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STABLE SOLUTIONS FOR THE BILAPLACIAN WITH EXPONENTIAL NONLINEARITY*

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Abstract. Let $\lambda^* > 0$ denote the largest possible value of λ such that $\{\Delta^2 u = \lambda e^u$ in B, $u = \frac{\partial u}{\partial n} = 0$ on $\partial B\}$ has a solution, where B is the unit ball in $\mathbb{R}^N$ and n is the exterior unit normal vector. We show that for $\lambda = \lambda^*$ this problem possesses a unique *weak* solution u^*. We prove that u^* is smooth if $N \le 12$ and singular when $N \ge 13$, in which case $u^*(r) = -4\log r + \log(8(N-2)(N-4)/\lambda^*) + o(1)$ as $r \to 0$. We also consider the problem with general constant Dirichlet boundary conditions.

Key words. biharmonic, singular solutions, stability

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1. Introduction.

We study the fourth order problem

$$\begin{cases} \Delta^2 u = \lambda e^u & \text{in } B, \\ u = a & \text{on } \partial B, \\ \dfrac{\partial u}{\partial n} = b & \text{on } \partial B, \end{cases} \tag{1}$$

where $a, b \in \mathbb{R}$, B is the unit ball in $\mathbb{R}^N$, $N \ge 1$, n is the exterior unit normal vector, and $\lambda \ge 0$ is a parameter.

Recently higher order equations have attracted the interest of many researchers. In particular, fourth order equations with an exponential nonlinearity have been studied in four dimensions in a setting analogous to Liouville's equation in [3, 12, 24] and in higher dimensions by [1, 2, 4, 5, 13].

We shall pay special attention to (1) in the case $a = b = 0$, as it is the natural fourth order analogue of the classical Gelfand problem

$$\begin{cases} -\Delta u = \lambda e^u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

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(Ω is a smooth bounded domain in $\mathbb{R}^N$) for which a vast literature exists [7, 8, 9, 10, 18, 19, 20, 21].

From the technical point of view, one of the basic tools in the analysis of (2) is the maximum principle. As pointed out in [2], in general domains the maximum principle for Δ^2 with Dirichlet boundary condition is not valid anymore. One of the reasons to study (1) in a ball is that a maximum principle holds in this situation; see [6]. In this simpler setting, though there are some similarities between the two problems, several tools that are well suited for (2) no longer seem to work for (1).

As a start, let us introduce the class of weak solutions we shall be working with: we say that $u \in H^2(B)$ is a weak solution to (1) if $e^u \in L^1(B)$, $u = a$ on ∂B, $\frac{\partial u}{\partial n} = b$ on ∂B, and

$$\int_B \Delta u \Delta \varphi = \lambda \int_B e^u \varphi \quad \forall \varphi \in C_0^\infty(B).$$

The following basic result is a straightforward adaptation of Theorem 3 in [2].

THEOREM 1.1 (see [2]). *There exists λ^* such that if $0 \le \lambda < \lambda^*$ then* (1) *has a minimal smooth solution u_λ and if $\lambda > \lambda^*$ then* (1) *has no weak solution.*

The limit $u^* = \lim_{\lambda \nearrow \lambda^*} u_\lambda$ exists pointwise, belongs to $H^2(B)$, and is a weak solution to (1). *It is called the extremal solution.*

The functions u_λ, $0 \le \lambda < \lambda^*$, and u^* are radially symmetric and radially decreasing.

The branch of minimal solutions of (1) has an important property; namely, u_λ is stable in the sense that

$$\int_B (\Delta \varphi)^2 \ge \lambda \int_B e^{u_\lambda} \varphi^2 \quad \forall \varphi \in C_0^\infty(B); \tag{3}$$

see [2, Proposition 37].

The authors in [2] pose several questions, some of which we address in this work. First we show that the extremal solution u^* is the unique solution to (1) in the class of weak solutions. Actually the statement is stronger, asserting that for $\lambda = \lambda^*$ there are no strict supersolutions.

THEOREM 1.2. *If*

$$v \in H^2(B),\ e^v \in L^1(B),\ v|_{\partial B} = a,\ \tfrac{\partial v}{\partial n}|_{\partial B} \le b, \tag{4}$$

and

$$\int_B \Delta v \Delta \varphi \ge \lambda^* \int_B e^v \varphi \quad \forall\, \varphi \in C_0^\infty(B),\ \varphi \ge 0, \tag{5}$$

then $v = u^*$. In particular, for $\lambda = \lambda^*$ problem (1) *has a unique weak solution.*

This result is analogous to the work of Martel [19] for more general versions of (2), where the exponential function is replaced by a positive, increasing, convex, and superlinear function.

Next, we discuss the regularity of the extremal solution u^*. In dimensions $N = 5, \ldots, 16$ the authors of [2] find, with a computer assisted proof, a radial singular solution U_σ to (1) with $a = b = 0$ associated to a parameter $\lambda_\sigma > 8(N-2)(N-4)$. They show that $\lambda_\sigma < \lambda^*$ if $N \le 10$ and claim to have numerical evidence that this holds for $N \le 12$. They leave open the question of whether u^* is singular in dimension $N \le 12$. We prove the following theorem.

THEOREM 1.3. *If $N \le 12$ then the extremal solution u^* of* (1) *is smooth.*

The method introduced in [10, 20] to prove the boundedness of u^* in low dimensions for (2) seems not useful for (1), thus requiring a new strategy. A first indication that the borderline dimension for the boundedness of u^* is 12 is Rellich's inequality [23], which states that if $N \ge 5$ then

$$\int_{\mathbb{R}^N} (\Delta\varphi)^2 \ge \frac{N^2(N-4)^2}{16} \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^4} \qquad \forall \varphi \in C_0^\infty(\mathbb{R}^N), \tag{6}$$

where the constant $N^2(N-4)^2/16$ is known to be optimal. The proof of Theorem 1.3 is based on the observation that if u^* is singular then $\lambda^* e^{u^*} \sim 8(N-2)(N-4)|x|^{-4}$ near the origin. But $8(N-2)(N-4) > N^2(N-4)^2/16$ if $N \le 12$, which would contradict the stability condition (3).

In view of Theorem 1.3, it is natural to ask whether u^* is singular in dimension $N \ge 13$. If $a = b = 0$, we prove the following theorem.

THEOREM 1.4. *Let $N \ge 13$ and $a = b = 0$. Then the extremal solution u^* to* (1) *is unbounded.*

For general boundary values, it seems more difficult to determine the dimensions for which the extremal solution is singular. We observe first that given any $a, b \in \mathbb{R}$, u^* is the extremal solution of (1) if and only if $u^* - a$ is the extremal solution of the same equation with boundary condition $u = 0$ on ∂B. In particular, if $\lambda^*(a,b)$ denotes the extremal parameter for problem (1), one has that $\lambda^*(a,b) = e^{-a}\lambda^*(0,b)$. So the value of a is irrelevant. But one may ask if Theorem 1.4 still holds for any $N \ge 13$ and any $b \in \mathbb{R}$. The situation turns out to be somewhat more complicated.

PROPOSITION 1.5.

(a) *Fix $N \ge 13$ and take any $a \in \mathbb{R}$. Assume $b \ge -4$. There exists a critical parameter $b^{max} > 0$, depending only on N, such that the extremal solution u^* is singular if and only if $b \le b^{max}$.*

(b) *Fix $b \ge -4$ and take any $a \in \mathbb{R}$. There exists a critical dimension $N^{min} \ge 13$, depending only on b, such that the extremal solution u^* to* (1) *is singular if $N \ge N^{min}$.*

Remark 1.6.

- We have not investigated the case $b < -4$.
- If follows from item (a) that for $b \in [-4, 0]$, the extremal solution is singular if and only if $N \ge 13$.
- It also follows from item (a) that there exist values of b for which $N^{min} > 13$. We do not know whether u^* remains bounded for $13 \le N < N^{min}$.

Our proof of Theorem 1.4 is related to an idea that Brezis and Vázquez applied to the Gelfand problem and is based on a characterization of singular *energy* solutions through linearized stability (see Theorem 3.1 in [8]). In our context we show the following.

PROPOSITION 1.7. *Assume that $u \in H^2(B)$ is an unbounded weak solution of* (1) *satisfying the stability condition*

$$\lambda \int_B e^u \varphi^2 \le \int_B (\Delta\varphi)^2 \quad \forall \varphi \in C_0^\infty(B). \tag{7}$$

Then $\lambda = \lambda^*$ and $u = u^*$.

In the proof of Theorem 1.4 we do not use Proposition 1.7 directly but some variants of it—see Lemma 2.6 and Remark 2.7—because we do not have at our disposal an explicit solution to (1). Instead, we show that it is enough to find a sufficiently good approximation to u^*. When $N \geq 32$ we are able to construct such an approximation by hand. However, for $13 \leq N \leq 31$ we resort to a computer assisted generation and verification.

Only in very few situations may one take advantage of Proposition 1.7 directly. For instance, for problem (1) with $a = 0$ and $b = -4$ we have an explicit solution

$$\bar{u}(x) = -4 \log |x|$$

associated to $\bar{\lambda} = 8(N-2)(N-4)$. Thanks to Rellich's inequality (6) the solution $\bar{u}$ satisfies condition (7) when $N \geq 13$. Therefore, by Theorem 1.3 and a direct application of Proposition 1.7 we obtain Theorem 1.4 in the case $b = -4$.

In [2] the authors say that a radial weak solution u to (1) is weakly singular if

$$\lim_{r \to 0} r u'(r) \text{ exists.}$$

For example, the singular solutions U_σ of [2] verify this condition.

As a corollary of Theorem 1.2 we show the following.

PROPOSITION 1.8. *The extremal solution u^* to* (1) *with $b \geq -4$ is always weakly singular.*

A weakly singular solution either is smooth or exhibits a log-type singularity at the origin. More precisely, if u is a nonsmooth weakly singular solution of (1) with parameter λ, then (see [2])

$$\lim_{r \to 0} u(r) + 4 \log r = \log \frac{8(N-2)(N-4)}{\lambda},$$
$$\lim_{r \to 0} r u'(r) = -4.$$

In section 2 we describe the comparison principles we use later. Section 3 is devoted to the proof of the uniqueness of u^* and Propositions 1.7 and 1.8. We prove Theorem 1.3, the boundedness of u^* in low dimensions, in section 4. The argument for Theorem 1.4 is contained in section 5 for the case $N \geq 32$ and section 6 for $13 \leq N \leq 31$. In section 7 we give the proof of Proposition 1.5.

Notation.

- B_R is the ball of radius R in $\mathbb{R}^N$ centered at the origin. $B = B_1$.
- n is the exterior unit normal vector to B_R.
- All inequalities or equalities for functions in L^p spaces are understood to be a.e.

