - C(n-1))}{r_0^{n-1}(2r_0^{n-3} - C(n-1))}.$$

Therefore, we have $\nu^2 > 0$ if and only if $r_0^{n-3} > C(n-1)$; this implies that $\kappa^2 > 0$ if and only if $(5-n)r_0^{n-3} - C(n-1) > 0$, and it occurs only when $n = 4$ and $r_0 > 6GM$. Hence, we have stable circular orbits only in a four-dimensional spacetime as was expected.

IV. EINSTEIN-GAUSS-BONNET THEORY WITH $\Lambda = 0$

Taking $\Lambda = 0$ in (16) with $T_{\mu\nu} = 0$, the equation of metric function f is

$$rf' - (n-3)(1-f)\left[-1 + \frac{\alpha(n-4)}{r^2}(2rf' - (n-5)\right.$$
$$\left. \times (1-f))\right] = 0, \qquad (28)$$

which has the general solution

$$f_\pm(r) = 1 + \frac{r^2}{2\tilde{\alpha}}\left[1 \pm \sqrt{1 + \frac{4\tilde{\alpha}C}{r^{n-1}}}\right], \qquad (29)$$

where $\tilde{\alpha} = (n-3)(n-4)\alpha$ and C is an integration constant. The sign of C is chosen analyzing the behavior of f for $|\alpha| \to 0$. Let f_+ be the function f where the sign of square root is positive, and let f_- be the other case. When α is small, the asymptotic behavior of f_- is given by

$$f_- \approx 1 - \frac{C}{r^{n-3}}. \tag{30}$$

So, considering the Gauss-Bonnet term as a perturbation of the n-dimensional Schwarzschild solution, C must be a positive constant.

When α is small, the asymptotic behavior of f_+ is given by

$$f_+ \approx 1 + \frac{r^2}{\tilde{\alpha}} + \frac{C}{r^{n-3}}. \tag{31}$$

For $\alpha > 0$, it corresponds to Schwarzschild–anti-de Sitter spacetime with negative gravitational mass, with the standard energy definition in background anti-de Sitter [19].

Then there are two families of solutions corresponding to the sign in front of the square root that appears in (29). Following the definition given in Ref. [20], the family with the minus (plus) sign will be called the general relativity (GR) branch (non-GR branch) solution. The spacetime structure of these solutions of the EGB theory was studied in [21].

A. The GR branch solution

By choosing the solution $f = f_-$ of (28) from (26), we find

$$\begin{aligned}
\nu^2 &= \frac{r_0^n(1 - \sqrt{\beta}) + Cr_0\tilde{\alpha}(5-n)}{\tilde{\alpha}(r_0^3 C(n-1) - 2r_0^n\sqrt{\beta})}, \\
\kappa^2 &= \frac{A_1 + A_2 + A_3}{-2r_0^n\beta\sqrt{\beta}\tilde{\alpha}(-2r_0^n\sqrt{\beta} + r_0^3 C(n-1))},
\end{aligned} \tag{32}$$

where $\beta = 1 + 4\tilde{\alpha}Cr_0^{1-n}$ and

$$\begin{aligned}
A_1 &= 8r_0^{2n}(1 - \sqrt{\beta}) + r_0^{n+3}C(n^2 - 1)(1 - \sqrt{\beta}), \\
A_2 &= 2r_0^{n+1}[32 - \sqrt{\beta}(n^2 - 8n + 39)] + 2r_0^4 C^2\tilde{\alpha}(n-1) \\
&\quad \times [n + 7 - \sqrt{\beta}(n+3)], \\
A_3 &= 4r_0^2 C^2\tilde{\alpha}^2[32 - 2\sqrt{\beta}\tilde{\alpha}C(n-5)(n-9)] \\
&\quad - 8r_0^{5-n}C\tilde{\alpha}(n-5)(n-1).
\end{aligned} \tag{33}$$

The behavior of the functions ν^2 and κ^2 is not simple, but it is possible to find n, $\tilde{\alpha}$, C, and r_0 such that ν^2 and κ^2 are both positive; i.e., there are stable circular orbits. For example, we have stable circular orbits when $n = 5$, $\tilde{\alpha} = 2$, $C = 1$, and $0.2 \leq r_0 \leq 2.0$; $n = 6$, $\tilde{\alpha} = 20$, $C = 1$, and

$0.2 \leq r_0 \leq 2.0$; $n = 7$, $\tilde{\alpha} = 20$, $C = 1$, and $0.7 \leq r_0 \leq 1.9$; $n = 8$, $\tilde{\alpha} = 20$, $C = 1$, and $1.07 \leq r_0 \leq 1.47$; and $n = 9$, $\tilde{\alpha} = 80$, $C = 1$. and $1.33 \leq r_0 \leq 1.46$. In all of these examples, we have $f_-(r_0) > 0$ in the specified region. Hence, the Gauss-Bonnet term can stabilize circular orbits in some higher dimension spacetimes.

