Real trajectories in the semiclassical coherent state propagator

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The semiclassical approximation to the coherent state propagator requires complex classical trajectories in order to satisfy the associated boundary conditions, but finding these trajectories in practice is a difficult task that may compromise the applicability of the approximation. In this work several approximations to the coherent state propagator are derived that make use only of real trajectories, which are easier to handle and have a more direct physical interpretation. It is verified in a particular example that these real trajectories approximations may have excellent accuracy.

I. INTRODUCTION

The path integral representation of the coherent state propagator $K(z_1, z_2, T) = \langle z_2|e^{-iHT/\hbar}|z_1\rangle$, where the $|z_i\rangle$ are the usual harmonic oscillator coherent states, appeared in the works of Klauder and collaborators[1,2] and of Weissman.[3] A semiclassical, or stationary phase, approximation leads to classical trajectories satisfying Hamilton equations of motion but subject to special boundary conditions that can only be satisfied in a complex phase space. Aguiar and Baranger also considered this problem[4] and discovered an extra term, which they called $\mathcal{I}$, in the semiclassical approximation that had been overlooked in previous studies and that turns out to be essential for a correct theory[5] (the semiclassical spin propagator has a similar term,[6] known as the Solari-Kochetov correction). Numerical calculations involving complex trajectories in the semiclassical coherent state propagator have been done for a variety of systems: Adachi considered a one-dimensional and time-dependent problem with chaotic dynamics;[7] Rubin and Klauder[2], as well as Xavier and Aguiar,[8] have treated 1D bound systems; one dimensional tunnelling was considered in [9] and also in [10]; Van Voorhis and Heller presented calculations for one and two dimensions[11] and for the N-dimensional Henon-Heiles potential;[12] Ribeiro *et al* have worked with the 2D chaotic Nelson potential;[13] (numerical applications involving the spin coherent states have also appeared[14]). Semiclassical approximations based on complex trajectories for the coordinate wave function, i.e. for the mixed representation $K(x, z, T) = \langle x|e^{-iHT/\hbar}|z\rangle$, were also developed, initially for the one-dimensional case[15,16,17] and then generalized to many dimensions.[18] The actual calculation of complex trajectories involves two difficulties: first, the effective dimensionality of the phase space is doubled, since both real and imaginary parts of position and momentum must be computed; second, the boundary conditions are defined part at initial time and part at final time, and finding the appropriate classical trajectory becomes a difficult problem known as 'root search'. Therefore approximations that make use only of real trajectories are certainly desirable.

Since the propagator $K(z_1, z_2, T)$ is a function of time, any complex trajectory that satisfies the boundary conditions at time T must belong to a whole 'branch' of trajectories, parametrized by T. In general, for a given system and for fixed values of z_1 and z_2 there are several such branches. In practice, once a solution is found for a particular value of T, one may obtain all elements of the same branch by making small steps forward or backward in time and using appropriate iteractive procedures. It may happen that for a certain value of time a relevant complex trajectory has a small (or even null) imaginary part, and in that case its branch was called 'nearly real' by Van Voorhis and Heller.[11,12] It is possible that more than one 'nearly real' branch contribute to the semiclassical propagator for a given time, and thus one may consider only these branches and still accurately reproduce interference effects.

A similar analysis can be made for the mixed propagator $K(x, z, T)$, but in this case one usually holds T fixed and considers x as a parameter. Varying x thus produces a 'family' of trajectories, and again there may exist several such families. However, there is always a value of x for which the involved trajectory is real, and its family was called the 'main family' by Aguiar *et al.*[16] Using the main contribution alone is sometimes a very good approximation, but it can not reproduce interference because only one trajectory enters the calculation at a time. On the other hand, as already noted, finding all the necessary complex trajectories (i.e. performing the 'root search') is usually a difficult problem, specially in more than one dimension. Therefore the possibility was considered[16,18] of employing only real trajectories in the semiclassical approximation to $K(x, z, T)$. This was done by approximating the complex trajectories by real ones, that are compatible with the quantum uncertainties and satisfy less restrictive boundary conditions. The final real trajectories approximations are in principle less accurate than the original complex one, but they are much simpler and sometimes have practically the same accuracy.[16,18]

The purpose of the present work is to present semiclassical approximations to $K(z_1, z_2, T)$ that are based only on real classical trajectories, thus making the calculation much more tractable. One method that accomplishes exactly this is the 'cellular dynamics', initially developed by Heller[19] (see also [20]) and later generalized and applied to the stadium billiard with great success.[21] This technique has shown to be accurate even for long times,[21,22] and it is actually very close in spirit to the present work, in the sense that the contribution of a complex classical trajectory is expanded to second order in the vicinity of a real one. However, Heller's starting point is the Van-Vleck-Gutzwiller formula for the semiclassical propagator,[23] while we start from the formulation of Baranger *et al*,[5] and our results are slightly different from those of Heller. We also consider a variety of boundary conditions that the real trajectories may satisfy, something not discussed at length in.[21]

Another approach to the semiclassical coherent state propagator that is based on real trajectories is the so-called Initial Value Representations, such as that of Herman and Kluk.[24] Recent reviews of this method can be found in [25]. Initial value methods are usually easy to apply and reasonably accurate for long times, but they require a numerical integration over all possible initial conditions. Since the present method requires only a few trajectories, at least for short times, it provides a much clearer physical picture.

This article is divided as follows. In the next section we give a brief account of the semiclassical approximation to the coherent state propagator $K(z_1, z_2, T)$ and the complex trajectories. In section III we present the approximations that are based on the real trajectories defined by z_1 or by z_2. Real trajectories that satisfy mixed boundary conditions are investigated in section IV. We present an application to a nonlinear oscillator in sections V and VI and we conclude in section VII.

II. THE SEMICLASSICAL COHERENT STATE PROPAGATOR

The coherent states of a harmonic oscillator of mass m and angular frequency ω are defined by

$$|z\rangle = \exp\{z a^\dagger - z^* a\}|0\rangle, \tag{2.1}$$

where $|0\rangle$ is the oscillator ground state. The operators $a^\dagger$ and a are respectively creation and annihilation operators, related to position Q and momentum P by

$$a = \frac{1}{\sqrt{2}}\left(\frac{Q}{b} + i\frac{P}{c}\right), \quad a^\dagger = \frac{1}{\sqrt{2}}\left(\frac{Q}{b} - i\frac{P}{c}\right). \tag{2.2}$$

The parameters b and c define natural scales of the problem, and are such that $bc = \hbar$ and $c/b = m\omega$. It is easy to see that if we write

$$z = \frac{1}{\sqrt{2}}\left(\frac{q}{b} + i\frac{p}{c}\right) \tag{2.3}$$

then x and p are average values,

$$\langle z|Q|z\rangle = q, \quad \langle z|P|z\rangle = p. \tag{2.4}$$

The parameters b and c are related to quantum uncertainties,

$$\Delta Q = \frac{b}{\sqrt{2}}, \quad \Delta P = \frac{c}{\sqrt{2}}, \tag{2.5}$$

and we see that coherent states are minimum uncertainty states.

These coherent states are never orthogonal,

$$\langle z_2|z_1\rangle = \exp\left\{-\frac{1}{2}|z_1|^2 - \frac{1}{2}|z_2|^2 + z_1 z_2^*\right\}, \tag{2.6}$$

and in the position representation they are Gaussians,

$$\langle x|z\rangle = \pi^{-\frac{1}{4}} b^{-\frac{1}{2}} \exp\left\{-\frac{(x-q)^2}{2b^2} + \frac{i}{\hbar} p(x-q)\right\}. \tag{2.7}$$

In terms of the usual basis of number states $|n\rangle$, defined such that $a^\dagger a|n\rangle = n|n\rangle$, the coherent states may be written as

$$|z\rangle = e^{-|z|^2/2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle. \tag{2.8}$$

It is easy to see that they are eigenstates of the annihilation operator, $a|z\rangle = z|z\rangle$.

In order to write the semiclassical approximation to the quantum coherent state propagator

$$K(z_1, z_2, T) = \langle z_2|e^{-iHT/\hbar}|z_1\rangle, \tag{2.9}$$

we must consider a complex version of the phase space, i.e. we must make use of a coordinate $q(t)$ and a momentum $p(t)$ that are complex numbers. Following the approach of Ref. [5] we define

$$u(t) = \frac{1}{\sqrt{2}}\left(\frac{q(t)}{b} + i\frac{p(t)}{c}\right), \quad v(t) = \frac{1}{\sqrt{2}}\left(\frac{q(t)}{b} - i\frac{p(t)}{c}\right). \tag{2.10}$$

It is of fundamental importance to realize that $v(t)$ is not the complex conjugate of $u(t)$. In terms of these variables the boundary conditions become

$$u(0) = u' = z_1, \quad v(T) = v'' = z_2^*. \tag{2.11}$$

There is nothing special about the values $u(T) = u''$ and $v(0) = v'$, they are to be determined dynamically. We use hereafter a prime (double prime) to denote initial (final) values, in order to simplify the formulas and stay close to the notation of [5].