2. Comparison principles.

LEMMA 2.1 (Boggio's principle [6]). *If $u \in C^4(\overline{B}_R)$ satisfies*

$$\begin{cases} \Delta^2 u \geq 0 & \text{in } B_R, \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial B_R, \end{cases}$$

then $u \geq 0$ in B_R.

LEMMA 2.2. *Let $u \in L^1(B_R)$ and suppose that*

$$\int_{B_R} u \Delta^2 \varphi \geq 0$$

for all $\varphi \in C^4(\overline{B}_R)$ *such that* $\varphi \geq 0$ *in* B_R, $\varphi|_{\partial B_R} = 0 = \frac{\partial \varphi}{\partial n}|_{\partial B_R}$. *Then* $u \geq 0$ *in* B_R. *Moreover,* $u \equiv 0$ *or* $u > 0$ *a.e. in* B_R.

For a proof see Lemma 17 in [2].

LEMMA 2.3. *If* $u \in H^2(B_R)$ *is radial,* $\Delta^2 u \geq 0$ *in* B_R *in the weak sense, that is,*

$$\int_{B_R} \Delta u \Delta \varphi \geq 0 \quad \forall \varphi \in C_0^\infty(B_R),\ \varphi \geq 0,$$

and $u|_{\partial B_R} \geq 0$, $\frac{\partial u}{\partial n}|_{\partial B_R} \leq 0$, *then* $u \geq 0$ *in* B_R.

Proof. We deal only with the case $R = 1$ for simplicity. Solve

$$\begin{cases} \Delta^2 u_1 = \Delta^2 u & \text{in } B_1, \\ u_1 = \dfrac{\partial u_1}{\partial n} = 0 & \text{on } \partial B_1 \end{cases}$$

in the sense $u_1 \in H_0^2(B_1)$ and $\int_{B_1} \Delta u_1 \Delta \varphi = \int_{B_1} \Delta u \Delta \varphi$ for all $\varphi \in C_0^\infty(B_1)$. Then $u_1 \geq 0$ in B_1 by Lemma 2.2.

Let $u_2 = u - u_1$ so that $\Delta^2 u_2 = 0$ in B_1. Define $f = \Delta u_2$. Then $\Delta f = 0$ in B_1 and since f is radial we find that f is constant. It follows that $u_2 = ar^2 + b$. Using the boundary conditions we deduce $a + b \geq 0$ and $a \leq 0$, which imply $u_2 \geq 0$. ∎

Similarly, we have the following lemma.

LEMMA 2.4. *If* $u \in H^2(B_R)$ *and* $\Delta^2 u \geq 0$ *in* B_R *in the weak sense, that is,*

$$\int_{B_R} \Delta u \Delta \varphi \geq 0 \quad \forall \varphi \in C_0^\infty(B_R),\ \varphi \geq 0,$$

and $u|_{\partial B_R} = 0$, $\frac{\partial u}{\partial n}|_{\partial B_R} \leq 0$, *then* $u \geq 0$ *in* B_R.

The next lemma is a consequence of a decomposition lemma of Moreau [22]. For a proof see [14, 15].

LEMMA 2.5. *Let* $u \in H_0^2(B_R)$. *Then there exist unique* $w, v \in H_0^2(B_R)$ *such that* $u = w + v$, $w \geq 0$, $\Delta^2 v \leq 0$ *in* B_R *and* $\int_{B_R} \Delta w \Delta v = 0$.

We need the following comparison principle.

LEMMA 2.6. *Let* u_1, $u_2 \in H^2(B_R)$ *with* e^{u_1}, $e^{u_2} \in L^1(B_R)$. *Assume that*

$$\Delta^2 u_1 \leq \lambda e^{u_1} \quad \text{in } B_R$$

in the sense that

$$\int_{B_R} \Delta u_1 \Delta \varphi \leq \lambda \int_{B_R} e^{u_1} \varphi \quad \forall \varphi \in C_0^\infty(B_R),\ \varphi \geq 0, \tag{8}$$

and $\Delta^2 u_2 \geq \lambda e^{u_2}$ *in* B_R *in the similar weak sense. Suppose also*

$$u_1|_{\partial B_R} = u_2|_{\partial B_R} \quad \text{and} \quad \frac{\partial u_1}{\partial n}|_{\partial B_R} = \frac{\partial u_2}{\partial n}|_{\partial B_R}.$$

Assume, furthermore, that u_1 *is stable in the sense that*

$$\lambda \int_{B_R} e^{u_1} \varphi^2 \leq \int_{B_R} (\Delta \varphi)^2 \quad \forall \varphi \in C_0^\infty(B_R). \tag{9}$$

Then

$$u_1 \leq u_2 \quad \text{in } B_R.$$

Proof. Let $u = u_1 - u_2$. By Lemma 2.5 there exist $w, v \in H_0^2(B_R)$ such that $u = w + v$, $w \geq 0$ and $\Delta^2 v \leq 0$. Observe that $v \leq 0$ so $w \geq u_1 - u_2$.

By hypothesis we have for all $\varphi \in C_0^\infty(B_R)$, $\varphi \geq 0$,

$$\int_{B_R} \Delta(u_1 - u_2)\Delta\varphi \leq \lambda \int_{B_R} (e^{u_1} - e^{u_2})\varphi \leq \lambda \int_{B_R \cap [u_1 \geq u_2]} (e^{u_1} - e^{u_2})\varphi$$

and by density this holds also for w:

$$\begin{aligned} (10) \quad \int_{B_R} (\Delta w)^2 &= \int_{B_R} \Delta(u_1 - u_2)\Delta w \\ &\leq \lambda \int_{B_R \cap [u_1 \geq u_2]} (e^{u_1} - e^{u_2}) w = \lambda \int_{B_R} (e^{u_1} - e^{u_2}) w, \end{aligned}$$

where the first equality holds because $\int_{B_R} \Delta w \Delta v = 0$. By density we deduce from (9)

$$(11) \qquad \lambda \int_{B_R} e^{u_1} w^2 \leq \int_{B_R} (\Delta w)^2.$$

Combining (10) and (11), we obtain

$$\int_{B_R} e^{u_1} w^2 \leq \int_{B_R} (e^{u_1} - e^{u_2}) w.$$

Since $u_1 - u_2 \leq w$ the previous inequality implies

$$(12) \qquad 0 \leq \int_{B_R} (e^{u_1} - e^{u_2} - e^{u_1}(u_1 - u_2)) w.$$

But by convexity of the exponential function $e^{u_1} - e^{u_2} - e^{u_1}(u_1 - u_2) \leq 0$, and we deduce from (12) that $(e^{u_1} - e^{u_2} - e^{u_1}(u_1 - u_2))w = 0$. Recalling that $u_1 - u_2 \leq w$ we deduce that $u_1 \leq u_2$. ∎

Remark 2.7. The following variant of Lemma 2.6 also holds.

Let $u_1,\ u_2 \in H^2(B_R)$ be *radial* with $e^{u_1},\ e^{u_2} \in L^1(B_R)$. Assume $\Delta^2 u_1 \leq \lambda e^{u_1}$ in B_R in the sense of (8) and $\Delta^2 u_2 \geq \lambda e^{u_2}$ in B_R. Suppose $u_1|_{\partial B_R} \leq u_2|_{\partial B_R}$ and $\frac{\partial u_1}{\partial n}|_{\partial B_R} \geq \frac{\partial u_2}{\partial n}|_{\partial B_R}$ and that the stability condition (9) holds. Then $u_1 \leq u_2$ in B_R.

Proof. We solve for $\tilde{u} \in H_0^2(B_R)$ such that

$$\int_{B_R} \Delta\tilde{u}\Delta\varphi = \int_{B_R} \Delta(u_1 - u_2)\Delta\varphi \quad \forall \varphi \in C_0^\infty(B_R).$$

By Lemma 2.3 it follows that $\tilde{u} \geq u_1 - u_2$. Next we apply the decomposition of Lemma 2.5 to $\tilde{u}$, that is, $\tilde{u} = w + v$ with $w, v \in H_0^2(B_R)$, $w \geq 0$, $\Delta^2 v \leq 0$ in B_R, and $\int_{B_R} \Delta w \Delta v = 0$. Then the argument follows that of Lemma 2.6. ∎

Finally, in several places we will need the method of sub- and supersolutions in the context of weak solutions.

LEMMA 2.8. *Let $\lambda > 0$ and assume that there exists $\bar{u} \in H^2(B_R)$ such that $e^{\bar{u}} \in L^1(B_R)$,*

$$\int_{B_R} \Delta\bar{u}\Delta\varphi \geq \lambda \int_{B_R} e^{\bar{u}}\varphi \quad \forall \varphi \in C_0^\infty(B_R), \varphi \geq 0,$$

and

$$\bar{u} = a, \quad \frac{\partial \bar{u}}{\partial n} \le b \quad \text{on } \partial B_1.$$

Then there exists a weak solution to (1) *such that* $u \le \bar{u}$.

The proof is similar to that of Lemma 19 in [2].

3. Uniqueness of the extremal solution: Proof of Theorem 1.2.

Proof of Theorem 1.2. Suppose that $v \in H^2(B)$ satisfies (4), (5), and $v \not\equiv u^*$. Notice that we do not need v to be radial. The idea of the proof is as follows.

Step 1. The function

$$u_0 = \frac{1}{2}(u^* + v)$$

is a supersolution to the problem

$$\begin{cases} \Delta^2 u = \lambda^* e^u + \mu \eta e^u & \text{in } B, \\ u = a & \text{on } \partial B, \\ \dfrac{\partial u}{\partial n} = b & \text{on } \partial B \end{cases} \tag{13}$$

for some $\mu = \mu_0 > 0$, where $\eta \in C_0^\infty(B)$, $0 \le \eta \le 1$, is a fixed radial cut-off function such that

$$\eta(x) = 1 \quad \text{for } |x| \le \tfrac{1}{2}, \qquad \eta(x) = 0 \quad \text{for } |x| \ge \tfrac{3}{4}.$$

Step 2. Using a solution to (13) we construct, for some $\lambda > \lambda^*$, a supersolution to (1). This provides a solution u_λ for some $\lambda > \lambda^*$, which is a contradiction.