In Fig. 1, we show the epicyclic and vertical frequencies for several values of n and $C = 1$, $1 \leq r_0 \leq 3$. For the first two figures $\tilde{\alpha} = 20$ and for the third $\tilde{\alpha} = 80$. In the first graphic, n take the values $n = 9$ (top curve), 8, 7, 6, and 5 (bottom curve). In the second graphic, counting along the line $r_0 = 2$, n takes the values 5 (top curve), 6, 7, 8, and 9 (bottom curve). There is no stable circular orbit when $n = 9$. For the third graphic, counting along the same line, $n = 6$ (top line), 5, 7, 8, 9, and 10 (bottom line). In this case, there is no stable circular orbit when $n = 10$, but for $n = 9$ we have stability.

The vertical frequency ν^2 is always positive no matter the value of n, but the range of r_0 where κ^2 is positive decreases when the dimension n increases.

For $n \geq 10$ and $\tilde{\alpha} > 0$, and also for $n \geq 5$ and $\tilde{\alpha} < 0$, we calculate the functions ν^2 and κ^2 into a grid of points with $0.1 \leq r_0 \leq 2$ (step 0.1), $5 \leq n \leq 20$ and $0.01 \leq \tilde{\alpha} \leq 100$ (step 0.01 into $[0.01, 0.1]$, step 0.1 into $[0.1, 1]$, step 1 into $[1, 100]$), and $C = 1$. In the case $\alpha < 0$, we work with $-100 \leq \tilde{\alpha} \leq -0.01$, with the same steps for the corresponding intervals. In all of these numerical examples, we obtain $\nu^2\kappa^2 < 0$, and, therefore, no circular orbit is stable.

B. The non-GR branch solution

For completeness we study the case when $f = f_+$ is the solution of (28). In this case, the constants ν^2 and κ^2 are given as in (32) and (33) replacing $-\sqrt{\beta}$ by $\sqrt{\beta}$.

Also in this case, the stability depends on the sign of α. When $\alpha < 0$, in the expression for $\nu^2 = N_\nu/D_\nu$, we have a numerator $N_\nu = r_0^n[(1 + \sqrt{\beta})r_0^n - (n-5)C\tilde{\alpha}r_0]$, nonnegative, and a denominator $D_\nu = \tilde{\alpha}(r_0^3 C(n-1) + 2r_0^n\sqrt{\beta})$, always negative. So ν^2 is not positive, and hence there are no stable circular orbits.

The parameters $n(\geq 5)$ and C are positive. When $\alpha > 0$, the denominators of ν^2 and κ^2 are always positive ($D_\kappa = 2\beta\sqrt{\beta}r_0^n D_\nu$). The same happens with the functions f_+ and β and with the coefficients of r_0 in $\bar{A}_1$ and $\bar{A}_2$ (note that $\bar{A}_i$ is the function A_i after replacing $-\sqrt{\beta}$ by $\sqrt{\beta}$, $i = 1, 2, 3$). These positive coefficients ensure that, for r_0 sufficiently large, the numerator of ν^2, $N_\nu = [(1 + \sqrt{\beta})r_0^n - (n-5)C\tilde{\alpha}r_0]$, and the numerator of κ^2, $N_\kappa = [\bar{A}_1 + \bar{A}_2 + \bar{A}_3]$, are positive. Hence, we conclude that it is possible to find stable circular orbits for $n \geq 5$ and $\alpha > 0$.