The canonical coherent state propagator is

$$K_{\rm sc}(z_1, z_2, T) = \mathcal{N}\sum_{\rm c.t.}\sqrt{\frac{i}{\hbar}\frac{\partial^2 S}{\partial u'\partial v''}}\exp\left\{\frac{i}{\hbar}(S+\mathcal{I})\right\}, \tag{2.12}$$

where $\mathcal{N} = \exp\{-\frac{1}{2}|z_1|^2 - \frac{1}{2}|z_2|^2\}$ is a normalization factor, the summation is over all classical trajectories satisfying the boundary conditions, and the complex action is given by

$$S(u', v'', T) = \int_0^T dt\left[\frac{i\hbar}{2}(\dot{u}v - \dot{v}u) - \mathcal{H}\right] - \frac{i\hbar}{2}(u'v' - u''v''). \tag{2.13}$$

This is related to the usual Hamilton action

$$S_H = \int_0^T (p\dot{q} - \mathcal{H})dt \tag{2.14}$$

by

$$S = S_H - \frac{q'p' - q''p''}{2} - \frac{i\hbar}{2}(u'v' - u''v''). \tag{2.15}$$

The Hamiltonian that governs the classical movement according to the usual Hamilton equations

$$\dot{q} = \frac{\partial\mathcal{H}}{\partial p}, \quad \dot{p} = -\frac{\partial\mathcal{H}}{\partial q}, \tag{2.16}$$

is the average value of the quantum Hamiltonian in coherent states,

$$\mathcal{H} = \langle z|H|z\rangle, \tag{2.17}$$

which is sometimes called the smoothed Hamiltonian. The quantity $\mathcal{I}$ is related to its second derivative,

$$\mathcal{I} = \frac{1}{2}\int_0^T \frac{\partial^2\mathcal{H}}{\partial u\partial v}dt. \tag{2.18}$$

The prefactor in (2.12) can be written only in terms of the complex tangent matrix. The classical tangent matrix of a certain trajectory is the linear application that relates initial and final displacements about it. We take into account the quantum uncertainties to define it as follows:

$$\begin{pmatrix} \delta q''/b \\ \delta p''/c \end{pmatrix} = \begin{pmatrix} m_{qq} & m_{qp} \\ m_{pq} & m_{pp} \end{pmatrix} \begin{pmatrix} \delta q'/b \\ \delta p'/c \end{pmatrix}. \tag{2.19}$$

The complex tangent matrix, on the other hand, is defined as

$$\begin{pmatrix} \delta u'' \\ \delta v'' \end{pmatrix} = \begin{pmatrix} M_{uu} & M_{uv} \\ M_{vu} & M_{vv} \end{pmatrix} \begin{pmatrix} \delta u' \\ \delta v' \end{pmatrix}. \tag{2.20}$$

The relation between the matrix elements of these different representations is as follows:

$$2M_{uu} = m_{qq} + m_{pp} + im_{pq} - im_{qp}, \tag{2.21}$$
$$2M_{uv} = m_{qq} - m_{pp} + im_{pq} + im_{qp}, \tag{2.22}$$
$$2M_{vu} = m_{qq} - m_{pp} - im_{pq} - im_{qp}, \tag{2.23}$$
$$2M_{vv} = m_{qq} + m_{pp} - im_{pq} + im_{qp}. \tag{2.24}$$

It is possible to show that the second derivative of the complex action is given by

$$\frac{i}{\hbar} \frac{\partial^2 S}{\partial u' \partial v''} = \frac{1}{M_{vv}}, \tag{2.25}$$

and therefore the semiclassical coherent state propagator becomes

$$K_{\rm sc}(z_1, z_2, T) = \sum_{\rm c.t.} \frac{\mathcal{N}}{\sqrt{M_{vv}}} \exp\left\{\frac{i}{\hbar}(\mathcal{I} + S)\right\}. \tag{2.26}$$

Upon fixing z_1 and z_2, the squared modulus of this propagator may be interpreted as a time dependent transition probability. On the other hand, if we fix z_1 and T and consider z_2 as a variable then $|K(z_1, z_2, T)|^2$ is a phase space representation, a Husimi function, of the evolved state $e^{-iHT/\hbar}|z_1\rangle$.

If it happens that M_{vv} tends to zero for a certain combination of (z_1, z_2, T), then we see that the semiclassical approximation (2.26) diverges. This is called a phase space caustic[2,7,12,16] and the quadratic approximation used in the derivation of (2.26) is not valid in its vicinity. In order to obtain an uniform approximation that remains valid at caustics it is necessary to employ a conjugate application of the Bargmann representation, as discussed in.[26] We shall not be concerned with caustics in this work.

III. THE 'LEAVING' AND THE 'ARRIVING' TRAJECTORIES

We have seen that the classical trajectories entering the semiclassical propagator are determined by mixed boundary conditions. The initial position and momentum q' and p' are not the real numbers q_1 and p_1, but rather some complex numbers such that $u' = z_1$. Conversely, the final values q'', p'' are not q_2, p_2 but are such that $v'' = z_2^*$. It is in general not a easy task to find such trajectories in practice, even for simple systems. However, it may happen that the complex trajectory is close enough to a real one so that we may still obtain a reasonable result by expanding the propagator to second order in the vicinity of this real trajectory.[16,18] We investigate this problem with some detail in the next sections.

A. Leaving

Let us suppose a certain complex classical trajectory that is to be used in the calculation of the semiclassical propagator, and let us assume it is not very different from the real trajectory that starts at the point (q_1, p_1). We call this the 'leaving' trajectory because it leaves the phase space point corresponding to the initial coherent state. After a time T the position and the momentum will be some real numbers (q_f, p_f), generally different from the pair (q_2, p_2). We will expand the complex action up to second order around this trajectory. If q' is the initial complex position and p' is the initial complex momentum, we may write

$$q' = q_1 + \Delta q_1, \quad p' = p_1 + \Delta p_1, \tag{3.1}$$

where Δq_1 and Δp_1 are assumed to be small (complex) quantities. Moreover, if q'' is the final complex position and p'' is the final complex momentum, we may write in a similar way

$$q'' = q_f + \Delta q_f, \quad p'' = p_f + \Delta p_f. \tag{3.2}$$

Therefore we have the approximation

$$\begin{aligned} S(u', v'', T) \approx S(z_1, v_r, T) &+ \left.\frac{\partial S}{\partial q'}\right|_r \Delta q_1 + \left.\frac{\partial S}{\partial p'}\right|_r \Delta p_1 \\ &+ \frac{1}{2} \left.\frac{\partial^2 S}{\partial q'^2}\right|_r \Delta q_1^2 + \left.\frac{\partial^2 S}{\partial q' \partial p'}\right|_r \Delta q_1 \Delta p_1 + \frac{1}{2} \left.\frac{\partial^2 S}{\partial p'^2}\right|_r \Delta p_1^2, \end{aligned} \tag{3.3}$$

where the subscript r means that the quantity must be evaluated at the real trajectory (therefore $v'_r = b^{-1} q_1 - ic^{-1} p_1$ and $u''_r = b^{-1} q_f + ic^{-1} p_f$). In order to obtain the derivatives of the action, we resort to equations (2.10) and (2.15). Noticing that

$$\left.\frac{\partial S_H}{\partial q'}\right|_r = -p_i + \frac{\partial S_H}{\partial q''}\frac{\partial q''}{\partial q'}, \quad \left.\frac{\partial S_H}{\partial p'}\right|_r = \frac{\partial S_H}{\partial q''}\frac{\partial q''}{\partial p'}, \tag{3.4}$$

one can obtain the derivatives of the total action, which are given by

$$\left.\frac{\partial S}{\partial q'}\right|_r = -\frac{ic}{\sqrt{2}}[v'_r + (m_{qq} - i m_{pq}) u''_r], \quad \left.\frac{\partial S}{\partial p'}\right|_r = \frac{b}{\sqrt{2}}[v'_r - (m_{pp} + i m_{qp}) u''_r]. \tag{3.5}$$