Proof of Step 1. Observe that given $0 < R < 1$ we must have for some $c_0 = c_0(R) > 0$

$$v(x) \ge u^*(x) + c_0, \quad |x| \le R. \tag{14}$$

To prove this we recall the Green's function for Δ^2 with Dirichlet boundary conditions

$$\begin{cases} \Delta_x^2 G(x,y) = \delta_y, & x \in B, \\ G(x,y) = 0, & x \in \partial B, \\ \dfrac{\partial G}{\partial n}(x,y) = 0, & x \in \partial B, \end{cases}$$

where δ_y is the Dirac mass at $y \in B$. Boggio gave an explicit formula for $G(x,y)$ which was used in [16] to prove that in dimension $N \ge 5$ (the case $1 \le N \le 4$ can be treated similarly)

$$G(x,y) \sim |x-y|^{4-N} \min\left(1, \frac{d(x)^2 d(y)^2}{|x-y|^4}\right), \tag{15}$$

where

$$d(x) = \text{dist}(x, \partial B) = 1 - |x|$$

and $a \sim b$ means that for some constant $C > 0$ we have $C^{-1}a \le b \le Ca$ (uniformly for $x, y \in B$). Formula (15) yields

$$G(x,y) \ge c d(x)^2 d(y)^2 \tag{16}$$

for some $c > 0$ and this in turn implies that for smooth functions $\tilde v$ and $\tilde u$ such that $\tilde v - \tilde u \in H_0^2(B)$ and $\Delta^2(\tilde v - \tilde u) \ge 0$,

$$\begin{aligned}\tilde v(y) - \tilde u(y) &= \int_{\partial B} \Big(\frac{\partial \Delta_x G}{\partial n_x}(x,y)(\tilde v - \tilde u) - \Delta_x G(x,y) \frac{\partial(\tilde v - \tilde u)}{\partial n} \Big)\, dx \\ &\quad + \int_B G(x,y) \Delta^2 (\tilde v - \tilde u)\, dx \\ &\ge c d(y)^2 \int_B (\Delta^2 \tilde v - \Delta^2 \tilde u) d(x)^2\, dx.\end{aligned}$$

Using a standard approximation procedure, we conclude that

$$v(y) - u^*(y) \ge c d(y)^2 \lambda^* \int_B (e^v - e^{u^*}) d(x)^2\, dx.$$

Since $v \ge u^*$, $v \not\equiv u^*$ we deduce (14).

Let $u_0 = (u^* + v)/2$. Then by Taylor's theorem

$$e^v = e^{u_0} + (v - u_0)e^{u_0} + \frac{1}{2}(v-u_0)^2 e^{u_0} + \frac{1}{6}(v-u_0)^3 e^{u_0} + \frac{1}{24}(v-u_0)^4 e^{\xi_2} \tag{17}$$

for some $u_0 \le \xi_2 \le v$ and

$$e^{u^*} = e^{u_0} + (u^* - u_0)e^{u_0} + \frac{1}{2}(u^*-u_0)^2 e^{u_0} + \frac{1}{6}(u^*-u_0)^3 e^{u_0} + \frac{1}{24}(u^*-u_0)^4 e^{\xi_1} \tag{18}$$

for some $u^* \le \xi_1 \le u_0$. Adding (17) and (18) yields

$$\frac{1}{2}(e^v + e^{u^*}) \ge e^{u_0} + \frac{1}{8}(v - u^*)^2 e^{u_0}. \tag{19}$$

From (14) with $R = 3/4$ and (19) we see that $u_0 = (u^* + v)/2$ is a supersolution of (13) with $\mu_0 := c_0/8$.

Proof of Step 2. Let us show now how to obtain a weak supersolution of (1) for some $\lambda > \lambda^*$. Given $\mu > 0$, let u denote the minimal solution to (13). Define φ_1 as the solution to

$$\begin{cases} \Delta^2 \varphi_1 = \mu \eta e^u & \text{in } B, \\ \varphi_1 = 0 & \text{on } \partial B, \\ \dfrac{\partial \varphi_1}{\partial n} = 0 & \text{on } \partial B, \end{cases}$$

and φ_2 as the solution to

$$\begin{cases} \Delta^2 \varphi_2 = 0 & \text{in } B, \\ \varphi_2 = a & \text{on } \partial B, \\ \dfrac{\partial \varphi_2}{\partial n} = b & \text{on } \partial B. \end{cases}$$

If $N \geq 5$ (the case $1 \leq N \leq 4$ can be treated similarly), relation (16) yields

$$\varphi_1(x) \geq c_1 d(x)^2 \quad \forall x \in B, \tag{20}$$

for some $c_1 > 0$. But u is a radial solution of (13) and therefore it is smooth in $B \setminus B_{1/4}$. Thus

$$u(x) \leq M\varphi_1 + \varphi_2 \quad \forall x \in B_{1/2}, \tag{21}$$

for some $M > 0$. Therefore, from (20) and (21), for $\lambda > \lambda^*$ with $\lambda - \lambda^*$ sufficiently small we have

$$\left(\frac{\lambda}{\lambda^*} - 1\right) u \leq \varphi_1 + \left(\frac{\lambda}{\lambda^*} - 1\right) \varphi_2 \quad \text{in } B.$$

Let $w = \frac{\lambda}{\lambda^*} u - \varphi_1 - (\frac{\lambda}{\lambda^*} - 1)\varphi_2$. The inequality just stated guarantees that $w \leq u$. Moreover,

$$\Delta^2 w = \lambda e^u + \frac{\lambda\mu}{\lambda^*}\eta e^u - \mu\eta e^u \geq \lambda e^u \geq \lambda e^w \quad \text{in } B$$

and

$$w = a, \quad \frac{\partial w}{\partial n} = b \quad \text{on } \partial B.$$

Therefore, w is a supersolution to (1) for λ. By the method of sub- and supersolutions a solution to (1) exists for some $\lambda > \lambda^*$, which is a contradiction. ∎

Proof of Proposition 1.7. Let $\lambda > 0$ and $u \in H^2(B)$ be a weak unbounded solution of (1). If $\lambda < \lambda^*$ from Lemma 2.6 we find that $u \leq u_\lambda$, where u_λ is the minimal solution. This is impossible because u_λ is smooth and u is unbounded. If $\lambda = \lambda^*$ then necessarily $u = u^*$ by Theorem 1.2. ∎

Proof of Proposition 1.8. Let u denote the extremal solution of (1) with $b \geq -4$. If u is smooth, then the result is trivial. So we restrict our attention to the case where u is singular. By Theorem 1.3 we have, in particular, that $N \geq 13$. We may also assume that $a = 0$. If $b = -4$ by Theorem 1.2 we know that if $N \geq 13$, then $u = -4 \log |x|$ so that the desired conclusion holds. Henceforth we assume $b > -4$ in this section.

For $\rho > 0$ define

$$u_\rho(r) = u(\rho r) + 4 \log \rho$$

so that

$$\Delta^2 u_\rho = \lambda^* e^{u_\rho} \qquad \text{in } B_{1/\rho}.$$

Then

$$\frac{du_\rho}{d\rho}\Big|_{\rho=1, r=1} = u'(1) + 4 > 0.$$

Hence, there is $\delta > 0$ such that

$$u_\rho(r) < u(r) \quad \forall 1 - \delta < r \leq 1,\ 1 - \delta < \rho \leq 1.$$

This implies

$$u_\rho(r) < u(r) \quad \forall 0 < r \leq 1,\, 1-\delta < \rho \leq 1. \tag{22}$$

Otherwise set

$$r_0 = \sup\{\, 0 < r < 1 \,|\, u_\rho(r) \geq u(r)\,\}.$$

This definition yields

$$u_\rho(r_0) = u(r_0) \quad \text{and} \quad u'_\rho(r_0) \leq u'(r_0). \tag{23}$$

Write $\alpha = u(r_0)$, $\beta = u'(r_0)$. Then u satisfies

$$\begin{cases} \Delta^2 u = \lambda e^u & \text{on } B_{r_0}, \\ u(r_0) = \alpha, \\ u'(r_0) = \beta. \end{cases} \tag{24}$$

Observe that u is an unbounded $H^2(B_{r_0})$ solution to (24), which is also stable. Thus Proposition 1.7 shows that u is the extremal solution to this problem. On the other hand, u_ρ is a supersolution to (24), since $u'_\rho(r_0) \leq \beta$ by (23). We may now use Theorem 1.2 and we deduce that

$$u(r) = u_\rho(r) \quad \forall 0 < r \leq r_0,$$

which in turn implies by standard ODE theory that

$$u(r) = u_\rho(r) \quad \forall 0 < r \leq 1,$$

which is a contradiction to (22). This proves estimate (22).

From (22) we see that

$$\frac{du_\rho}{d\rho}\Big|_{\rho=1}(r) \geq 0 \quad \forall 0 < r \leq 1. \tag{25}$$

But

$$\frac{du_\rho}{d\rho}\Big|_{\rho=1}(r) = u'(r)r + 4 \quad \forall 0 < r \leq 1,$$

and this together with (25) implies

$$\frac{du_\rho}{d\rho}(r) = \frac{1}{\rho}(u'(\rho r)\rho r + 4) \geq 0 \quad \forall 0 < r \leq \frac{1}{\rho},\, 0 < \rho \leq 1, \tag{26}$$

which means that $u_\rho(r)$ is nondecreasing in ρ. We wish to show that $\lim_{\rho\to 0} u_\rho(r)$ exists for all $0 < r \leq 1$. For this we shall show

$$u_\rho(r) \geq -4\log(r) + \log\left(\frac{8(N-2)(N-4)}{\lambda^*}\right) \quad \forall 0 < r \leq \frac{1}{\rho},\, 0 < \rho \leq 1. \tag{27}$$

Set

$$u_0(r) = -4\log(r) + \log\left(\frac{8(N-2)(N-4)}{\lambda^*}\right),$$

and suppose that (27) is not true for some $0 < \rho < 1$. Let

$$r_1 = \sup \{ 0 < r < 1/\rho \,|\, u_\rho(r) < u_0(r) \}.$$

Observe that

$$\lambda^* > 8(N-2)(N-4). \tag{28}$$

Otherwise $w = -4 \ln r$ would be a strict supersolution of the equation satisfied by u, which is not possible by Theorem 1.2. In particular, $r_1 < 1/\rho$ and

$$u_\rho(r_1) = u_0(r_1) \quad \text{and} \quad u_\rho'(r_1) \geq u_0'(r_1).$$

It follows that u_0 is a supersolution of

$$\begin{cases} \Delta^2 u = \lambda^* e^u & \text{in } B_{r_1}, \\ u = A & \text{on } \partial B_{r_1}, \\ \dfrac{\partial u}{\partial n} = B & \text{on } \partial B_{r_1}, \end{cases} \tag{29}$$

with $A = u_\rho(r_1)$ and $B = u_\rho'(r_1)$. Since u_ρ is a singular stable solution of (29), it is the extremal solution of the problem by Proposition 1.7. By Theorem 1.2, there is no strict supersolution of (29), and we conclude that $u_\rho \equiv u_0$ first for $0 < r < r_1$ and then for $0 < r \leq 1/\rho$. This is impossible for $\rho > 0$ because $u_\rho(1/\rho) = 4 \log \rho$ and $u_0(1/\rho) < 4 \log \rho + \log(\frac{8(N-2)(N-4)}{\lambda^*}) < u_\rho(1/\rho)$ by (28). This proves (27).