Figure 2 shows the functions ν^2 and κ^2, for $C = 1$, $\tilde{\alpha} = 20$, $1 \leq r \leq 7$, and $n = 10$ (top line), 9, 8, 7, 6, and 5 (bottom line). We have stability for $r_0 > r_n$, where r_n depends on the dimension n.
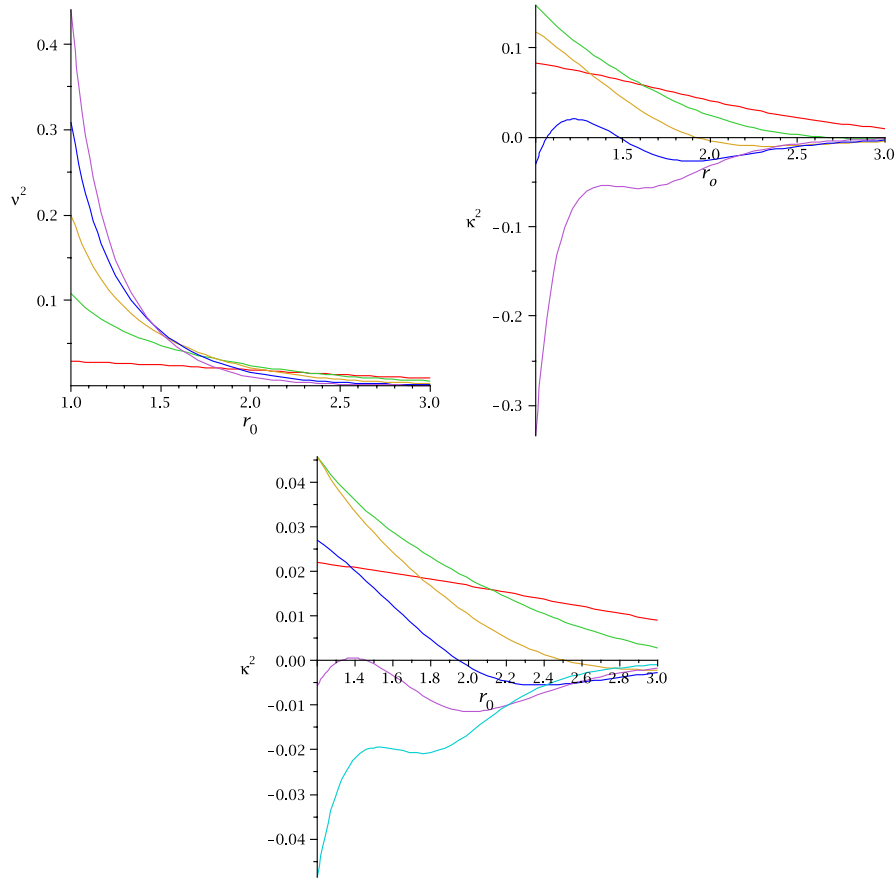
FIG. 1 (color online). GR branch, $\Lambda = 0$ case. Epicyclic and vertical frequencies for several values of n and $C = 1$, $1 \leq r_0 \leq 3$. For the first two figures $\tilde{\alpha} = 20$ and for the third $\tilde{\alpha} = 80$. In the first graphic n take the values $n = 9$ (top curve), 8, 7, 6, and 5 (bottom curve). In the second, counting along the line $r_0 = 2$, n takes the values 5 (top curve), 6, 7, 8, and 9 (bottom curve). For the third graphic, counting along the same line, $n = 6$ (top line), 5, 7, 8, 9, and 10 (bottom line). We see that the region of stability shrinks and disappears for large n.
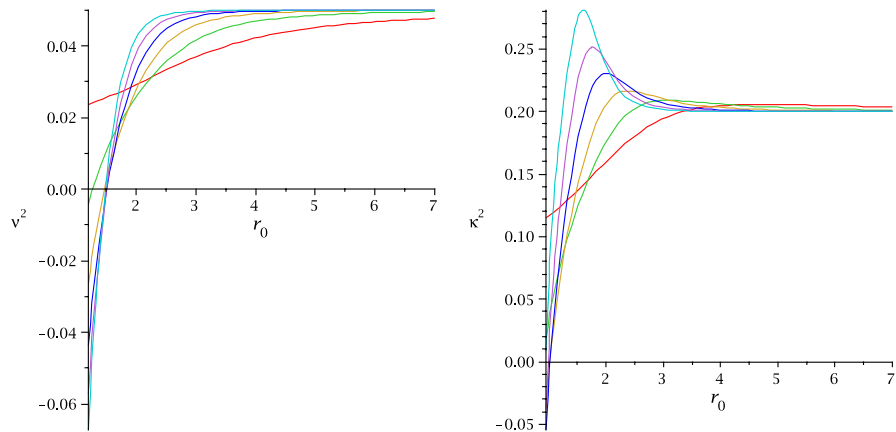

FIG. 2 (color online). Non-GR branch, $\Lambda = 0$ case. Graphics of the functions ν^2 and κ^2 for $\tilde{\alpha} = 20$, $C = 1$ and $1 \leq r_0 \leq 7$, and $n = 10$ (top line), 9, 8, 7, 6, and 5 (bottom line). They have a large region of stability.