From the definition of the tangent matrix we have

$$\frac{\partial q''}{\partial q'} = m_{qq}, \quad \frac{\partial q''}{\partial p'} = \frac{b}{c} m_{qp}, \quad \frac{\partial p''}{\partial q'} = \frac{c}{b} m_{pq}, \quad \frac{\partial p''}{\partial p'} = m_{pp}, \tag{3.6}$$

which determines, to first order, the final differences in terms of the initial ones:

$$\Delta q_2 = m_{qq} \Delta q_1 + \frac{b}{c} m_{qp} \Delta p_1, \quad \Delta p_2 = \frac{c}{b} m_{pq} \Delta q_1 + m_{pp} \Delta p_1. \tag{3.7}$$

On the other hand, the boundary conditions

$$\frac{q'}{b} + i\frac{p'}{c} = \frac{q_1}{b} + i\frac{p_1}{c}, \quad \frac{q''}{b} - i\frac{p''}{c} = \frac{q_2}{b} - i\frac{p_2}{c}, \tag{3.8}$$

provide the secondary relations

$$b^{-1}[\Delta q_2 + (q_f - q_2)] = ic^{-1}[\Delta p_2 + (p_f - p_2)], \quad b^{-1} \Delta q_1 = -ic^{-1} \Delta p_1. \tag{3.9}$$

Solving for Δq_1 and Δp_1 in terms of $(q_f - q_2)$ and $(p_f - p_2)$ we have

$$\Delta q_1 = -M_{vv}^{-1}[(q_f - q_2) - ib(p_f - p_2)/c], \quad \Delta p_1 = ic \Delta q_1 / b. \tag{3.10}$$

Substituting this in (3.3) one can see that the first order terms give

$$\left.\frac{\partial S}{\partial q'}\right|_r \Delta q_1 + \left.\frac{\partial S}{\partial p'}\right|_r \Delta p_1 = -\frac{i\hbar}{\sqrt{2}} u''_r \left[\frac{q_f - q_2}{b} - i\frac{p_f - p_2}{c}\right] = -i\hbar u''_r (v''_r - z_2^*). \tag{3.11}$$

It is easy to take derivatives of equation (3.5) in order to calculate the quadratic terms. In so doing we neglect derivatives of the tangent matrix elements, because this would be a higher order correction. Adding up all quadratic terms and making the proper identifications, we see that it can be related to the difference $(v_r'' - z_2^*)$ as

$$\text{quadratic terms} = -\frac{i\hbar}{2} M_{uv} M_{vv}^{-1} (v_r'' - z_2^*)^2. \tag{3.12}$$

Therefore the final result is the following:

$$K_{q_1 p_1}(z_1, z_2, T) = \frac{\mathcal{N}}{\sqrt{(M_{vv})_r}} \exp\left\{ \frac{i}{\hbar}(\mathcal{I}_r + S_r) + u_r''(v_r'' - z_2^*) + \frac{1}{2}\frac{M_{uv}}{M_{vv}}(v_r'' - z_2^*)^2 \right\}. \tag{3.13}$$

The subscript in $K_{q_1 p_1}$ denotes that this formula was obtained using the 'leaving' trajectory. Notice that the prefactor and the extra term were not expanded but simply evaluated at the real trajectory, which is consistent with the original quadratic derivation of the semiclassical approximation. It is also important to remember that even though the action S_r is evaluated at a real trajectory, it continues to be a complex number.

The expression (3.13) depends quadratically on the difference between the final value of the variable v along the real trajectory and the value that it would have in the complex trajectory. If by some reason the situation is such that v_r'' and z_2^* coincide, then this formula and the original one (2.12) will give the same result. One may argue that it is possible to obtain the same expression by expanding the action as

$$S \approx S_r + \left.\frac{\partial S}{\partial v''}\right|_r (v_r'' - z_2^*) + \frac{1}{2} \left.\frac{\partial^2 S}{\partial v''^2}\right|_r (v_r'' - z_2^*)^2. \tag{3.14}$$

This is certainly true and actually an easy calculation. We have chosen the long way of using the position/momentum variables because this will be the only possibility in the next section.

B. Arriving

What we call the 'arriving' trajectory is the real trajectory that starts in a certain initial point (q_i, p_i) and after a time T arrives at the point (q_2, p_2). We can use this trajectory to approximate the semiclassical propagator in the very same way that we did with the 'leaving' trajectory. Similar to the previous arguments, we write

$$q' = q_i + \Delta q_1, \quad p' = p_i + \Delta p_1, \quad q'' = q_2 + \Delta q_2, \quad p'' = p_2 + \Delta p_2. \tag{3.15}$$

Inverting equation (2.19) we see that

$$\frac{\partial q'}{\partial q''} = m_{pp}, \quad \frac{\partial q'}{\partial p''} = -\frac{b}{c} m_{qp}, \quad \frac{\partial p'}{\partial q''} = -\frac{c}{b} m_{pq}, \quad \frac{\partial p'}{\partial p''} = m_{qq}. \tag{3.16}$$

Using these relations we can write the initial differences in terms of the final ones, analogously to what we did in (3.7). Using the boundary conditions it is possible to show that

$$\Delta q_2 = -M_{vv}^{-1}[(q_i - q_1) + ib(p_i - p_1)/c], \quad \Delta p_2 = -ic\Delta q_2/b. \tag{3.17}$$

The first derivatives of the action are in this case given by

$$\left.\frac{\partial S}{\partial q''}\right|_r = -\frac{ic}{\sqrt{2}}[u_r'' + (m_{pp} - i m_{pq})v_r'], \quad \left.\frac{\partial S}{\partial p''}\right|_r = -\frac{b}{\sqrt{2}}[u_r'' - (m_{qq} + i m_{qp})v_r']. \tag{3.18}$$

We now expand the complex action to second order around this real trajectory. After simplifications, we obtain

$$K_{q_2 p_2}(z_1, z_2, T) = \frac{\mathcal{N}}{\sqrt{(M_{vv})_r}} \exp\left\{ \frac{i}{\hbar}(\mathcal{I}_r + S_r) + v_r'(u_r' - z_1) + \frac{1}{2}\frac{M_{vu}}{M_{vv}}(u_r' - z_1)^2 \right\}, \tag{3.19}$$

where the meaning of the subscript is evident. This time the expression depends on the difference between the initial value of the variable u in the real trajectory and the value that it would have in the complex one. Its interpretation is quite close to that of (3.13).

IV. OTHER POSSIBLE REAL TRAJECTORIES

In the previous section we saw that we may expand the semiclassical propagator in the vicinity of the real trajectories determined by the initial or by the final labels, (q_1, p_1) and (q_2, p_2), which we called the 'leaving' and the 'arriving' trajectories respectively. Although these are probably the most natural real trajectories approximations, we can devise four more possibilities that are also interesting. Of course one may use any real trajectory to build an approximation –in fact, in principle it should be possible to find the 'best' choice by a variational approach, but this seems to be a highly nontrivial problem–, but the idea here is to obtain explicit formulas for the most natural cases. These are the four trajectories that are determined by pairwise combination of the coherent state labels.

We shall present a detailed calculation for the case when the trajectories determined by the pair (q_1, q_2) are used. All other cases can be treated in a very similar way, and for them we shall be less explicit.