By (26) and (27) we see that

$$v(r) = \lim_{\rho \to 0} u_\rho(r) \quad \text{exists } \forall 0 < r < +\infty,$$

where the convergence is uniform (even in C^k for any k) on compact sets of $\mathbb{R}^N \setminus \{0\}$. Moreover, v satisfies

$$\Delta^2 v = \lambda^* e^v \quad \text{in } \mathbb{R}^N \setminus \{0\}. \tag{30}$$

Then for any $r > 0$

$$v(r) = \lim_{\rho \to 0} u_\rho(r) = \lim_{\rho \to 0} u(\rho r) + 4 \log(\rho r) - 4 \log(r) = v(1) - 4 \log(r).$$

Hence, using (30) we obtain

$$v(r) = -4 \log r + \log \left(\frac{8(N-2)(N-4)}{\lambda^*} \right) = u_0(r).$$

But then

$$u_\rho'(r) = u'(\rho r)\rho \to -4 \quad \text{as } \rho \to 0,$$

and therefore, with $r = 1$

$$\rho u'(\rho) \to -4 \quad \text{as } \rho \to 0. \qquad \square \tag{31}$$

4. Proof of Theorem 1.3. First we will show the following lemma.

LEMMA 4.1. *Suppose that the extremal solution* u^* *to* (1) *is singular. Then for any* $\sigma > 0$ *there exists* $0 < R < 1$ *such that*

$$u^*(x) \geq (1-\sigma)\log\left(\frac{1}{|x|^4}\right) \quad \forall\, |x| < R. \tag{32}$$

Proof. Assume by contradiction that (32) is false. Then there exist $\sigma > 0$ and a sequence $x_k \in B$ with $x_k \to 0$ such that

$$u^*(x_k) < (1-\sigma)\log\left(\frac{1}{|x_k|^4}\right). \tag{33}$$

Let $s_k = |x_k|$ and choose $0 < \lambda_k < \lambda^*$ such that

$$\max_{\overline{B}} u_{\lambda_k} = u_{\lambda_k}(0) = \log\left(\frac{1}{s_k^4}\right). \tag{34}$$

Note that $\lambda_k \to \lambda^*$; otherwise u_{λ_k} would remain bounded. Let

$$v_k(x) = \frac{u_{\lambda_k}(s_k x)}{\log(\frac{1}{s_k^4})}, \qquad x \in B_k \equiv \frac{1}{s_k} B.$$

Then $0 \leq v_k \leq 1$, $v_k(0) = 1$,

$$\begin{aligned} \Delta^2 v_k(x) &= \lambda_k \frac{s_k^4}{\log(\frac{1}{s_k^4})} e^{u_{\lambda_k}(s_k x)} \\ &\leq \frac{\lambda_k}{\log(\frac{1}{s_k^4})} \to 0 \quad \text{in } B_k \end{aligned}$$

by (34). By elliptic regularity $v_k \to v$ uniformly on compact sets of $\mathbb{R}^N$ to a function v satisfying $0 \leq v \leq 1$, $v(0) = 1$, $\Delta^2 v = 0$ in $\mathbb{R}^N$. By Liouville's theorem for biharmonic functions [17] we conclude that v is constant and therefore $v \equiv 1$.

Since $|x_k| = s_k$ we deduce that

$$\frac{u_{\lambda_k}(x_k)}{\log(\frac{1}{s_k^4})} \to 1,$$

which contradicts (33). ∎

Proof of Theorem 1.3. We write for simplicity $u = u^*$, $\lambda = \lambda^*$. Assume by contradiction that u^* is unbounded and $5 \leq N \leq 12$. If $N \leq 4$ the problem is subcritical, and the boundedness of u^* can be proved by other means: no singular solutions exist for positive λ (see [2]), though in dimension $N = 4$, a family of solutions (u_λ) can blow up as $\lambda \to 0$ (see [24]).

For $\varepsilon > 0$ let $\psi = |x|^{\frac{4-N}{2}+\varepsilon}$ and let $\eta \in C_0^\infty(\mathbb{R}^N)$ with $\eta \equiv 1$ in $B_{1/2}$ and $\mathrm{supp}(\eta) \subseteq$ B. Observe that

$$(\Delta\psi)^2 = (H_N + O(\varepsilon))|x|^{-N+2\varepsilon}, \quad \text{where } H_N = \frac{N^2(N-4)^2}{16}.$$

Using a standard approximation argument as in the proof of Lemma 2.6, we can use $\psi\eta$ as a test function in (9) and we obtain

$$\int_B (\Delta\psi)^2 + O(1) \geq \lambda \int_B e^u \psi^2,$$

since the contribution of the integrals outside a fixed ball around the origin remains bounded as $\varepsilon \to 0$ (here $O(1)$ denotes a bounded function as $\varepsilon \to 0$).

This implies

$$\lambda \int_B e^u |x|^{4-N+2\varepsilon} \leq (H_N + O(\varepsilon)) \int_B |x|^{-N+2\varepsilon} = \omega_N \frac{H_N}{2\varepsilon} + O(1), \tag{35}$$

where ω_N is the surface area of the unit $N-1$ dimensional sphere S^{N-1}. In particular, $\int_B e^u |x|^{4-N+2\varepsilon} < +\infty$.

For $\varepsilon > 0$ we define $\varphi = |x|^{4-N+2\varepsilon}$. Note that away from the origin

$$\Delta^2 \varphi = \varepsilon k_N |x|^{-N+2\varepsilon}, \quad \text{where } k_N = 4(N-2)(N-4) + O(\varepsilon). \tag{36}$$

Let φ_j solve

$$\begin{cases} \Delta^2 \varphi_j = \varepsilon k_N \min(|x|^{-N+2\varepsilon}, j) & \text{in } B, \\ \varphi_j = \dfrac{\partial \varphi_j}{\partial n} = 0 & \text{on } \partial B. \end{cases} \tag{37}$$

Then $\varphi_j \uparrow \varphi$ as $j \to +\infty$. Using (35) and (37)

$$\begin{aligned} \varepsilon k_N \int_B u \ \min(|x|^{-N+2\varepsilon}, j) = \int_B u \Delta^2 \varphi_j &= \lambda \int_B e^u \varphi_j \\ &\leq \lambda \int_B e^u \varphi \\ &\leq \omega_N \frac{H_N}{2\varepsilon} + O(1), \end{aligned}$$

where $O(1)$ is bounded as $\varepsilon \to 0$ independently of j. Letting $j \to +\infty$ yields

$$\varepsilon k_N \int_B u \ |x|^{-N+2\varepsilon} \leq \omega_N \frac{H_N}{2\varepsilon} + O(1), \tag{38}$$

showing that the integral on the left-hand side is finite. On the other hand, by (32)

$$\varepsilon k_N \int_B u \ |x|^{-N+2\varepsilon} \geq \varepsilon k_N \omega_N (1-\sigma) \int_0^1 \log\left(\frac{1}{r^4}\right) r^{-1+2\varepsilon} \, dr = k_N \omega_N (1-\sigma) \frac{1}{\varepsilon}. \tag{39}$$

Combining (38) and (39), we obtain

$$(1-\sigma) k_N \leq \frac{H_N}{2} + O(\varepsilon).$$

Letting $\varepsilon \to 0$ and then $\sigma \to 0$, we have

$$8(N-2)(N-4) \leq H_N = \frac{N^2 (N-4)^2}{16}.$$

This is valid only if $N \geq 13$, which is a contradiction. □

Remark 4.2. The conclusion of Theorem 1.3 can be obtained also from Proposition 1.8. However, that proposition depends crucially on the radial symmetry of the solutions, while the argument in this section can be generalized to other domains.

5. The extremal solution is singular in large dimensions. In this section we take $a = b = 0$ and prove Theorem 1.4 for $N \geq 32$.

The idea for the proof of Theorem 1.4 is to estimate accurately from above the function $\lambda^* e^{u^*}$, and to deduce that the operator $\Delta^2 - \lambda^* e^{u^*}$ has a strictly positive first eigenvalue (in the $H_0^2(B)$ sense). Then, necessarily, u^* is singular.

Upper bounds for both λ^* and u^* are obtained by finding suitable sub- and supersolutions. For example, if for some λ_1 there exists a supersolution, then $\lambda^* \geq \lambda_1$. If for some λ_2 one can exhibit a stable singular subsolution u, then $\lambda^* \leq \lambda_2$. Otherwise, $\lambda_2 < \lambda^*$, and one can then prove that the minimal solution u_{λ_2} is above u, which is impossible. The bound for u^* also requires a stable singular subsolution.

It turns out that in dimension $N \geq 32$ we can construct the necessary subsolutions and verify their stability by hand. For dimensions $13 \leq N \leq 31$ it seems difficult to find these subsolutions explicitly. We adopt then an approach that involves a computer assisted construction of subsolutions and verification of the desired inequalities. We present this part in the next section.

Lemma 5.1. *Assume $N \geq 13$. Then $u^* \leq \bar{u} = -4\log|x|$ in B_1.*

Proof. Define $\bar{u}(x) = -4\log|x|$. Then $\bar{u}$ satisfies

$$\begin{cases} \Delta^2 \bar{u} = 8(N-2)(N-4)e^{\bar{u}} & \text{in } \mathbb{R}^N, \\ \bar{u} = 0 & \text{on } \partial B_1, \\ \dfrac{\partial \bar{u}}{\partial n} = -4 & \text{on } \partial B_1. \end{cases}$$

Observe that since $\bar{u}$ is a supersolution to (1) with $a = b = 0$ we deduce immediately that $\lambda^* \geq 8(N-2)(N-4)$.

In the case $\lambda^* = 8(N-2)(N-4)$ we have $u_\lambda \leq \bar{u}$ for all $0 \leq \lambda < \lambda^*$ because $\bar{u}$ is a supersolution, and therefore $u^* \leq \bar{u}$ holds. Alternatively, one can invoke Theorem 3 in [2] to conclude that we always have $\lambda^* > 8(N-2)(N-4)$.