V. EINSTEIN-GAUSS-BONNET THEORY WITH $\Lambda \neq 0$

Taking $\alpha = 0$ and $\Lambda \neq 0$ in (16) with $T_{\mu\nu} = 0$, the metric function obeys the equation

$$rf' - (n-3)(1-f) + \frac{2\Lambda r^2}{n-2} = 0, \qquad (34)$$

which has the general solution

$$f(r) = 1 - \frac{C}{r^{n-3}} - \frac{2\Lambda r^2}{(n-1)(n-2)}, \qquad (35)$$

and the integration constant C will be considered positive as before. The frequencies (26) in this case are $\nu^2 = N_1/D_1$ and $\kappa^2 = N_2/D_2$, where

$$\begin{aligned}
N_1 &= Cr_0^2(n-1)(n-2)(n-3)^2 - 4\Lambda r_0^{n+1}, \\
D_1 &= r_0^4(n-1)(n-2)(2r_0^{n-3} - C(n-1)), \\
N_2 &= Cr_0^2(n-2)(n-3)^2((5-n)r_0^{n-3} - C(n-1)) \quad (36) \\
&\quad - 2\Lambda r_0^{n+1}(8r_0^{n-3} - C(n^2-1)), \\
D_2 &= r_0^{n+1}(2r_0^{n-3} - C(n-1))(n-1)(n-2).
\end{aligned}$$

When $\Lambda < 0$, the numerator N_1 of ν^2 is positive for all r_0, and its denominator D_1 is positive if and only if $r_0^{n-3} > C(n-1)/2$. Note that D_2 change of sign with D_1. The function f is positive when $r_0^{n-3} > C$, and this condition is satisfied when $D_1 > 0$. So it is necessary to find the values of r_0 for which the numerator N_2 of κ^2 is positive. For better visualization, we write $\Lambda = -\lambda$ in N_2:

$$\begin{aligned}
N_2 &= 16\lambda r_0^{2n-2} - 2C\lambda(n^2-1)r_0^{n+1} - C(n-2)(n-3) \\
&\quad \times (n-5)r_0^{n-1} - C^2(n-2)(n-3)(n-1)r_0^2.
\end{aligned}$$

Analyzing the variation of the signs of the polynomial N_2, we can see that there is at least one positive root for N_2. Adding the fact that, for large r_0, N_2 is positive, we can conclude that it is possible to find a value $\bar{r}$ such that $\nu^2 > 0$ and $\kappa^2 > 0$ for $r_0 > \bar{r}$. Therefore, when $\Lambda < 0$ and for $n \geq 4$ there exist stable circular orbits.

When $\Lambda > 0$, in order to have $f > 0$ it is necessary that, at least,

$$2r^{n-3} - C(n-1) > 0. \qquad (37)$$

Under this condition the denominators D_1 and D_2 are positive. Now it is necessary that the numerators N_1 and N_2 be also positive. The polynomial N_1 is positive when

$$Cr_0^2(n-1)(n-2)(n-3)^2 - 4\Lambda r_0^{n+1} > 0. \qquad (38)$$

In four dimensions we have $N_1 > 0$ and $N_2 > 0$ when we take, for example, $C = 0.1$, $\Lambda = 0.05$, and $0.32 \leq r_0 \leq 0.55$. But when $n \geq 5$ the numerators N_1 and N_2 cannot be positive at the same time. We can see this by doing an analysis of the polynomial N_2. Using (37) and (38) we obtain

$$\begin{aligned}
N_2 &< -2\Lambda r_0^{n+1}(4C(n-1) - C(n^2-1)) - Cr_0^2(n-2) \\
&\quad \times (n-3)^2((5-n)C(n-1)/2 - C(n-1)) \\
&= C(n-1)(n-3)[4\Lambda r_0^{n+1} - Cr_0^2(n-1) \\
&\quad \times (n-2)(n-3)^2]/2 < 0. \qquad (39)
\end{aligned}$$

So for $n \geq 5$ and $\Lambda > 0$ there is no stable circular orbit. It is interesting to note that the addition of a negative cosmological constant stabilizes formerly not stable circular orbits for any $n \geq 4$.