A. From q_1 to q_2

Let us consider a trajectory which satisfies the following boundary conditions: it leaves q_1 at time zero and arrives at q_2 at time T. Its initial and final momenta, p_i and p_f, remain unknown, but are real numbers. Differently from the previous section, now there may be more than one trajectory satisfying these requirements. We write

$$q' = q_1 + \Delta q_1, \quad q'' = q_2 + \Delta q_2, \quad p' = p_i + \Delta p_i, \quad p'' = p_f + \Delta p_f. \tag{4.1}$$

The initial and final momenta are regarded as functions of the initial and final positions. Therefore we may write

$$\Delta p_i = \left.\frac{\partial p'}{\partial q'}\right|_r \Delta q_1 + \left.\frac{\partial p'}{\partial q''}\right|_r \Delta q_2, \quad \Delta p_f = \left.\frac{\partial p''}{\partial q'}\right|_r \Delta q_1 + \left.\frac{\partial p''}{\partial q''}\right|_r \Delta q_2, \tag{4.2}$$

where again the subscript r means that the quantity must be evaluated at the real trajectory. On the other hand the boundary conditions $u' = z_1$ and $v'' = z_2^*$ imply that

$$\Delta p_i = \frac{ic}{b}\Delta q_1 - (p_i - p_1), \quad \Delta p_f = -\frac{ic}{b}\Delta q_2 - (p_f - p_2). \tag{4.3}$$

Since we are considering q' and q'' as independent variables, the partial derivatives in (4.2) are given by

$$\frac{\partial p'}{\partial q'} = -\frac{c}{b}\frac{m_{qq}}{m_{qp}}, \quad \frac{\partial p'}{\partial q''} = \frac{c}{b}\frac{1}{m_{qp}}, \quad \frac{\partial p''}{\partial q'} = -\frac{c}{b}\frac{1}{m_{qp}}, \quad \frac{\partial p''}{\partial q''} = \frac{c}{b}\frac{m_{pp}}{m_{qp}}, \tag{4.4}$$

where we have used that $m_{qq}m_{pp} - m_{qp}m_{pq} = 1$. Substituting this in (4.2) and using (4.3) we have

$$\frac{\Delta q_1}{b} = \frac{m_{qp}}{c}\frac{[(p_f - p_2) - M_2(p_i - p_1)]}{1 - M_1 M_2}, \quad \frac{\Delta q_2}{b} = \frac{m_{qp}}{c}\frac{[M_1(p_f - p_2) - (p_i - p_1)]}{1 - M_1 M_2}, \tag{4.5}$$

where we have defined the complex numbers

$$M_1 = m_{qq} + i m_{qp}, \quad M_2 = m_{pp} + i m_{qp}. \tag{4.6}$$

We now expand the complex action around this real trajectory up to second order,

$$\begin{aligned} S \approx S_r &+ \left.\frac{\partial S}{\partial q'}\right|_r \Delta q_1 + \left.\frac{\partial S}{\partial q''}\right|_r \Delta q_2 \\ &+ \frac{1}{2}\left.\frac{\partial^2 S}{\partial q'^2}\right|_r \Delta q_1^2 + \left.\frac{\partial^2 S}{\partial q' \partial q''}\right|_r \Delta q_1 \Delta q_2 + \frac{1}{2}\left.\frac{\partial^2 S}{\partial q''^2}\right|_r \Delta q_2^2. \end{aligned} \tag{4.7}$$

Noticing that

$$\left.\frac{\partial S_H}{\partial q'}\right|_r = -p_i, \quad \left.\frac{\partial S_H}{\partial q''}\right|_r = p_f, \tag{4.8}$$

we can obtain the derivatives of the total action,

$$\left.\frac{\partial S}{\partial q'}\right|_r = \frac{c}{\sqrt{2}m_{qp}}[u''_r - v'_r(m_{qq} + im_{qp})], \quad \left.\frac{\partial S}{\partial q''}\right|_r = \frac{c}{\sqrt{2}m_{qp}}[v'_r - u''_r(m_{pp} + im_{qp})]. \tag{4.9}$$

After simplifications, the linear terms can be written as

$$\text{linear terms} = -\frac{b}{\sqrt{2}}[v'(p_i - p_1) - u''(p_f - p_2)]. \tag{4.10}$$

We now calculate the second derivatives and substitute (4.5) in (4.7). After many simplifications, the final result can be shown to be

$$K_{q_1 q_2}(z_1, z_2, T) = \sum_{\text{c.t}} \frac{\mathcal{N}}{\sqrt{(M_{vv})_r}} \exp\left\{\frac{i}{\hbar}(\mathcal{I}_r + S_r) + \frac{iz_2}{\sqrt{2}c}(p_f - p_2) - \frac{iz_1^*}{\sqrt{2}c}(p_i - p_1)\right\}$$
$$\times \exp\left\{-\frac{A_1}{2c^2}(p_i - p_1)^2 - \frac{A_2}{2c^2}(p_f - p_2)^2 - \frac{A_{12}}{2c^2}(p_i - p_1)(p_f - p_2)\right\}, \tag{4.11}$$

where

$$A_1 = 1 - \frac{1}{2}\left(\frac{1 - M_1^* M_2}{1 - M_1 M_2}\right), \quad A_2 = 1 - \frac{1}{2}\left(\frac{1 - M_2^* M_1}{1 - M_1 M_2}\right), \quad A_{12} = \frac{2im_{qp}}{1 - M_1 M_2}. \tag{4.12}$$

This expression is more complicated that the ones we obtained in section III. This is so because the classical trajectories involved are determined by mixed boundary conditions, i.e. their initial and final positions. Its structure is nevertheless still the same: it depends on differences between the values of the variables in the real trajectories and the corresponding coherent state labels. The most important property of this formula is that the initial momentum p_i is not known *a priori*. It must be determined as a function of the given parameters, and in fact there may be more than one possible value for it. Notice that since p_i and also p_f depend nontrivially on z_1, z_2 and T this formula is not a simple Gaussian as it may seem at first. Once again, even though the function S_r is evaluated at a real classical trajectory, it will in general be a complex number.

Notice that the differences $p_i - p_1$ and $p_f - p_2$ are always divided by the momentum uncertainty c. Therefore only classical trajectories whose initial momentum is within a distance c from p_1 may be important for the semiclassical propagator. The same reasoning applies to the final momentum. We see that the real trajectories to be used in this formalism must be compatible with the quantum uncertainty principle.

As a simple illustration of this formula, let us consider a harmonic oscillator of unit mass and angular frequency $\omega = c/b$. An initial condition (q', p') leads, after a time T, to the final values

$$q'' = q'\cos(\omega T) + \frac{p'}{\omega}\sin(\omega T), \quad p'' = -\omega q'\sin(\omega T) + p'\cos(\omega T). \tag{4.13}$$

If we impose that the trajectory must start in q_1 and end in q_2 then it is easy to see that there is only one possibility that satisfies these boundary conditions, for which

$$p_i = \frac{\omega(q_2 - q_1\cos(\omega T))}{\sin(\omega T)}, \quad p_f = \frac{\omega(q_2\cos(\omega T) - q_1)}{\sin(\omega T)}. \tag{4.14}$$

In this case we have $m_{qp} = \sin(\omega T)$ and $M_1 = M_2 = e^{i\omega T}$, which leads to $A_1 = A_2 = 1$ and $A_{12} = -e^{-i\omega T}$. The $e^{i\mathcal{I}_r/\hbar}$ term cancels the prefactor. Finally, using that $S_r = e^{-i\omega T}(q_1/b + ip_i/c)(q_2/b - ip_f/c)/2i$ we obtain

$$K_{q_1 q_2}(z_1, z_2, T) = \exp\{-\frac{1}{2}(|z_1|^2 + |z_2|^2) + e^{-i\omega T} z_1 z_2^*\}, \tag{4.15}$$

which is precisely the exact result. This comes as no surprise since the exact action in this case is of second order to begin with and thus all semiclassical approximations we consider in this work will be exact.

B. From q_1 to p_2

We now consider the real trajectory that starts in $q' = q_1$ with a certain momentum $p' = p_i$ and, after a time T, is in a final point $q'' = q_f$ with the momentum $p'' = p_2$. We therefore treat q' and p'' as independent variables, in which case we have the following partial derivatives:

$$\frac{\partial p'}{\partial q'} = -\frac{c}{b}\frac{m_{pq}}{m_{pp}}, \quad \frac{\partial p'}{\partial p''} = \frac{\partial q''}{\partial q'} = \frac{1}{m_{pp}}, \quad \frac{\partial q''}{\partial p''} = \frac{b}{c}\frac{m_{qp}}{m_{pp}}. \tag{4.16}$$

We may calculate the action's first derivatives,

$$\left.\frac{\partial S}{\partial q'}\right|_r = -\frac{ic}{\sqrt{2}m_{pp}}[u''_r + v'_r(m_{pp} - im_{pq})], \quad \left.\frac{\partial S}{\partial p''}\right|_r = \frac{b}{\sqrt{2}m_{pp}}[v'_r - u''_r(m_{pp} + im_{qp})], \tag{4.17}$$

and after writing

$$q' = q_1 + \Delta q_1, \quad q'' = q_f + \Delta q_f, \quad p' = p_i + \Delta p_i, \quad p'' = p_2 + \Delta p_2, \tag{4.18}$$

we may also obtain, using an expansion analogous to (4.2) and the boundary conditions, the relations

$$\frac{\Delta q_1}{b} = -\frac{im_{pp}}{c}\frac{[M_2(p_i - p_1) - ic(q_f - q_2)/b]}{1 + M_2 M_3^*}, \tag{4.19}$$

$$\frac{\Delta p_2}{c} = -\frac{im_{pp}}{b}\frac{[M_3(q_f - q_2) - ib/c(p_i - p_1)]}{1 + M_2 M_3^*}, \tag{4.20}$$

where M_2 has already been defined and $M_3 = m_{pp} + im_{pq}$.