Suppose now that $\lambda^* > 8(N-2)(N-4)$. We prove that $u_\lambda \leq \bar{u}$ for all $8(N-2)(N-4) < \lambda < \lambda^*$. Fix such λ and assume by contradiction that $u_\lambda \leq \bar{u}$ is not true. Note that for $r < 1$ and sufficiently close to 1 we have $u_\lambda(r) < \bar{u}(r)$ because $u_\lambda'(1) = 0$ while $\bar{u}'(1) = -4$. Let

$$R_1 = \inf\{\, 0 \leq R \leq 1 \mid u_\lambda < \bar{u} \text{ in } (R,1)\,\}.$$

Then $0 < R_1 < 1$, $u_\lambda(R_1) = \bar{u}(R_1)$, and $u_\lambda'(R_1) \leq \bar{u}'(R_1)$. So u_λ is a supersolution to the problem

$$\begin{cases} \Delta^2 u = 8(N-2)(N-4)e^{u} & \text{in } B_{R_1}, \\ u = u_\lambda(R_1) & \text{on } \partial B_{R_1}, \\ \dfrac{\partial u}{\partial n} = u_\lambda'(R_1) & \text{on } \partial B_{R_1}, \end{cases} \tag{40}$$

while $\bar{u}$ is a subsolution to (40). Moreover it is stable for this problem, since from Rellich's inequality (6) and $8(N-2)(N-4) \leq N^2(N-4)^2/16$ for $N \geq 13$, we have

$$8(N-2)(N-4)\int_{B_{R_1}} e^{\bar{u}}\varphi^2 \leq \frac{N^2(N-4)^2}{16}\int_{\mathbb{R}^N}\frac{\varphi^2}{|x|^4} \leq \int_{\mathbb{R}^N}(\Delta\varphi)^2 \quad \forall \varphi \in C_0^\infty(B_{R_1}).$$

By Remark 2.7 we deduce that $\bar{u} \leq u_\lambda$ in B_{R_1}, which is impossible. ∎

An upper bound for λ^* is obtained by considering again a stable, singular subsolution to the problem (with another parameter, though).

LEMMA 5.2. *For $N \geq 32$ we have*

$$\lambda^* \leq 8(N-2)(N-4)e^2. \tag{41}$$

Proof. Consider $w = 2(1-r^2)$ and define

$$u = \bar{u} - w,$$

where $\bar{u}(x) = -4\log|x|$. Then

$$\Delta^2 u = 8(N-2)(N-4)\frac{1}{r^4} = 8(N-2)(N-4)e^{\bar{u}} = 8(N-2)(N-4)e^{u+w} \\ \leq 8(N-2)(N-4)e^2 e^u.$$

Also $u(1) = u'(1) = 0$, so u is a subsolution to (1) with parameter $\lambda_0 = 8(N-2)(N-4)e^2$.

For $N \geq 32$ we have $\lambda_0 \leq N^2(N-4)^2/16$. Then by (6) u is a stable subsolution of (1) with $\lambda = \lambda_0$. If $\lambda^* > \lambda_0 = 8(N-2)(N-4)e^2$ the minimal solution u_{λ_0} to (1) with parameter λ_0 exists and is smooth. From Lemma 2.6 we find $u \leq u_{\lambda_0}$ which is impossible because u is singular and u_{λ_0} is bounded. Thus we have proved (41) for $N \geq 32$. ∎

Proof of Theorem 1.4 *in the case $N \geq 32$.* Combining Lemmas 5.1 and 5.2, we have that if $N \geq 32$ then $\lambda^* e^{u^*} \leq r^{-4}\, 8(N-2)(N-4)e^2 \leq r^{-4} N^2(N-4)^2/16$. This and (6) show that

$$\inf_{\varphi \in C_0^\infty(B)} \frac{\int_B (\Delta\varphi)^2 - \lambda^* \int_B e^{u^*}\varphi^2}{\int_B \varphi^2} > 0,$$

which is not possible if u^* is bounded. ∎

6. A computer assisted proof for dimensions $13 \leq N \leq 31$. Throughout this section we assume $a = b = 0$. As was mentioned in the previous section, the proof of Theorem 1.4 relies on precise estimates for u^* and λ^*. We present first some conditions under which it is possible to find these estimates. Later we show how to meet such conditions with a computer assisted verification.

The first lemma is analogous to Lemma 5.2.

LEMMA 6.1. *Suppose there exist $\varepsilon > 0$, $\lambda > 0$, and a radial function $u \in H^2(B) \cap W^{4,\infty}_{loc}(B \setminus \{0\})$ such that*

$$\Delta^2 u \leq \lambda e^u \quad \forall 0 < r < 1,$$

$$|u(1)| \leq \varepsilon, \quad \left|\frac{\partial u}{\partial n}(1)\right| \leq \varepsilon,$$

$$u \notin L^\infty(B),$$

$$\lambda e^\varepsilon \int_B e^u \varphi^2 \leq \int_B (\Delta\varphi)^2 \quad \forall \varphi \in C_0^\infty(B). \tag{42}$$

Then

$$\lambda^* \leq \lambda e^{2\varepsilon}.$$

Proof. Let

$$\psi(r) = \varepsilon r^2 - 2\varepsilon \tag{43}$$

so that

$$\Delta^2 \psi \equiv 0, \quad \psi(1) = -\varepsilon, \quad \psi'(1) = 2\varepsilon,$$

and

$$-2\varepsilon \le \psi(r) \le -\varepsilon \quad \forall 0 \le r \le 1.$$

It follows that

$$\Delta^2(u+\psi) \le \lambda e^u = \lambda e^{-\psi} e^{u+\psi} \le \lambda e^{2\varepsilon} e^{u+\psi}.$$

On the boundary we have $u(1) + \psi(1) \le 0$, $u'(1) + \psi'(1) \ge 0$. Thus $u + \psi$ is a singular subsolution to the equation with parameter $\lambda e^{2\varepsilon}$. Moreover, since $\psi \le -\varepsilon$ we have $\lambda e^{2\varepsilon} e^{u+\psi} \le \lambda e^{\varepsilon} e^u$, and hence, from (42) we see that $u + \psi$ is stable for the problem with parameter $\lambda e^{2\varepsilon}$. If $\lambda e^{2\varepsilon} < \lambda^*$ then the minimal solution associated to the parameter $\lambda e^{2\varepsilon}$ would be above $u + \psi$, which is impossible because u is singular. ∎

LEMMA 6.2. *Suppose we can find $\varepsilon > 0$, $\lambda > 0$, and $u \in H^2(B) \cap W^{4,\infty}_{loc}(B \setminus \{0\})$ such that*

$$\Delta^2 u \ge \lambda e^u \quad \forall 0 < r < 1,$$
$$|u(1)| \le \varepsilon, \quad \left|\frac{\partial u}{\partial n}(1)\right| \le \varepsilon.$$

Then

$$\lambda e^{-2\varepsilon} \le \lambda^*.$$

Proof. Let ψ be given by (43). Then $u - \psi$ is a supersolution to the problem with parameter $\lambda e^{-2\varepsilon}$. ∎

The next result is the main tool to guarantee that u^* is singular. The proof, as in Lemma 5.1, is based on an upper estimate of u^* by a stable singular subsolution.

LEMMA 6.3. *Suppose there exist ε_0, $\varepsilon > 0$, $\lambda_a > 0$, and a radial function $u \in H^2(B) \cap W^{4,\infty}_{loc}(B \setminus \{0\})$ such that*

$$\Delta^2 u \le (\lambda_a + \varepsilon_0) e^u \quad \forall 0 < r < 1, \tag{44}$$

$$\Delta^2 u \ge (\lambda_a - \varepsilon_0) e^u \quad \forall 0 < r < 1, \tag{45}$$

$$|u(1)| \le \varepsilon, \quad \left|\frac{\partial u}{\partial n}(1)\right| \le \varepsilon, \tag{46}$$

$$u \notin L^\infty(B), \tag{47}$$

$$\beta_0 \int_B e^u \varphi^2 \le \int_B (\Delta\varphi)^2 \quad \forall \varphi \in C_0^\infty(B), \tag{48}$$

where

$$\beta_0 = \frac{(\lambda_a + \varepsilon_0)^3}{(\lambda_a - \varepsilon_0)^2} e^{9\varepsilon}. \tag{49}$$

Then u^* is singular and

$$(\lambda_a - \varepsilon_0)e^{-2\varepsilon} \le \lambda^* \le (\lambda_a + \varepsilon_0)e^{2\varepsilon}. \tag{50}$$

Proof. By Lemmas 6.1 and 6.2 we have (50). Let

$$\delta = \log\left(\frac{\lambda_a + \varepsilon_0}{\lambda_a - \varepsilon_0}\right) + 3\varepsilon$$

and define

$$\varphi(r) = -\frac{\delta}{4} r^4 + 2\delta.$$

We claim that

$$u^* \le u + \varphi \quad \text{in } B_1. \tag{51}$$

To prove this, we shall show that for $\lambda < \lambda^*$

$$u_\lambda \le u + \varphi \quad \text{in } B_1. \tag{52}$$

Indeed, we have

$$\begin{aligned} &\Delta^2 \varphi = -\delta 2N(N+2), \\ &\varphi(r) \ge \delta \quad \forall 0 \le r \le 1, \\ &\varphi(1) \ge \delta \ge \varepsilon, \quad \varphi'(1) = -\delta \le -\varepsilon, \end{aligned}$$

and therefore

$$\begin{aligned} \Delta^2(u+\varphi) &\le (\lambda_a + \varepsilon_0)e^u + \Delta^2\varphi \le (\lambda_a + \varepsilon_0)e^u = (\lambda_a + \varepsilon_0)e^{-\varphi}e^{u+\varphi} \\ &\le (\lambda_a + \varepsilon_0)e^{-\delta}e^{u+\varphi}. \end{aligned} \tag{53}$$

By (50) and the choice of δ

$$(\lambda_a + \varepsilon_0)e^{-\delta} = (\lambda_a - \varepsilon_0)e^{-3\varepsilon} < \lambda^*. \tag{54}$$

To prove (52) it suffices to consider λ in the interval $(\lambda_a - \varepsilon_0)e^{-3\varepsilon} < \lambda < \lambda^*$. Fix such λ and assume that (52) is not true. Write

$$\bar{u} = u + \varphi$$

and let

$$R_1 = \sup\{\, 0 \le R \le 1 \,|\, u_\lambda(R) = \bar{u}(R) \,\}.$$

Then $0 < R_1 < 1$ and $u_\lambda(R_1) = \bar{u}(R_1)$. Since $u'_\lambda(1) = 0$ and $\bar{u}'(1) < 0$ we must have $u'_\lambda(R_1) \le \bar{u}'(R_1)$. Then u_λ is a solution to the problem

$$\begin{cases} \Delta^2 u = \lambda e^u & \text{in } B_{R_1}, \\ u = u_\lambda(R_1) & \text{on } \partial B_{R_1}, \\ \dfrac{\partial u}{\partial n} = u'_\lambda(R_1) & \text{on } \partial B_{R_1}, \end{cases}$$

while, thanks to (53) and (54), $\bar{u}$ is a subsolution to the same problem. Moreover, $\bar{u}$ is stable thanks to (48) since, by Lemma 6.1,

$$\lambda < \lambda^* \le (\lambda_a + \varepsilon_0)e^{2\varepsilon} \tag{55}$$

and hence

$$\lambda e^{\bar{u}} \le (\lambda_a + \varepsilon_0)e^{2\varepsilon}e^{2\delta}e^u \le \beta_0 e^u.$$

We deduce $\bar{u} \le u_\lambda$ in B_{R_1} which is impossible, since $\bar{u}$ is singular while u_λ is smooth. This establishes (51).