In the following we analyze the stability of the circular orbits when both Λ and α are not null. In this case the equation of metric function is written as

$$\begin{aligned}
rf' - (n-3)(1-f) &+ \frac{2\Lambda r^2}{n-2} + \frac{\alpha(n-4)(n-3)(1-f)}{r^2} \\
&\times (2rf' - (n-5)(1-f)) = 0, \qquad (40)
\end{aligned}$$

which has the general solution

$$f(r)\pm = 1 + \frac{r^2}{2\tilde{\alpha}}\left[1 \pm \sqrt{1 + \frac{4\tilde{\alpha}}{(n-1)(n-2)}\left(2\Lambda + \frac{C}{r^{n-1}}\right)}\right], \qquad (41)$$

where $\tilde{\alpha} = (n-3)(n-4)\alpha$ and C is a integration constant. As before, let f_+ be the function f when the sign of the square root is positive, and let f_- be the other case. In the limit $|\alpha| \to 0$, the function f_- reduces to

$$f_- \approx 1 - \frac{2\Lambda r^2}{(n-1)(n-2)} - \frac{C}{(n-1)(n-2)r^{n-3}}. \qquad (42)$$

By considering the Gauss-Bonnet term as a perturbation of the Schwarzschild-de Sitter (Schwarzschild–anti-de Sitter) spacetime with $\Lambda > 0\,(\Lambda < 0)$, we conclude that C must be positive.

When α is small, the asymptotic behavior of f_+ is given by

$$f_+ \approx 1 + \frac{r^2}{\tilde{\alpha}} + \frac{2\Lambda r^2}{(n-1)(n-2)} + \frac{C}{(n-1)(n-2)r^{n-3}}. \qquad (43)$$

As we can see, here again, there are two families of solutions corresponding to the sign in front of the square root in (41). We define two branches following the same criterion, the GR branch and the non-GR branch.

A. The GR branch solution ($\Lambda \neq 0$)

The functions ν^2 and κ^2 in this case can be written as (26)

$$\nu^2 = \frac{r_0^n[(n-1)(n-2)(1-\sqrt{\beta_*}) + 8\Lambda\tilde{\alpha} - \tilde{\alpha}Cr_0^{1-n}(n-5)]}{\tilde{\alpha}(n-1)(-2r_0^n\sqrt{\beta_*}(n-2) + r_0^3 C)}, \qquad (44)$$

$$\kappa^2 = \frac{B_4 r_0^{n+3} + B_3 r_0^{n+1} + B_2 r_0^n + B_1 r_0^4 + B_0 r_0^2}{2\tilde{\alpha}(n-1)(-2r_0^n\sqrt{\beta_*}(n-2) + r_0^3 C)r_0^n[(n-1)(n-2) + 4\tilde{\alpha}(2\Lambda + Cr_0^{1-n})]}, \tag{45}$$

where

$$\beta_* = 1 + \frac{4\tilde{\alpha}}{(n-1)(n-2)}\left(2\Lambda + \frac{C}{r^{n-1}}\right) \tag{46}$$

and

$$B_4 = C(8\Lambda\tilde{\alpha} + (n-1)(n-2))(n^2 - 1)[-\sqrt{\beta_*} + 1],$$

$$B_3 = C[-32\tilde{\alpha}\sqrt{\beta_*}(n-1)(n-2) + 2\tilde{\alpha}(n^2 - 8n + 39)$$
$$\times (8\tilde{\alpha}\Lambda + (n-1)(n-2))],$$

$$B_2 = 8[[(-\sqrt{\beta_*} + 1)(n-1)(n-2) + 8\tilde{\alpha}\Lambda]$$
$$\times [(n-1)(n-2) + 8\tilde{\alpha}\Lambda]],$$

$$B_1 = 2C^2\tilde{\alpha}(n-1)[-\sqrt{\beta_*}(5-n) + n + 3],$$

$$B_0 = 4\tilde{\alpha}^2 C^2[(n-5)(n-9)]. \tag{47}$$

First, when $\Lambda < 0$ it is possible to find stable circular orbits. For example, if $\tilde{\alpha} = \pm 1$, $\Lambda = -1$, and $C = 1$, then γ is stable when $r_0 \geq 1.2$ ($n \geq 5$). Note that $f_- = 0$ if and only if

$$p(r) = 2(-\Lambda)r^{n+4} + (n-1)(n-2)r^{n+2}$$
$$+ (n-1)(n-2)\tilde{\alpha}r^n - Cr^5 = 0. \tag{48}$$