After calculating the action's second derivatives, the final result is

$$K_{q_1p_2}(z_1, z_2, T) = \sum_{\text{c.t}} \frac{\mathcal{N}}{\sqrt{(M_{vv})_r}} \exp\left\{\frac{i}{\hbar}(\mathcal{I}_r + S_r) - \frac{z_2}{\sqrt{2}b}(q_f - q_2) - \frac{iz_1^*}{\sqrt{2}c}(p_i - p_1)\right\}$$
$$\times \exp\left\{-\frac{B_1}{2c^2}(p_i - p_1)^2 - \frac{B_2}{2b^2}(q_f - q_2)^2 + \frac{B_{12}}{\hbar}(p_i - p_1)(q_f - q_2)\right\}, \tag{4.21}$$

where the coefficients are given by

$$B_1 = 1 - \frac{1}{2}\left(\frac{1 - M_2 M_3}{1 + M_2 M_3^*}\right), \quad B_2 = 1 - \frac{1}{2}\left(\frac{1 - M_2^* M_3^*}{1 + M_2 M_3^*}\right), \quad B_{12} = \frac{im_{pp}}{1 + M_2 M_3^*}. \tag{4.22}$$

We see that the semiclassical propagator obtained is quite similar in structure to the one presented in the previous subsection. Only this time we have position and momentum in a more equal footing. As $p_i - p_1$ is always divided by c and $q_f - q_2$ is always divided by b, we see that again the quantum uncertainties play a fundamental role in selecting the relevant classical trajectories.

C. From p_1 to q_2

It is also possible to fix the initial momentum as p_1 and then search for an initial position q_i such that the final position is q_2. In that case the final momentum will be some p_f. Proceeding in complete analogy with the previous cases, we take p' and q'' to be independent variables and calculate derivatives of q', p'' and S with respect to them. After obtaining the values of Δp_1 and Δq_2 in terms of $(q_i - q_1)$ and $(p_f - p_2)$ and expanding the action to second order, the final result will be

$$K_{p_1q_2}(z_1, z_2, T) = \sum_{\text{c.t}} \frac{\mathcal{N}}{\sqrt{(M_{vv})_r}} \exp\left\{\frac{i}{\hbar}(\mathcal{I}_r + S_r) + \frac{iz_2}{\sqrt{2}c}(p_f - p_2) - \frac{z_1^*}{\sqrt{2}b}(q_i - q_1)\right\}$$
$$\times \exp\left\{-\frac{C_1}{2b^2}(q_i - q_1)^2 - \frac{C_2}{2c^2}(p_f - p_2)^2 - \frac{C_{12}}{\hbar}(q_i - q_1)(p_f - p_2)\right\}, \tag{4.23}$$

where the coefficients are given by

$$C_1 = 1 - \frac{1}{2}\left(\frac{1 - M_1^* M_4^*}{1 + M_1 M_4^*}\right), \quad C_2 = 1 - \frac{1}{2}\left(\frac{1 - M_1 M_4}{1 + M_1 M_4^*}\right), \quad C_{12} = \frac{im_{qq}}{1 + M_1 M_4^*}, \tag{4.24}$$

with $M_4 = m_{qq} + im_{pq}$.

D. From p_1 to p_2

Finally, we consider the trajectory determined by the pair (p_1, p_2). This has initial and final positions q_i and q_f, respectively. The procedure to obtain the semiclassical approximation is certainly clear by now, so it will not be repeated in any detail. The final result in this case will be

$$K_{p_1p_2}(z_1, z_2, T) = \sum_{\text{c.t}} \frac{\mathcal{N}}{\sqrt{(M_{vv})_r}} \exp\left\{\frac{i}{\hbar}(\mathcal{I}_r + S_r) - \frac{z_1^*}{\sqrt{2}b}(q_i - q_1) - \frac{z_2}{\sqrt{2}b}(q_f - q_2)\right\}$$
$$\times \exp\left\{-\frac{D_1}{2b^2}(q_i - q_1)^2 - \frac{D_2}{2b^2}(q_f - q_2)^2 - \frac{D_{12}}{b^2}(q_i - q_1)(q_f - q_2)\right\}, \qquad (4.25)$$

where the coefficients are given by

$$D_1 = 1 - \frac{1}{2}\left(\frac{1 - M_3 M_4^*}{1 - M_3^* M_4^*}\right), \quad D_2 = 1 - \frac{1}{2}\left(\frac{1 - M_3^* M_4}{1 - M_3^* M_4^*}\right), \quad D_{12} = \frac{i m_{pq}}{1 - M_3^* M_4^*}. \qquad (4.26)$$

E. Summary of section IV

In this section we have obtained four different semiclassical approximations to the quantum coherent state propagator that are based only on real trajectories. The trajectories considered share the property that they are not determined by initial or final values, but satisfy mixed boundary conditions. Therefore finding them in practice is not trivial, but is certainly easier than finding the original complex ones. All the semiclassical propagators obtained are in principle able to reproduce quantum effects such as interference, since there may be more than one classical trajectory involved. They will be affected by caustics just like the original formula (2.26), but the location of such caustics will change because $(M_{vv})_r$ is different for each one of them.

Which one of the several formulas obtained here and in section III is more accurate will depend on the particular problem at hand. We have considered only initial and final coherent states with the same value of the parameter b, but a generalization of the semiclassical propagator was presented[27] for more general b's, and the calculations presented here may be adapted to that case with no essential difficulty. Let us suppose for a moment that the initial coherent state $|z_1\rangle$ has a position uncertainty b_1 while $|z_2\rangle$ has a position uncertainty b_2. If these numbers are small that means the states are very narrow in the position representation, while having a large uncertainty in momentum. In that case we conjecture that an approximation in the spirit of $K_{q_1q_2}$ would be the most effective one. If b_1 is small but b_2 is large, than $K_{q_1p_2}$ would be a better candidate, and so on. Of course for the free particle and the harmonic oscillator they are all exact, regardless of the values of b_1 and b_2.

In the next section, we present an application of the formalism just presented to a nonlinear system. The purpose is not to attempt an exhaustive investigation of the several possibilities, but rather to illustrate the method in a simple case. We have chosen a system for which many analytical results are possible so that the main properties of the theory do not disappear under numerical calculations.

V. APPLICATION TO A NONLINEAR OSCILLATOR: SHORT TIME

We consider the nonlinear Hamiltonian

$$H = \hbar\omega(a^\dagger a)^2 = \frac{1}{\hbar\omega}\frac{(p^2 + \omega^2 q^2 - \hbar\omega)^2}{4}, \qquad (5.1)$$

which is diagonal in the usual number basis,

$$H|n\rangle = E_n|n\rangle = \hbar\omega n^2|n\rangle. \qquad (5.2)$$

The quantum propagator in this case is quite simple:

$$K(z_1, z_2, T) = \langle z_2|e^{-iHT/\hbar}|z_1\rangle = \mathcal{N}\sum_{n=0}^{\infty}\frac{(z_1 z_2^*)^n}{n!}e^{-in^2\omega T}. \qquad (5.3)$$

We shall be interested, for simplicity, only in the diagonal case

$$K(z_1,z_1,T)=e^{-|z_1|^2}\sum_{n=0}^{\infty}\frac{|z_1|^{2n}}{n!}e^{-in^2\omega T}, \tag{5.4}$$

whose squared modulus is the return probability,

$$P(z_1,T)=|K(z_1,z_1,T)|^2. \tag{5.5}$$

This function is periodic with period $T_r=2\pi/\omega$. In the semiclassical limit the term that is responsible for the largest contribution to the sum in (5.4) is $n_0\approx|z_1|^2$. If we linearize the exponent in the vicinity of this term we have

$$P(z_1,T)\approx e^{-2|z_1|^2}\left|\sum_{n\approx n_0}\frac{|z_1|^{2n}}{n!}e^{2in_0n\omega T}\right|^2. \tag{5.6}$$

Notice that expression (5.6) has a distinct time scale,

$$T_c=\frac{\pi}{n_0\omega}. \tag{5.7}$$

The quantities T_r and T_c are usually called revival time and classical time.[28]

For short times, we can approximate $|K(z_1,T)|^2\approx 1-\left(\langle H^2\rangle-\langle H\rangle^2\right)T^2/\hbar^2$, where $\langle\cdot\rangle$ denotes an average value in the state $|z_1\rangle$. For the system in question, this gives

$$P(z_1,T)\approx 1-\left(4|z_1|^6+6|z_1|^4+|z_1|^2\right)\omega^2T^2. \tag{5.8}$$