From (51) and (55) we have

$$\lambda^* e^{u^*} \le \beta_0 e^{-\varepsilon} e^u$$

and therefore

$$\inf_{\varphi \in C_0^\infty(B)} \frac{\int_B (\Delta\varphi)^2 - \lambda^* e^{u^*}\varphi^2}{\int_B \varphi^2} > 0.$$

This is not possible if u^* is a smooth solution. □

For each dimension $13 \le N \le 31$ we construct u satisfying (44)–(48) of the form

$$u(r) = \begin{cases} -4\log r + \log\left(\frac{8(N-2)(N-4)}{\lambda}\right) & \text{for } 0 < r < r_0, \\ \tilde{u}(r) & \text{for } r_0 \le r \le 1, \end{cases} \tag{56}$$

where $\tilde{u}$ is explicitly given. Thus u satisfies (47) automatically.

Numerically it is better to work with the change of variables

$$w(s) = u(e^s) + 4s, \quad -\infty < s < 0,$$

which transforms the equation $\Delta^2 u = \lambda e^u$ into

$$Lw + 8(N-2)(N-4) = \lambda e^w, \quad -\infty < s < 0,$$

where

$$Lw = \frac{d^4w}{ds^4} + 2(N-4)\frac{d^3w}{ds^3} + (N^2 - 10N + 20)\frac{d^2w}{ds^2} - 2(N-2)(N-4)\frac{dw}{ds}.$$

The boundary conditions $u(1) = 0$, $u'(1) = 0$ then yield

$$w(0) = 0, \quad w'(0) = 4.$$

Regarding the behavior of w as $s \to -\infty$, observe that

$$u(r) = -4\log r + \log\left(\frac{8(N-2)(N-4)}{\lambda}\right) \quad \text{for } r < r_0$$

if and only if

$$w(s) = \log\frac{8(N-2)(N-4)}{\lambda} \quad \forall s < \log r_0.$$

The steps we perform are the following.

(1) We fix $x_0 < 0$ and using numerical software we follow a branch of solutions to

$$\begin{cases} L\hat{w} + 8(N-2)(N-4) = \lambda e^{\hat{w}}, \quad x_0 < s < 0, \\ \hat{w}(0) = 0, \quad \hat{w}'(0) = t, \\ \hat{w}(x_0) = \log \dfrac{8(N-2)(N-4)}{\lambda}, \quad \dfrac{d^2\hat{w}}{ds^2}(x_0) = 0, \quad \dfrac{d^3\hat{w}}{ds^3}(x_0) = 0 \end{cases}$$

as t increases from 0 to 4. The numerical solution $(\hat{w}, \hat{\lambda})$ we are interested in corresponds to the case $t = 4$. The five boundary conditions are due to the fact that we are solving a fourth order equation with an unknown parameter λ.

(2) Based on $\hat{w}$, $\hat{\lambda}$ we construct a C^3 function w which is constant for $s \le x_0$ and piecewise polynomial for $x_0 \le s \le 0$. More precisely, we first divide the interval $[x_0, 0]$ into smaller intervals of length h. Then we generate a cubic spline approximation g_{fl} with floating point coefficients of $\frac{d^4\hat{w}}{ds^4}$. From g_{fl} we generate a piecewise cubic polynomial g_{ra} which uses rational coefficients and we integrate it four times to obtain w, where the constants of integration are such that $\frac{d^j w}{ds^j}(x_0) = 0$, $1 \le j \le 3$, and $w(x_0)$ is a rational approximation of $\log(8(N-2)(N-4)/\lambda)$. Thus w is a piecewise polynomial function that in each interval is of degree 7 with rational coefficients, and which is globally C^3. We also let λ be a rational approximation of $\hat{\lambda}$. With these choices note that $Lw + 8(N-2)(N-4) - \lambda e^w$ is a small constant (not necessarily 0) for $s \le x_0$.

(3) The conditions (44) and (45) we need to check for u are equivalent to the following inequalities for w:

$$Lw + 8(N-2)(N-4) - (\lambda + \varepsilon_0)e^w \le 0, \quad -\infty < s < 0, \tag{57}$$

$$Lw + 8(N-2)(N-4) - (\lambda - \varepsilon_0)e^w \ge 0, \quad -\infty < s < 0. \tag{58}$$

Using a program in Maple we verify that w satisfies (57) and (58). This is done by evaluating a second order Taylor approximation of $Lw + 8(N-2)(N-4) - (\lambda+\varepsilon_0)e^w$ at sufficiently close mesh points. All arithmetic computations are done with rational numbers and thus obtain exact results. The exponential function is approximated by a Taylor polynomial of degree 14, and the difference with the real value is controlled.

More precisely, we write

$$f(s) = Lw + 8(N-2)(N-4) - (\lambda + \varepsilon_0)e^w,$$
$$\tilde{f}(s) = Lw + 8(N-2)(N-4) - (\lambda + \varepsilon_0)T(w),$$

where T is the Taylor polynomial of order 14 of the exponential function around 0. Applying Taylor's formula to f at y_j, we have for $s \in [y_j, y_{j+h}]$

$$\begin{aligned} f(s) &\le f(y_j) + |f'(y_j)|h + \frac{1}{2}Mh^2 \\ &\le \tilde{f}(y_j) + |\tilde{f}'(y_j)|h + \frac{1}{2}Mh^2 + |f(y_j) - \tilde{f}(y_j)| + |f'(y_j) - \tilde{f}'(y_j)|h \\ &\le \tilde{f}(y_j) + |\tilde{f}'(y_j)|h + \frac{1}{2}Mh^2 + E_1 + E_2 h, \end{aligned}$$

where

M is a bound for $|f''|$ in $[y_j, y_j + h]$,
E_1 is such that $(\lambda + \varepsilon_0)|e^w - T(w)| \le E_1$ in $[y_j, y_j + h]$,
E_2 is such that $(\lambda + \varepsilon_0)|(e^w - T'(w))w'| \le E_2$ in $[y_j, y_j + h]$.

So, inequality (57) will be verified on each interval $[y_j, y_j+h]$ where w is a polynomial as soon as

$$\tilde{f}(y_j) + |\tilde{f}'(y_j)|h + \frac{1}{2}Mh^2 + E_1 + E_2 h \le 0. \tag{59}$$

When more accuracy is desired, instead of (59) one can verify that

$$\tilde{f}(x_i) + |\tilde{f}'(x_i)|\frac{h}{m} + \frac{1}{2}M\left(\frac{h}{m}\right)^2 + E_1 + E_2\frac{h}{m} \le 0,$$

where $(x_i)_{i=1\dots m+1}$ are $m+1$ equally spaced points in $[y_j, y_j+h]$.

We obtain exact values for the upper bounds M, E_1, E_2 as follows. First note that $f'' = Lw'' - (\lambda+\varepsilon_0)e^w((w')^2 + w'')$. On $[y_j, y_j+h]$, we have $w(s) = \sum_{i=0}^7 a_i(s-y_j)^i$ and we estimate $|w(s)| \le \sum_{i=0}^7 |a_i|h^i$ for $s \in [y_j, y_j+h]$. Similarly,

$$\left|\frac{d^\ell w}{ds^\ell}(s)\right| \le \sum_{i=\ell}^{7} i(i-1)\dots(i-\ell+1)|a_i|h^{i-\ell} \quad \forall s \in [y_j, y_j+h]. \tag{60}$$

The exponential is estimated by $e^w \le e^1 \le 3$, since our numerical data satisfies the rough bounds $-3/2 \le w \le 1$. Using this information and (60) yields a rational upper bound M. E_1 is estimated using Taylor's formula:

$$E_1 = (\lambda+\varepsilon_0)\frac{(3/2)^{15}}{15!}.$$

Similarly, $E_2 = (\lambda+\varepsilon_0)\frac{(3/2)^{14}}{14!}B_1$, where B_1 is the right-hand side of (60) when $\ell = 1$.

(4) We show that the operator $\Delta^2 - \beta e^u$ where $u(r) = w(\log r) - 4\log r$, satisfies condition (48) for some $\beta \ge \beta_0$ where β_0 is given by (49). In dimension $N \ge 13$ the operator $\Delta^2 - \beta e^u$ has indeed a positive eigenfunction in $H_0^2(B)$ with finite eigenvalue if β is not too large. The reason is that near the origin

$$\beta e^u = \frac{c}{|x|^4},$$

where c is a number close to $8(N-2)(N-4)\beta/\lambda$. If β is not too large compared to λ, then $c < N^2(N-4)^2/16$, and hence, using (6), $\Delta^2 - \beta e^u$ is coercive in $H_0^2(B_{r_0})$ (this holds under even weaker conditions; see [11]). It follows that there exists a first eigenfunction $\varphi_1 \in H_0^2(B)$ for the operator $\Delta^2 - \beta e^u$ with a finite first eigenvalue μ_1; that is,

$$\begin{aligned} &\Delta^2\varphi_1 - \beta e^u\varphi_1 = \mu_1\varphi_1 \quad \text{in } B,\\ &\varphi_1 > 0 \quad \text{in } B,\\ &\varphi_1 \in H_0^2(B). \end{aligned}$$

Moreover, μ_1 can be characterized as

$$\mu_1 = \inf_{\varphi \in C_0^\infty(B)} \frac{\int_B (\Delta\varphi)^2 - \beta e^u\varphi^2}{\int_B \varphi^2}$$

and is the smallest number for which a positive eigenfunction in $H_0^2(\Omega)$ exists.