Analyzing the variation of signs in the coefficients of p, we can see that, if $\Lambda < 0$, then the number of variations is one and there is only one positive value $\bar{r}$ such that $p(\bar{r}) = 0$. Also, for $r_0 > \bar{r}$ we have $f(r_0) > 0$. If $\Lambda > 0$, the existence of a root of $p(r) = 0$ depends on the relation among $\tilde{\alpha}$, Λ, and C. In the cases $\tilde{\alpha} = \pm 1$, $\Lambda = -1$, and $C = 1$, the polynomial $p(r) = 2r^{n+4} + (n-1)(n-2)(r^2 - 1)r^n - r^5$ has only one positive root. Hence, we have a horizon (at $r_H < 1$) with the zone of stability outside this horizon.

Figure 3 shows the behavior of the functions ν^2 and κ^2 for $\Lambda = -1$, $C = 1$, $\tilde{\alpha} = 1$, and $1 \leq r_0 \leq 3$, and $n = 5$ (top curve), 6, 7, 8, 9, and 10 (bottom curve). We see a small region of instability for large n. Similar behavior is found for the same parameters, but $\tilde{\alpha} = -1$.

The cases where $\Lambda > 0$ were numerically analyzed. For example, we have stable circular orbits when $n = 5$, $\tilde{\alpha} = C = \Lambda = 1$, and $0.1 \leq r_0 \leq 0.5$; $n = 6$, $\tilde{\alpha} = C = \Lambda = 1$, and $0.2 \leq r_0 \leq 0.55$; $n = 7$, $\tilde{\alpha} = 2$, $C = \Lambda = 1$, and $0.3 \leq r_0 \leq 0.55$; $n = 8$, $\tilde{\alpha} = 9$, $C = \Lambda = 1$, and $0.45 \leq r_0 \leq 0.55$; and $n = 9$, $\tilde{\alpha} = 100$, $C = 1$, $\Lambda = 0.05$, and $0.62 \leq r_0 \leq 0.83$. In all of these examples we have $f_-(r_0) > 0$ into the fixed intervals.

For the other cases with $\Lambda > 0$ ($n \geq 10$ and $\alpha > 0$ or $n \geq 5$ and $\alpha < 0$), we calculate the functions ν^2 and κ^2 into a grid of points $(r_0, \Lambda, \tilde{\alpha}, n)$, and at each point of the grid these functions are not positive at the same time. The grid is built with $0.1 \leq r_0 \leq 2$ (step 0.1), $0 \leq \Lambda \leq 2$ (step 0.5), $5 \leq n \leq 20$ and $0.01 \leq \tilde{\alpha} \leq 100$ (step 0.01 into $[0.01, 0.1]$, step 0.1 into $[0.1, 1]$, step 1 into $[1, 100]$), and $C = 1$. In the case where $\alpha < 0$, we work with $-100 \leq \tilde{\alpha} \leq -0.01$, with the same steps for the corresponding intervals.

In these cases ($\Lambda > 0$), each value of n has a different set of parameters $(r_0, \tilde{\alpha}, C)$ where both ν^2 and κ^2 are positive, and these differences makes it quite impossible to repeat the graphical analysis as in the $\Lambda < 0$ cases.

B. The non-GR branch solution ($\Lambda \neq 0$)

Again, for completeness, we describe the case where f_+ is the solution of (40). In this case, the functions ν^2 and κ^2 (26) are given as (45) replacing $-\sqrt{\beta_*}$ by $\sqrt{\beta_*}$.
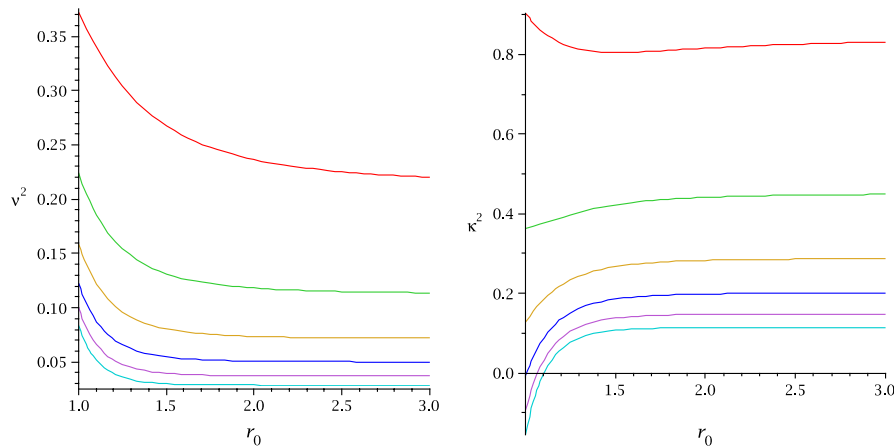