Let us write $z_1=(q+ip)/\sqrt{2}$ and take for simplicity the value $q=0$. Since the movement in phase space has circular symmetry, this choice is of no fundamental importance. The short-time expansion (5.8) becomes simply

$$P(z_1,T)\approx 1-\frac{1}{2}\left(p^6+3p^4+p^2\right)T^2. \tag{5.9}$$

Let us now turn to the semiclassical approximation. From now on we set $\hbar=\omega=1$, which implies $b=c=1$. The smoothed Hamiltonian associated with (5.1) is

$$\mathcal{H}=\frac{(p^2+q^2)(p^2+q^2+2)}{4}=uv(uv+1), \tag{5.10}$$

and the corresponding Hamilton equations are

$$\dot{q}=\frac{\partial\mathcal{H}}{\partial p}=\sigma p,\quad \dot{p}=-\frac{\partial\mathcal{H}}{\partial q}=-\sigma q, \tag{5.11}$$

where we have defined $\sigma=p^2+q^2+1$. If we note that $\{\sigma,\mathcal{H}\}=0$, and thus that σ is a constant of the motion, then it is clear that

$$q''=q'\cos(\sigma t)+p'\sin(\sigma t),\quad p''=p'\cos(\sigma t)-q'\sin(\sigma t). \tag{5.12}$$

We see that the classical trajectories have a period of motion that is energy-dependent and given by $2\pi/\sigma$. If we remember that $n_0+1/2=(q^2+p^2)/2$ we see that in the semiclassical limit this time scale becomes precisely T_c.

The tangent matrix that is associated with the classical trajectory that starts in (q',p') can be obtained by simply differentiating the equations of motion. We must remember that the angular frequency σ is not uniform. The result is

$$\begin{pmatrix} m_{qq} & m_{qp}\\ m_{pq} & m_{pp}\end{pmatrix}=\begin{pmatrix}\cos(\sigma t) & \sin(\sigma t)\\ -\sin(\sigma t) & \cos(\sigma t)\end{pmatrix}\begin{pmatrix}1+2q'p't & 2p'^2t\\ -2q'^2t & 1-2q'p't\end{pmatrix}. \tag{5.13}$$

The action of such a trajectory is easily seen to be

$$S=\frac{(\sigma-1)^2T}{4}-i\frac{(\sigma-1)}{2}, \tag{5.14}$$

while the extra term is

$$\mathcal{I}=(\sigma-1/2)T. \tag{5.15}$$

The result of this semiclassical approximation based on complex trajectories will be given in section VI.

A. The 'leaving' and the 'arriving' trajectories

The first possibility we consider is to approximate the return probability by using only the real trajectory that leaves the position $q = 0$ with momentum p. The tangent matrix for that trajectory is

$$\begin{pmatrix} m_{qq} & m_{qp} \\ m_{pq} & m_{pp} \end{pmatrix} = \begin{pmatrix} \cos(\sigma t) & \sin(\sigma t) \\ -\sin(\sigma t) & \cos(\sigma t) \end{pmatrix} \begin{pmatrix} 1 & 2p^2 T \\ 0 & 1 \end{pmatrix}, \tag{5.16}$$

which gives the values

$$M_{uv} = -ip^2 T e^{-i\sigma T}, \quad M_{vv} = (1 + ip^2 T) e^{i\sigma T}. \tag{5.17}$$

The angular frequency is $\sigma = p^2 + 1$ and the final points in phase space are $q_f = p\sin(\sigma T)$ and $p_f = p\cos(\sigma T)$. This corresponds to

$$u_r'' = (v_r'')^* = \frac{ip}{\sqrt{2}} e^{-i\sigma T}. \tag{5.18}$$

Inserting all this information, together with (5.14) and (5.15), into the formula (3.13) we have

$$K_{q_1 p_1}(z_1, T) = \frac{1}{\sqrt{1 + ip^2 T}} \exp\left\{ \frac{iT}{4}(p^4 + 2p^2) - ip^2 e^{-i\sigma T/2} \sin(\sigma T/2) \right\}$$
$$\times \exp\left\{ \frac{ip^4 T}{(1 + ip^2 T)} e^{-i\sigma T} \sin^2(\sigma T/2) \right\}. \tag{5.19}$$

The first thing we note is that for $p = 0$ we obtain the exact result $K = 1$. Moreover, in the short time regime we can expand (5.19) and obtain

$$|K_{q_1 p_1}(z_1, T)|^2 \approx 1 - \frac{1}{2}\left(p^6 + 3p^4 + p^2\right) T^2 \quad \text{(short times)}, \tag{5.20}$$

which again reproduces the exact calculation. For later times we should not expect exact agreement.

Let us now turn to the 'arriving' trajectory, the one that starts in q_i, p_i and arrives at te position $q = 0$ with momentum p after a time T. The tangent matrix is less trivial than in the previous case, but in the end we get

$$M_{vu} = -ip^2 T e^{-i\sigma T}, \quad M_{vv} = (1 + ip^2 T) e^{i\sigma T}. \tag{5.21}$$

Since $q^2 + p^2$ is a conserved quantity, we have $\sigma = q_i^2 + p_i^2 + 1 = p^2 + 1$. After the whole calculation is done, we find out that $K_{q_2 p_2} = K_{q_1 p_1}$. This indicates that perhaps these two approximations will always have the same content of information, something that is not completely unexpected because of the dual role of $|z_1\rangle$ and $|z_2\rangle$ in the quantum propagator.

B. The $q_1 \to q_2$ possibility

In that case the trajectories that enter the approximation have initial momentum given by the equation

$$p_i \sin[(p_i^2 + 1)T] = 0. \tag{5.22}$$

Of course one solution to this equation is

$$p_i (= p_f) = 0, \tag{5.23}$$

in which case the particle simply stay still and $\sigma = 1$. It is easy to see that for this trajectory the tangent matrix is very simple,

$$\begin{pmatrix} m_{qq} & m_{qp} \\ m_{pq} & m_{pp} \end{pmatrix} = \begin{pmatrix} \cos(T) & \sin(T) \\ -\sin(T) & \cos(T) \end{pmatrix}, \tag{5.24}$$

and therefore $M_1 = M_2 = M_{vv} = e^{iT}$. The contribution of this trajectory to the propagator is

$$K_0 = \exp\{-ip^2 \sin(T/2) e^{-iT/2}\}, \tag{5.25}$$

where we have used $S_r + \mathcal{I}_r = T/2$. Notice that for $p = 0$ we again have the exact result $K_0 = 1$. We also note that the function $|K_0|^2$ has a period of 2π, which of course corresponds to the quantum revival time.

We now turn to the other solutions of equation (5.22). They are of the form

$$p_i^2(n) = \frac{2n\pi}{T} - 1, \tag{5.26}$$

which leads to $\sigma_n = 2n\pi/T$. In this case we have less trivial trajectories, for which the tangent matrix is given by

$$\begin{pmatrix} m_{qq} & m_{qp} \\ m_{pq} & m_{pp} \end{pmatrix} = \begin{pmatrix} 1 & 2p_i^2 T \\ 0 & 1 \end{pmatrix}, \tag{5.27}$$

and we see that the prefactor is $M_{vv} = (1 + ip_i^2 T)$, while $M_1 = M_2 = -4p_i^4 T^2$. The action and the extra term are given by $S_r = (p_i^4 T - 2ip_i^2)/4$ and $\mathcal{I}_r = (p_i^2 + 1/2)T$. The coefficients in (4.11) are

$$A_1 = A_2 = 1 + \frac{ip_i^2 T}{2(1 + ip_i^2 T)}, \quad A_{12} = -\frac{1}{1 + ip_i^2 T}. \tag{5.28}$$

After many simplifications, we obtain

$$K_{q_1 q_2} = \sum_{n=0}^{\infty} K_n, \tag{5.29}$$

where the contribution of the trajectory with label n (different from zero) is given by

$$K_n = \frac{1}{\sqrt{1 + ip_i^2 T}} \exp\{-\frac{ip_i^2 T}{1 + ip_i^2 T}(p_i - p)^2 + \frac{iT}{4}(p_i^4 + 4p_i^2 + 2)\}. \tag{5.30}$$

Note that for short times $p_i(n)$ is very large, so K_n becomes negligible and K_0 gives the only contribution. However, it predicts the initial decay $|K_0|^2 \approx 1 - 2p^2 T^2$, which is very slow compared to the exact calculation (5.9). The two results agree only for very small values of the momentum p.