Thus to prove that (48) holds it suffices to verify that $\mu_1 \geq 0$ and for this it is enough to show the existence of a nonnegative $\varphi \in H_0^2(B)$, $\varphi \not\equiv 0$, such that

$$\begin{cases} \Delta^2\varphi - \beta e^u \varphi \geq 0 & \text{in } B, \\ \varphi = 0 & \text{on } \partial B, \\ \dfrac{\partial \varphi}{\partial n} \leq 0 & \text{on } \partial B. \end{cases} \tag{61}$$

Indeed, multiplication of (61) by φ_1 and integration by parts yield

$$\mu_1 \int_B \varphi\varphi_1 + \int_{\partial B} \frac{\partial \varphi}{\partial n} \Delta\varphi_1 \geq 0.$$

But $\Delta\varphi_1 \geq 0$ on ∂B and thus $\mu_1 \geq 0$. To achieve (61) we again change variables and define

$$\phi(s) = \varphi(e^s), \quad -\infty < s \leq 0.$$

Then we have to find $\phi \geq 0$, $\phi \not\equiv 0$, satisfying

$$\begin{cases} L\phi - \beta e^w \phi \geq 0 & \text{in } -\infty < s \leq 0, \\ \phi(0) = 0, \\ \phi'(0) \leq 0. \end{cases} \tag{62}$$

Regarding the behavior as $s \to -\infty$, we note that w is constant for $-\infty < s < x_0$, and therefore, if

$$L\phi - \beta e^w \phi \equiv 0, \quad -\infty < s \leq x_0,$$

then ϕ is a linear combination of exponential functions $e^{-\alpha s}$, where α must be a solution to

$$\alpha^4 - 2(N-4)\alpha^3 + (N^2 - 10N + 20)\alpha^2 + 2(N-2)(N-4)\alpha = \beta e^{w(x_0)},$$

where $\beta e^{w(x_0)}$ is close to $8(N-2)(N-4)\beta/\lambda$. If $N \geq 13$ the polynomial

$$\alpha^4 - 2(N-4)\alpha^3 + (N^2 - 10N + 20)\alpha^2 + 2(N-2)(N-4)\alpha - 8(N-2)(N-4)$$

has four distinct real roots, while if $N \leq 12$ there are two real roots and two complex conjugates. If $N \geq 13$ there is exactly one root in the interval $(0, (N-4)/2)$, two roots greater than $(N-4)/2$, and one negative. We know that $\varphi(r) = \phi(\log r) \sim r^{-\alpha}$ is in H^2, which forces $\alpha < (N-4)/2$. It follows that for $s < x_0$, ϕ is a combination of $e^{-\alpha_0 s}$, $e^{-\alpha_1 s}$ where $\alpha_0 > 0$, $\alpha_1 < 0$ are the two roots smaller than $\alpha < (N-4)/2$. For simplicity, however, we will look for ϕ such that $\phi(s) = Ce^{-\alpha_0 s}$ for $s < x_0$, where $C > 0$ is a constant. This restriction will mean that we will not be able to impose $\phi'(0) = 0$ at the end. This is not a problem because $\phi'(0) \leq 0$.

Notice that we need only the inequality in (62), and hence we need to choose $\alpha \in (0, N-4/2)$ such that

$$\alpha^4 - 2(N-4)\alpha^3 + (N^2 - 10N + 20)\alpha^2 + 2(N-2)(N-4)\alpha \geq \beta e^{w(x_0)}.$$

The precise choice we employed in each dimension is in a summary table at the end of this section.

To find a suitable function ϕ with the behavior $\phi(s) = Ce^{-\alpha s}$ for $s < x_0$ we set $\phi = \psi e^{-\alpha s}$ and solve the equation

$$T_\alpha \psi - \beta e^w \psi = f,$$

where the operator T_α is given by

$$\begin{aligned} T_\alpha \psi = \frac{d^4\psi}{ds^4} &+ (-4\alpha + 2(N-4))\frac{d^3\psi}{ds^3} + (6\alpha^2 - 6\alpha(N-4) + N^2 - 10N + 20)\frac{d^2 w}{ds^2} \\ &+ (-4\alpha^3 + 6\alpha^2(N-4) - 2\alpha(N^2 - 10N + 20) - 2(N-2)(N-4))\frac{d\psi}{ds} \\ &+ (\alpha^4 - 2\alpha^3(N-4) + \alpha^2(N^2 - 10N + 20) + 2\alpha(N-2)(N-4))\psi \end{aligned}$$

and f is some smooth function such that $f \geq 0$, $f \not\equiv 0$. Actually we choose $\bar\beta > \beta_0$ (where β_0 is given in (49)) and find $\bar\alpha$ satisfying approximately

$$\bar\alpha^4 - 2(N-4)\bar\alpha^3 + (N^2 - 10N + 20)\bar\alpha^2 + 2(N-2)(N-4)\bar\alpha = \bar\beta e^{w(x_0)}.$$

We solve numerically

$$\begin{aligned} T_{\bar\alpha}\hat\psi - \bar\beta e^w \hat\psi = f, &\quad x_0 < s < 0, \\ \hat\psi(x_0) = 1, \quad \hat\psi''(x_0) = 0, &\quad \hat\psi'''(x_0) = 0, \\ \hat\psi(0) = 0. & \end{aligned}$$

Using the same strategy as in (2) from the numerical approximation of $\frac{d^4\hat\psi}{ds^4}$ we compute a piecewise polynomial ψ of degree 7, which is globally C^3 and constant for $s \leq x_0$. The constant $\psi(x_0)$ is chosen so that $\psi(0) = 0$. We then use Maple to verify the inequalities

$$\begin{aligned} \psi \geq 0, &\quad x_0 \leq s \leq 0, \\ T_\alpha\psi - \beta e^w \psi \geq 0, &\quad x_0 \leq s \leq 0, \\ \psi'(0) \leq 0, & \end{aligned}$$

where $\beta_0 < \beta < \bar\beta$ and $0 < \alpha < (N-4)/2$ are suitably chosen.

At the URLs http://www.lamfa.u-picardie.fr/dupaigne/ and http://www.ime.unicamp.br/~msm/ we provide the data of the functions w and ψ defined as piecewise polynomials of degree 7 in $[x_0, 0]$ with rational coefficients for each dimension in $13 \leq N \leq 31$. We also give a rational approximation of the constants involved in the corresponding problems.

We use Maple to verify that w and ψ (with suitable extensions) are C^3 global functions and satisfy the corresponding inequalities, using only its capability to operate on arbitrary rational numbers. These operations are exact and are limited only by the memory of the computer and clearly slower than floating point operations. We chose Maple since it is a widely used software, but the reader can check the validity of our results with any other software (see, e.g., the open-source solution pari/gp).

The tests were conducted using Maple 9. See Table 1 for a summary of parameters and results.

Remark 6.4. (1) Although we work with λ rational, in Table 1 we prefer to display a decimal approximation of λ.

(2) In Table 1 we selected a "large" value of ε_0 in order to have a fast verification with Maple. By requiring more accuracy in the numerical calculations, using a

TABLE 1

N	λ	ε_0	ε	β	β	α
13	2438.6	1	$5 \cdot 10^{-7}$	2550	2500	3.9
14	2911.2	1	$3 \cdot 10^{-6}$	3100	3000	3.4
15	3423.8	1	$3 \cdot 10^{-6}$	3600	3500	3.1
16	3976.4	1	$1 \cdot 10^{-5}$	4100	4000	3.0
17	4568.8	1	$2 \cdot 10^{-4}$	4800	4600	3.0
18	5201.1	2	$2 \cdot 10^{-4}$	5400	5300	2.7
19	5873.2	2	$2 \cdot 10^{-4}$	6100	6000	2.7
20	6585.1	3	$7 \cdot 10^{-4}$	7000	6800	2.7
21	7336.7	3	$7 \cdot 10^{-4}$	7700	7500	2.6
22	8128.1	4	$1 \cdot 10^{-3}$	8600	8400	2.6
23	8959.1	4	$1 \cdot 10^{-3}$	9400	9200	2.5
24	9829.8	4	$1 \cdot 10^{-3}$	10400	10200	2.5
25	10740.1	4	$1 \cdot 10^{-3}$	11400	11200	2.5
26	11690.1	6	$2 \cdot 10^{-3}$	12400	12200	2.5
27	12679.7	7	$2 \cdot 10^{-3}$	13400	13200	2.4
28	13709.0	7	$2 \cdot 10^{-3}$	14500	14300	2.4
29	14777.8	7	$2 \cdot 10^{-3}$	15400	15200	2.4
30	15886.2	8	$2 \cdot 10^{-3}$	16600	16400	2.4
31	17034.3	10	$2 \cdot 10^{-3}$	17600	17500	2.3

TABLE 2

N	λ	ε_0	ε	λ^*_{min}	λ^*_{max}	β	β	α
13	2438.589	0.003	$5 \cdot 10^{-7}$	2438.583	2438.595	2550	2510	3.9
14	2911.194	0.003	$5 \cdot 10^{-7}$	2911.188	2911.200	3100	3000	3.4

smaller value of ε_0, and using more subintervals to verify the inequalities in the Maple program, it is possible to obtain better estimates of λ^*. For instance, using formulas (50), we obtained the results in Table 2.

The verification above, however, is required to check 1500 subintervals of each of the 4500 intervals of length 0.002, which amounts to substantial computer time.

7. Proof of Proposition 1.5. Throughout this section, we restrict our attention, as permitted, to the case $a = 0$.

(a) Let u denote the extremal solution of (1) with homogeneous Dirichlet boundary condition $a = b = 0$. We extend u on its maximal interval of existence $(0, \bar{R})$.

LEMMA 7.1. *$\bar{R} < \infty$ and $u(r) \sim \log(\bar{R} - r)^{-4}$ for $r \sim \bar{R}$.*

Proof. The fact that $\bar{R} < \infty$ can be readily deduced from section 2 of [1]. We present an alternative (and more quantitative) argument. We first observe that

$$u'' - \frac{1}{r}u' > 0 \qquad \forall r \in [1, \bar{R}). \tag{63}$$

Integrate indeed (1) over a ball of radius r to conclude that

$$0 < \lambda \int_{B_r} e^u = \int_{\partial B_r} \frac{\partial}{\partial r}\Delta u = \omega_N r^{N-1}\left(u''' + \frac{N-1}{r}\left(u'' - \frac{1}{r}u'\right)\right). \tag{64}$$

If $r = 1$, since u is nonnegative in $(0, 1)$ and $u(1) = u'(1) = 0$, we must have $u''(1) \geq 0$. In fact, $u''(1) > 0$. Otherwise, we would have $u''(1) = 0$ and $u'''(1) > 0$ by (64), contradicting $u > 0$ in $(0, 1)$. So, we may define

$$R = \sup\left\{r > 1 : u''(t) - \frac{1}{t}u'(t) > 0 \;\; \forall t \in [1, r)\right\},$$

and we just need to prove that $R = \bar{R}$. Assume this is not the case; then $u''(R) - \frac{1}{R}u'(R) = 0$ and $u'''(R) = \left(u'' - \frac{1}{R}u'\right)'(R) \le 0$. This contradicts (64) and we have just proved (63). In particular, we see that u is convex increasing on $(1, \bar{R})$.