FIG. 3 (color online). GR branch, $\Lambda \neq 0$ case. The functions ν^2 and κ^2 for $\Lambda = -1$, $C = 1$, $\tilde{\alpha} = 1$ and $1 \leq r_0 \leq 3$, and $n = 5$ (top curve), 6, 7, 8, 9, and 10 (bottom curve). For large n we see a small region of instability.
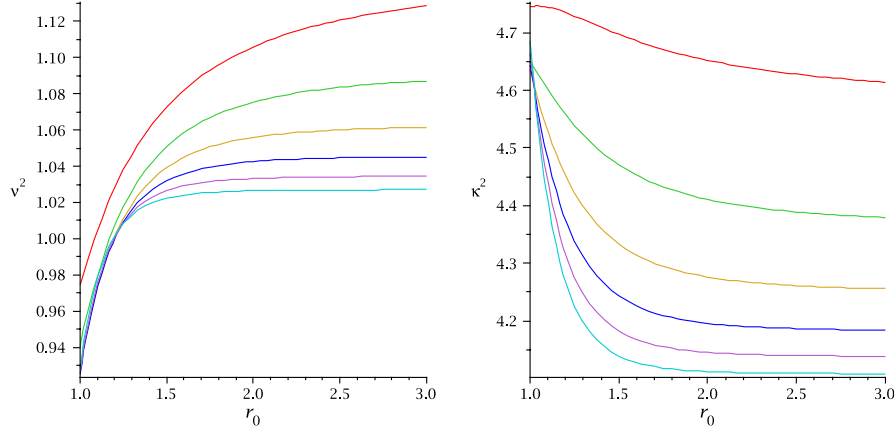
FIG. 4 (color online). Non-GR branch, $\Lambda \neq 0$ case. The frequencies ν^2 and κ^2 for $\Lambda = 1$, $C = 1$, $\tilde{\alpha} = 1$, $1 \leq r_0 \leq 3$, and $n = 5$ (top curve) 6, 7, 8, 9, and 10 (bottom curve). In this case we have stable orbits of arbitrary size.

The condition for the existence of f ($\beta_* > 0$) gives us $[(n - 1)(n - 2) + 8\Lambda\tilde{\alpha} + 4\tilde{\alpha}Cr_0^{1-n}] > 0$. For $\alpha < 0$, the numerator of ν^2, $N_\nu = r_0^n[(n - 1)(n - 2)(1 + \sqrt{\beta_*}) + 8\Lambda\tilde{\alpha} - \tilde{\alpha}Cr_0^{1-n}(n - 5)]$, is always positive, but the denominator $D_\nu = \tilde{\alpha}(n - 1)(+2r_0^n\sqrt{\beta_*}(n - 2) + r_0^3C)$ is negative, so $\nu^2 < 0$, and in this case no circular orbit is stable.

For $\alpha > 0$, we first analyze the case $\Lambda > 0$. In this case all parameters are positive, and the denominators of ν^2 and κ^2 are always positive. The same happens with f_+, β_*, $\tilde{B}_4$, $\tilde{B}_3$, $\tilde{B}_2$, and the expression $[(n - 1)(n - 2) \times (1 + \sqrt{\beta_*}) + 8\Lambda\tilde{\alpha}]$ (note that $\tilde{B}_i$ is the coefficient B_i after replacing $-\sqrt{\beta_*}$ by $\sqrt{\beta_*}$, $i = 2, 3, 4$). These positive coefficients ensure that, for r_0 sufficiently large, the numerators of ν^2 and κ^2 are positive. Hence, we conclude that there are stable circular orbits for $n \geq 5$, $\alpha > 0$, and $\Lambda > 0$.

If $\alpha > 0$ and $\Lambda < 0$, it is necessary to set conditions to ensure $\beta_* > 0$. It happens when $(n - 1)(n - 2) + 8\Lambda\tilde{\alpha} > 0$, which is true when $2|\Lambda|\tilde{\alpha} < 3$. In this case, from the expressions for ν^2 and κ^2, we can see that the same argument used in the case $\Lambda > 0$ is valid. Hence, we conclude that there are stable circular orbits for $n \geq 5$, $\alpha > 0$, and $\Lambda < 0$.