Concerning the contributions K_n, we see that for a given instant of time the value of n that will contribute the most is that for which $p_i(n)$ is as close as possible to p, because of the Gaussian decay in (5.30). If we impose $p_i^2(n) \approx p^2$ we have $T \approx 2n\pi/(p^2 + 1)$, which means that the return probability has a maximum at the classical period, in agreement with the exact result.

C. The $q_1 \to p_2$ possibility

If we impose that the classical trajectory starts in $q = 0$ with momentum p_i and ends at q_f with momentum p, we have

$$q_f = p_i \sin(\sigma T), \quad p = p_i \cos(\sigma T), \quad \sigma = p_i^2 + 1. \tag{5.31}$$

These transcendental equations have no explicit solution. If we confine ourselves to the short time regime, then we can write $p_i \approx p$ and $q_f \approx p(p^2 + 1)T$. The tangent matrix is given by

$$\begin{pmatrix} m_{qq} & m_{qp} \\ m_{pq} & m_{pp} \end{pmatrix} = \begin{pmatrix} \cos(\sigma t) & \sin(\sigma t) \\ -\sin(\sigma t) & \cos(\sigma t) \end{pmatrix} \begin{pmatrix} 1 & 2p_i^2 T \\ 0 & 1 \end{pmatrix}, \tag{5.32}$$

and we obtain $M_2 = e^{i\sigma T}(1 + 2ip_i^2 T)$ and $M_3 = e^{-i\sigma T} - 2p_i^2 T$. Substituting this in (4.21), we obtain

$$|K_{q_1 p_2}|^2 \approx 1 - \left(p^6 + \frac{5}{2}p^4 + p^2\right) T^2 \quad \text{(short times)}, \tag{5.33}$$

which decays faster than the exact result but is a better approximation than the one obtained using the $q_1 \to q_2$ trajectory. We see that the different approximations may lead to very different results.

D. The $p_1 \to q_2$ possibility

The equations of motion in this case are

$$0 = q_i \cos(\sigma T) + p\sin(\sigma T), \quad p_f = -q_i \sin(\sigma T) + p\cos(\sigma T), \quad \sigma = q_i^2 + p^2 + 1, \tag{5.34}$$

while the tangent matrix is

$$\begin{pmatrix} m_{qq} & m_{qp} \\ m_{pq} & m_{pp} \end{pmatrix} = \begin{pmatrix} \cos(\sigma t) & \sin(\sigma t) \\ -\sin(\sigma t) & \cos(\sigma t) \end{pmatrix} \begin{pmatrix} 1 + 2q_i pT & 2p^2 T \\ -2q_i^2 T & 1 - 2q_i pT \end{pmatrix}. \tag{5.35}$$

The situation here regarding solubility of the equations is even worse than in the previous case. Once again we restrict the analysis to the short time regime. Then it is possible to write the first equation as $q_i \approx -p\sigma T/2$ and find

$$q_i \approx -\frac{1}{2pT}\left(1 - \sqrt{1 - 4p^2T^2(p^2+1)}\right), \tag{5.36}$$

which we substitute in the first equation to find p_f. Carrying out the whole calculation will give in the end

$$|K_{p_1 q_2}|^2 \approx 1 - \frac{1}{2}\left(p^6 + 3p^4 + p^2\right)T^2 \quad \text{(short times)}, \tag{5.37}$$

which agrees with the exact result.

E. The $p_1 \to p_2$ possibility

Finally, in the last possibility we have

$$q_f = q_i \cos(\sigma T) + p\sin(\sigma T), \quad p = -q_i \sin(\sigma T) + p\cos(\sigma T), \quad \sigma = q_i^2 + p^2 + 1. \tag{5.38}$$

In the short time limit we have again

$$q_i \approx -\frac{1}{2pT}\left(1 - \sqrt{1 - 4p^2T^2(p^2+1)}\right), \tag{5.39}$$

and the final result is

$$|K_{p_1 p_2}|^2 \approx 1 - \frac{p^4}{2}T^2 \quad \text{(short times)}. \tag{5.40}$$

This is kind of intermediate between the result we found in subsection B and that of subsections C and D.

VI. APPLICATION TO A NONLINEAR OSCILLATOR: NUMERICAL RESULTS

Before we consider the semiclassical approximations based on real trajectories for longer times, let us see how well the original one (2.26) compares to the exact result. This has been considered in detail in [29], so we just present the result. Given the initial condition $u' = z_1$, for each time T we must find a value for v' such that $v'' = z_2^* = z_1^*$. This problem usually has more than one solution, and we must add their contributions coherently. In Fig.1 we see the return probability as a function of time (in units of T_c) for the case $p = 10$, which we have chosen to ensure that we are in the semiclassical limit. The corresponding classical period is $T_c \approx 0.062$. The exact and the semiclassical results are indistinguishable in this scale.

In the previous section we saw how the different approximations based on real trajectories performed in the short time regime. The exact result was reproduced only by the 'leaving' and the 'arriving' formulas and by $K_{p_1 q_2}$. We now turn to the less simple case of arbitrary T, when the classical trajectories and the associated propagators must be computed numerically.

Let us start with the propagator $K_{q_1p_1}$, which is based on a real periodic orbit. Its initial decay is exact, and we can see how well it does for later times in Fig.2. It is able to reproduce the height of the peaks with great accuracy, but not their widths. Since there are never more than one contribution for each time, it never displays any interference effects.

This is not the case for $K_{q_1q_2}$. We see from (5.29) and (5.30) that it consists in the sum of many contributions. We focus on the values $n = 1, 2, 3$. Their individual contributions are depicted in Fig.3. Notice that the second and third peaks overlap. When we calculate the total propagator, this gives rise to interference. The final result is indistinguishable from the exact one (for $T > T_c/2$, because we have ignored K_0 which gives a bad initial decay).

So far the propagators could be obtained analytically. Since the calculation of $K_{q_1p_2}$ depends on the solution of the transcendental equation (5.31), we must resort to numerical routines. Let us try to find solutions to the second equation in (5.31) in the vicinity of the first period, $T \approx T_c$. In Fig.4a we see that there are two solutions (solid lines) for $T < T_c$ and no solution at all for $T > T_c$. This is because the cosine function with a real argument is always less than unity, and thus p_i must always be greater than p. The complex solutions do not have this obstruction, as we can also see in Fig.4a (dashed line), where we plot the real part of the complex momentum that satisfies the boundary conditions (2.11). Therefore the semiclassical approximation based on the complex trajectory can reproduce the whole peak, while $K_{q_1p_2}$ is discontinuous.

In Fig.4b we see the squared modulus of the exact propagator and the values of $|K_{q_1p_2}|^2$ obtained using the two available real trajectories. Note that one should not add these results. They are independent and we may choose any of them, because both real trajectories are good approximations to the actual complex one (the real trajectories do not come from a saddle point approximation). As observed in [16,18], the mixed propagator $\langle \mathbf{x}|e^{-iHT/\hbar}|\mathbf{z}\rangle$ can also be discontinuous when calculated using real trajectories. But in that case there are caustics involved, while here we have an algebraic obstruction.

The discussion of the approximation $K_{p_1q_2}$ is quite similar to the above. The solutions to the first equation in (5.34) are shown in Fig.5a, where we also show the real part of the complex position that satisfies the boundary conditions (2.11). Again there is no solution for $T > T_c$ and the semiclassical propagator is discontinuous, as we appreciate in Fig.5b. The results are practically the same as in Fig. 4b.

Finally, the propagator $K_{p_1p_2}$. This time we solve numerically the conditions (5.38) and find that there is a single real trajectory for $T < T_c$ and no one for $T > T_c$. The final result is in Fig.6.

VII. CONCLUSIONS

Several approximations to the semiclassical coherent state propagator $\langle z_2|e^{-iHT/\hbar}|z_1\rangle$ were presented that are based solely on real classical trajectories. Two of these approximations do not involve mixed boundary conditions and thus are not hindered by the associated ‘root search’ problem. The remaining four possibilities are based on trajectories that are determined by initial and final data, but since they are real for all times they are simpler to determine than the original complex ones.

As a testing ground we have used the nonlinear system $H = (a^\dagger a)^2$. Only one of the approximations, namely $K_{q_1q_2}$, reproduced the exact result to the fine details. This is certainly due to the particular initial coherent state that was chosen, one corresponding to $q = 0$ and $p = 10$. Had we chosen for example $q = 10$ and $p = 0$ and then $K_{p_1p_2}$ would give excellent results. We could also consider a nondiagonal propagator, and in that case we would expect $K_{q_1p_2}$, for example, to improve its performance.