Since u is radial, (1) reduces to

$$u^{(4)} + \frac{2(N-1)}{r}u''' + \frac{(N-1)(N-3)}{r^2}u'' - \frac{(N-1)(N-3)}{r^3}u' = \lambda e^u. \tag{65}$$

Multiply by u':

$$u^{(4)}u' + \frac{2(N-1)}{r}u'''u' + \frac{(N-1)(N-3)}{r^2}u''u' - \frac{(N-1)(N-3)}{r^3}(u')^2 = \lambda(e^u)',$$

which we rewrite as

$$[(u'''u')' - u'''u''] + 2(N-1)\left[\left(\frac{1}{r}u''u'\right)' - u''\left(\frac{1}{r}u'\right)'\right] + (N-1)(N-3)\left(\frac{(u')^2}{2r^2}\right)' = \lambda(e^u)'.$$

By (63), it follows that for $r \in [1, \bar{R})$,

$$[(u'''u')' - u'''u''] + 2(N-1)\left(\frac{1}{r}u''u'\right)' + (N-1)(N-3)\left(\frac{(u')^2}{2r^2}\right)' \ge \lambda(e^u)'.$$

Integrating, we obtain for some constant A

$$u'''u' - \frac{(u'')^2}{2} + 2(N-1)\frac{1}{r}u''u' + \frac{(N-1)(N-3)}{2}\frac{(u')^2}{r^2} \ge \lambda e^u - A.$$

We multiply again by u':

$$\left[(u''(u')^2)' - u''\left((u')^2\right)'\right] - \frac{1}{2}(u'')^2u' + 2(N-1)\frac{1}{r}u''(u')^2 + \frac{(N-1)(N-3)}{2}\frac{1}{r^2}(u')^3 \ge (\lambda e^u - Au)'. \tag{66}$$

We deduce from (63) that

$$\frac{1}{r}u''(u')^2 = \frac{1}{2}\left(\frac{1}{r}(u')^3\right)' - \frac{1}{2}(u')^2\left(\frac{1}{r}u'\right)' \le \frac{1}{2}\left(\frac{1}{r}(u')^3\right)' \quad \text{and}$$

$$\frac{1}{r^2}(u')^3 \le \frac{1}{r}(u')^2u'' \le \frac{1}{2}\left(\frac{1}{r}(u')^3\right)'.$$

Using this information in (66), dropping nonpositive terms, and integrating, we obtain for some constant B

$$u''(u')^2 + \frac{(N^2-1)}{4}\frac{1}{r}(u')^3 \ge \lambda e^u - Au - B.$$

Applying (63) again, it follows that for $C = \frac{N^2-1}{4} + 1$

$$Cu''(u')^2 \ge \lambda e^u - Au - B,$$

which after multiplication by u' and integration provides positive constants c, C such that

$$(u')^4 \geq c(e^u - Au^2 - Bu - C).$$

At this point, we observe that since u is convex and increasing, u converges to $+\infty$ as r approaches $\bar{R}$. Hence, for r close enough to $\bar{R}$ and for $c > 0$ perhaps smaller,

$$u' \geq c\, e^{u/4}.$$

By Gronwall's lemma, $\bar{R}$ is finite and

$$u \leq -4\log(\bar{R} - r) + C \qquad \text{for } r \text{ close to } \bar{R}.$$

It remains to prove that $u \geq -4\log(\bar{R} - r) - C$. This time, we rewrite (1) as

$$\left[r^{N-1}(\Delta u)'\right]' = \lambda r^{N-1} e^u.$$

We multiply by $r^{N-1}(\Delta u)'$ and obtain

$$\frac{1}{2}\left[r^{2N-2}((\Delta u)')^2\right]' = \lambda r^{2N-2} e^u (\Delta u)' \leq Ce^u(\Delta u)' \leq C(e^u \Delta u)'.$$

Hence, for r close to $\bar{R}$ and C perhaps larger,

$$((\Delta u)')^2 \leq Ce^u \Delta u,$$

and so

$$\sqrt{\Delta u}(\Delta u)' \leq Ce^{u/2}\Delta u \leq C' e^{u/2} u'' \leq C'(e^{u/2}u')',$$

where we have used (63). Integrate to conclude that

$$(\Delta u)^{3/2} \leq Ce^{u/2}u'.$$

Solving for Δu and multiplying by $(u')^{1/3}$, we obtain in particular that

$$(u')^{1/3}u'' \leq Ce^{u/3}u'.$$

Integrating again, it follows that $(u')^{4/3} \leq Ce^{u/3}$, i.e.,

$$u' \leq Ce^{u/4}.$$

It then follows easily that (for r close to $\bar{R}$)

$$u \geq -4\log(\bar{R} - r) - C. \qquad \square$$

Proof of Proposition 1.5(a). Given $N \geq 13$, let b^{max} denote the supremum of all parameters $b \geq -4$ such that the corresponding extremal solution is singular. We first observe that

$$b^{max} > 0.$$

In fact, it follows from sections 5 and 6 that the extremal solution u associated to parameters $a = b = 0$ is strictly stable:

$$\inf_{\varphi \in C_0^\infty(B)} \frac{\int_B (\Delta\varphi)^2 - \lambda^* \int_B e^u \varphi^2}{\int_B \varphi^2} > 0. \tag{67}$$

Extend u as before on its maximal interval of existence $(0, \bar{R})$. Choosing $R \in (1, \bar{R})$ close to 1, we deduce that (67) still holds on the ball B_R. In particular, letting $v(x) = u(Rx) - u(R)$ for $x \in B$, we conclude that v is a singular stable solution of (1) with $a = 0$ and $b = Ru'(R) > 0$. By Proposition 1.7, we conclude that $b^{max} > 0$. We now prove that

$$b^{max} < \infty.$$

Assume this is not the case and let u_n denote the (singular) extremal solution associated to b_n, where $b_n \nearrow \infty$. We first observe that there exists $\rho_n \in (0,1)$ such that $u_n'(\rho_n) = 0$. Otherwise, u_n would remain monotone increasing on $(0,1)$ and hence bounded above by $u_n(1) = 0$. It would then follow from (1) and elliptic regularity that u_n is bounded. Let $v_n(x) = u_n(\rho_n x) - u_n(\rho_n)$ for $x \in B$ and observe that v_n solves (1) with $a = b = 0$ and some $\lambda = \lambda_n$. Clearly v_n is stable and singular. By Proposition 1.7, v_n coincides with u, the extremal solution of (1) with $a = b = 0$. By standard ODE theory, $v_n = u$ on $(0, \bar{R})$. In addition,

$$b_n = u_n'(1) = \frac{1}{\rho_n} v_n'\left(\frac{1}{\rho_n}\right) = \frac{1}{\rho_n} u'\left(\frac{1}{\rho_n}\right) \to +\infty,$$

which can happen only if $1/\rho_n \to \bar{R}$.

Now, since u_n is stable on B, $u = v_n$ is stable on B_{1/ρ_n}. Letting $n \to \infty$, we conclude that u is stable on $B_{\bar{R}}$. This clearly contradicts Lemma 7.1.

We have just proved that b^{max} is finite. It remains to prove that u^* is singular when $-4 \leq b \leq b^{max}$. We begin with the case $b = b^{max}$. Choose a sequence (b_n) converging to b^{max} and such that the corresponding extremal solution u_n is singular. Using the same notation as above, we find a sequence $\rho_n \in (0,1)$ such that

$$\frac{1}{\rho_n} u'\left(\frac{1}{\rho_n}\right) = b_n \to b^{max}.$$

Taking subsequences if necessary and passing to the limit as $n \to \infty$, we obtain for some $\rho \in (0,1)$

$$\frac{1}{\rho} u'\left(\frac{1}{\rho}\right) = b^{max}.$$

Furthermore, by construction of ρ_n, u is stable in B_{1/ρ_n} and hence in $B_{1/\rho}$. This implies that v defined for $x \in B$ by $v(x) = u(\frac{x}{\rho}) - u(\frac{1}{\rho})$ is a stable singular solution of (1) with $b = b^{max}$. By Proposition 1.7, we conclude that the extremal solution is singular when $b = b^{max}$.

When $b = -4$, as we have already mentioned in the introduction, u^* is singular for $N \geq 13$ as a direct consequence of Proposition 1.7 and Rellich's inequality.

So we are left with the case $-4 < b < b^{max}$. Let u_m^* denote the extremal solution when $b = b^{max}$, which is singular, and λ_m^* the corresponding parameter. For $0 < R < 1$ set

$$u_R(x) = u_m^*(Rx) - u_m^*(R).$$

Then

$$\Delta^2 u_R = \lambda_R e^{u_R}, \quad \text{where} \quad \lambda_R = \lambda_0^* R^4 e^{u_m^*(R)},$$

and $u_R = 0$ on ∂B, while

$$\frac{du_R}{dr}(1) = R\frac{du_m^*}{dr}(R).$$

By (31), note that

$$R\frac{du_m^*}{dr}(R) \to b^{max} \quad \text{as } R \to 1, \quad \text{and} \quad R\frac{du_m^*}{dr}(R) \to -4 \quad \text{as } R \to 0.$$

Thus, for any $-4 < b < b^{max}$ we have found a singular stable solution to (1) (with $a = 0$). By Proposition 1.7 the extremal solution to this problem is singular. ∎

Proof of Proposition 1.5(b). Let $b \geq -4$. Lemma 5.1 applies also for $b \geq -4$ and yields $u^* \leq \bar{u}$, where $\bar{u}(x) = -4\log|x|$. We now modify slightly the proof of Lemma 5.2. Indeed, consider $w = (4+b)(1-r^2)/2$ and define $u = \bar{u} - w$. Then

$$\begin{aligned}\Delta^2 u &= 8(N-2)(N-4)\frac{1}{r^4} = 8(N-2)(N-4)e^{\bar{u}} = 8(N-2)(N-4)e^{u+w} \\ &\leq 8(N-2)(N-4)e^{(4+b)/2}e^u.\end{aligned}$$

Also $u(1) = 0$ and $u'(1) = b$, so u is a subsolution to (1) with parameter $\lambda_0 = 8(N-2)(N-4)e^{(4+b)/2}$.

If N is sufficiently large, depending on b, we have $\lambda_0 < N^2(N-4)^2/16$. Then by (6) u is a stable subsolution of (1) with $\lambda = \lambda_0$. As in Lemma 5.2 this implies $\lambda^* \leq \lambda_0$.

Thus for large enough N we have $\lambda^* e^{u^*} \leq r^{-4}\,8(N-2)(N-4)e^{(4+b)/2} < r^{-4}N^2(N-4)^2/16$. This and (6) show that

$$\inf_{\varphi \in C_0^\infty(B)} \frac{\int_B (\Delta\varphi)^2 - \lambda^* \int_B e^{u^*}\varphi^2}{\int_B \varphi^2} > 0,$$

which is not possible if u^* is bounded.

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