Figure 4 shows the functions ν^2 and κ^2 for $\Lambda = 1$, $C = 1$, $\tilde{\alpha} = 1$, $1 \leq r_0 \leq 3$, and $n = 5$ (top curve) 6, 7, 8, 9, and 10 (bottom curve). In this case we have stable orbits of arbitrary size.

We find a similar behavior for the equivalent case with $\Lambda = -1$.

VI. SUMMARY AND FINAL REMARKS

An observationally viable theory of gravitation must allow for the existence of stable limited orbits. It is known that, in Newtonian gravity as well as in classical general relativity, these orbits exist only if the dimension of the usual space is three ($n = 4$). We analyze the linear stability of circular geodesics in an n-dimensional ($n \geq 4$) space-time with not null cosmological constant Λ, and we find that there exist cases in which these orbits are stable.

We also analyze the linear stability of circular geodesics for n-dimensional spherically symmetric static solutions of EGB gravity. Here we also find that there exist cases with stable circular orbits. The differential equation that gives us the metric function has two families of solutions corresponding to the sign in front of the square root in (29). Following the definition given in Ref. [20], the family with the minus (plus) sign is called the GR branch (non-GR branch) solution. The EGB part of our work is divided into several cases, depending on whether the cosmological constant Λ is zero or not and on the kind of solution used (GR or non-GR branch solution).

For the non-GR solution, the sign of the coupling parameter α is very important. When $\alpha < 0$, no matter the value of Λ or n, it is not possible to find stable circular orbits. Otherwise, when $\alpha > 0$, for all $n \geq 5$, it is possible to find stable circular orbits, no matter whether Λ is null, positive, or negative.

The results obtained for the GR solution are similar. When Λ is positive or null, and α is negative, the circular orbits are not stable. But we show values of $\alpha > 0$, $5 \leq n \leq 9$, $\Lambda \geq 0$, and r_0 such that the existence of stable circular orbits is possible. The great difference of this branch solution occurs in the case $\Lambda < 0$. In this case, for example, taking $\Lambda = -1$, $\alpha = \pm 1$, $n \geq 5$, and $r_0 \geq 1.2$, we have stability. The results above are summarized in Tables I, II, and III.

TABLE I. Stability of circular orbits in n-dimensional Newtonian gravity with or without a cosmological constant. The existence of some stable curves is denoted by SSC and their nonexistence by NSC.

Newtonian case	$n = 4$	$n = 5$	$n \geq 6$
$\Lambda = 0$	SSC	NSC	NSC
$\Lambda < 0$	SSC	SSC	SSC
$\Lambda > 0$	SSC	NSC	NSC

TABLE II. Stability of circular orbits in n-dimensional Einstein gravity with or without a cosmological constant. The existence of some stable curves is denoted by SSC and their nonexistence by NSC.

GR case	$n = 4$	$n \geq 5$
$\Lambda = 0$	SSC	NSC
$\Lambda < 0$	SSC	SSC
$\Lambda > 0$	SSC	NSC

TABLE III. Stability of circular orbits in Einstein-Gauss-Bonnet theory with or without a cosmological constant and different signs for the EGB coupling constant. The existence of some stable curves is denoted by SSC and their nonexistence by NSC.

EGB ($\alpha \neq 0$)	$\Lambda = 0$		$\Lambda > 0$		$\Lambda < 0$	
	$\alpha > 0$	$\alpha < 0$	$\alpha > 0$	$\alpha < 0$	$\alpha > 0$	$\alpha < 0$
Non-GR branch	SSC	NSC	SSC	NSC	SSC	NSC
GR branch	SSC	NSC	SSC	NSC	SSC	SSC

A similar analysis as the one presented in this article is under study for higher dimensions rotating black holes like the one presented in Refs. [22] (five-dimensional GR with magnetic field), [23] (n-dimensional GR), and [24] (five-dimensional EGB theory). Other possible generalizations and applications of the present work concern the study of gravitational radiation in higher dimensions in the case where circular orbits do exist. For gravitational radiation in n-dimensional spacetimes, see, for instance, [25]. Also the ultrarelativistic plunge into a multidimensional black hole was studied in [26]. A generalization of this calcula-tion to circular orbits may be relevant to improve our understanding of mini-black-hole production in TeV-gravity scenarios.

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