Straightforward extensions of this work include the already mentioned case of different position uncertainties (squeezed states) and also higher dimensional systems. It is also possible to fix the time T and the initial state $|z_1\rangle$ and to regard $|K(z_1, z_2, T)|^2$ as a Husimi function defined in the z_2 plane. This is technically more difficult than what we have presented here, because it involves finding classical trajectories –usually more than one– parametrized by points in the plane.

Similar results can be obtained for the semiclassical $SU(2)$, or spin, coherent state propagator, even though the introduction of position and momentum variables in that case is not as natural. The associated phase space is also two-dimensional, but since it has curvature the calculations may be a little more involved. The same may be said about the semiclassical $SU(1,1)$ coherent state propagator. Since these groups have wide applications, it would be interesting to also have the corresponding approximations based on real trajectories.

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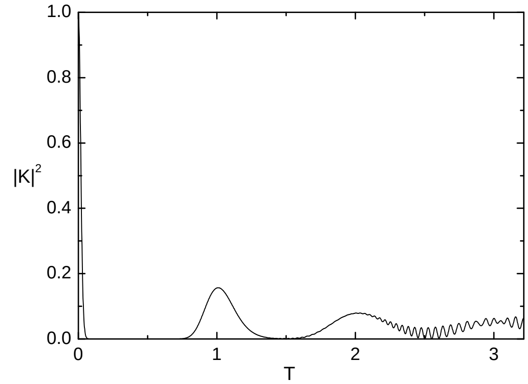

FIG. 1: Squared modulus of the exact propagator for $q = 0$ and $p = 10$. The semiclassical approximation based on complex trajectories is indistinguishable from it in this scale. Time is in units of the classical period.

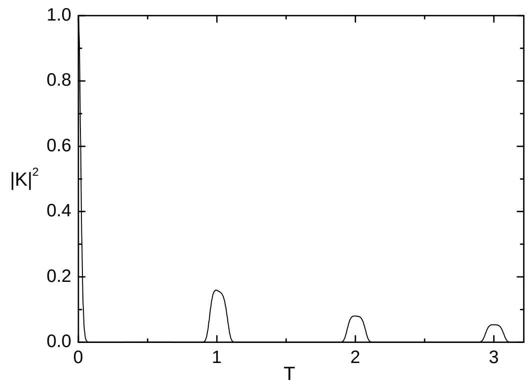

FIG. 2: The function $|K_{q_1 p_1}|^2$ as a function of time. It reproduces well the height of the peaks, but not their widths, and shows no interference.

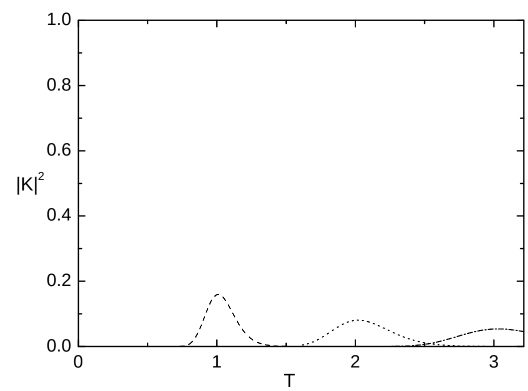

FIG. 3: The individual contributions $|K_n|^2$ for the approximation $K_{q_1 q_2}$. We show the cases $n = 1$, 2 and 3. When they are added there is interference, and the exact result of Fig.1 is reproduced with extraordinary accuracy for $T > T_c/2$ (we have not included K_0 in the calculation).

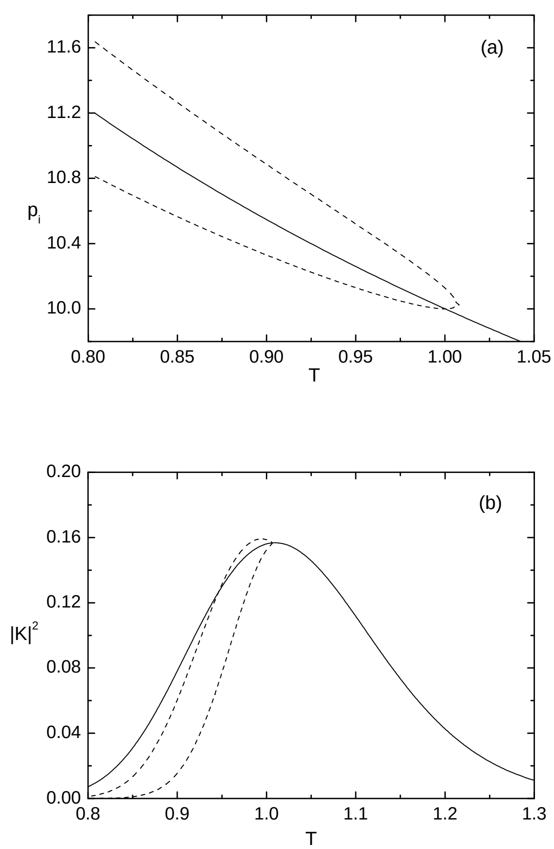

FIG. 4: Top: real solutions to the equation $p = p_i \cos((p_i^2+1)T)$ in the vicinity of the classical period (dashed lines). We also show the real part of the momentum for the complex trajectory (solid line). Bottom: approximation $|K_{q_1 p_2}|^2$ (dashed lines) compared to the exact result (solid line). Since there are no real trajectories for $T > T_c$, the propagator becomes truncated. For $T < T_c$ there are two possibilities.

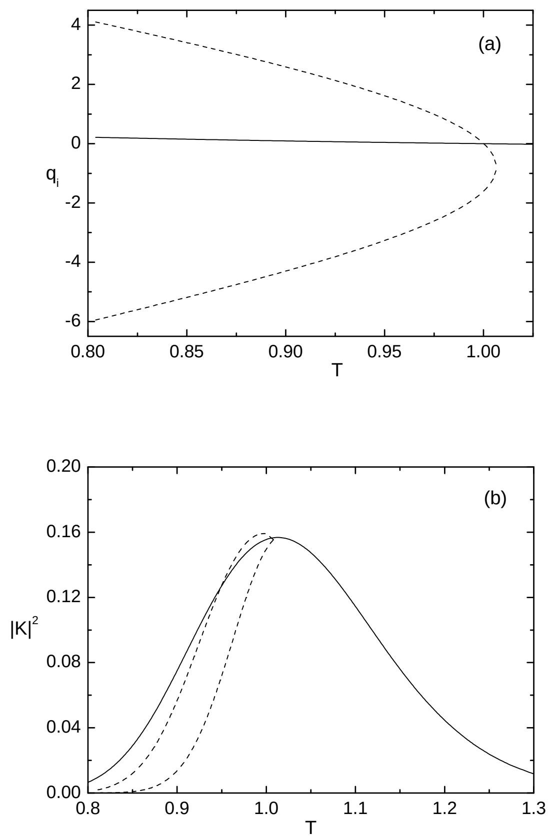

FIG. 5: Top: real solutions to the equation $0 = q_i \cos(\sigma T) + p\cos(\sigma T)$, where $\sigma = q_i^2 + p^2 + 1$, in the vicinity of the classical period (dashed lines). We also show the real part of the position for the complex trajectory (solid line). Bottom: approximation $|K_{p_1 q_2}|^2$ (dashed lines) compared to the exact result (solid line). The situation is analogous to the previous figure.

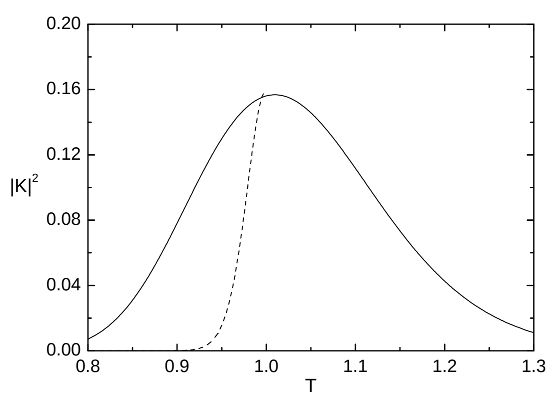

FIG. 6: Approximation $|K_{p_1 p_2}|^2$ (dashed line) compared to the exact result (solid line). This time only one real trajectory exists for $T < T_c$, but again the propagator is truncated.