A NEW APPROACH TO DETECTABILITY OF DISCRETE-TIME INFINITE MARKOV JUMP LINEAR SYSTEMS*

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Abstract. This paper deals with detectability for the class of discrete-time Markov jump linear systems (MJLS) with the underlying Markov chain having countably infinite state space. The formulation here relates the convergence of the output with that of the state variables, and due to the rather general setting, a novel point of view toward detectability is required. Our approach introduces invariant subspaces for the autonomous system and exhibits the role that they play. This allows us to show that detectability can be written equivalently in term of two conditions: stability of the autonomous system in a certain invariant space and convergence of general state trajectories to this invariant space under convergence of input and output variables. This, in turn, provides the tools to show that detectability here generalizes uniform observability ideas as well as previous detectability notions for MJLS with finite state Markov chain, and allows us to solve the jump-linear-quadratic control problem. In addition, it is shown for the MJLS with finite Markov state that the second condition is redundant and that detectability retrieves previously well-known concepts in their respective scenarios. Illustrative examples are included.

Key words. detectability, stochastic systems, Markov jump systems, infinite Markov state space, optimal control

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1. Introduction. Structural concepts such as observability and detectability have a solid ground in system theory, as the imposing literature for linear and linear-Gaussian systems conveys (see, e.g., [15]). For instance, in control problems, detectability firmly associates the solution for the optimal problems with stability of the corresponding controlled system, whereas, for filtering, it makes the system observations meaningful for state estimates by connecting convergence of the output with convergence of the state. Although the theory involving these concepts is quite developed and a number of results are available in the context of linear deterministic systems, there is still a great deal of research activity in this area (see, e.g., [13, 17] and references therein).

Among the most important properties of detectability for the linear deterministic scenario, we mention that

(i) detectability can be expressed in terms of the parameters of the autonomous version of the system, e.g., by requiring that nonobserved modes of the autonomous system are stable.

(ii) Detectability generalizes observability.

Another important but less acknowledged property is that

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(iii) detectability is a necessary and sufficient condition to guarantee convergence of the state from convergence of the output (under regular nonsingular linear state feedback controls).

Property (iii) ensures that the optimal control solution is stabilizing and makes output observations meaningful in filtering problems.

Due to its generic formulation, these properties constitute a paradigm for more general contexts. The challenge then is how to devise a detectability concept for a certain class of systems that allows one to employ the structure of the system to retrieve properties (i)–(iii).

In this spirit, the authors have recently developed a notion of detectability (called weak detectability) that generalizes previous detectability ideas for MJLS with finite Markov chain state, retrieves the properties (i)–(iii), and allows an associate observability matrix, in an extension to the well-known deterministic concepts, see [1] and [2]. In this process all but one[1] of the linear deterministic concepts are retrieved.

However, as far as the authors are aware, these ideas have no parallel in more complex scenarios such as the MJLS with countably infinite state space of the Markov chain. This is a rather general class of systems that includes the classes of finite MJLS and linear deterministic systems, as well as deterministic time varying systems. Previous works dealing with infinite MJLS are [7, 8, 9, 10].For this class of systems, up to this date there is no detectability concept that retrieves properties (i)–(iii) above. For instance, the stochastic notion in [7] can be expressed in terms of the autonomous system data, thus satisfying (i), but (ii) does not hold and only the sufficiency part of (iii) holds; in [4] we derive a detectability notion in the perspective of (iii) for which (ii) holds, but it does not satisfy (i).

These shortfalls come, in part, from the analytical complexity inherent to the infinite many Markov state contexts, and the loss of some friendlier structures of the simpler cases. In particular, the main difficulty arises from the fact demonstrated in this paper that converging input and output *do not* ensure convergence of state trajectory to the observed space; see Example 2 in connection. In the simpler case of finitely many Markov states, the above convergence relation holds, and apart from ensuring stability within the observed space, with detectability it guarantees convergence of the state trajectory to the origin. This is the mechanism that fails here, and in this regard we can conclude that any detectability concept with the perspective of (i) (stable nonobserved modes) by itself cannot provide the property in (iii) and thus, it cannot ensure that the optimal control is stabilizing.

In this paper, with the aim of studying detectability for MJLS with countably infinite state space of the Markov chain and to retrieve (i)–(iii), we introduce a novel point of view toward detectability by considering the paradigmatic property in (iii) as a general, direct, and intuitive notion of detectability, which relates the convergence of the input and output with that of the state variables. Then we introduce certain invariant subspaces for the autonomous system, which play a key role to relate detectability with stability and convergence of the state trajectory; this allows us to show that detectability here generalizes uniform observability ideas as well as previous detectability notions for MJLS with finite state Markov chain, and to solve the jump-linear-quadratic control problem. In order to show some nuances of the approach developed here, and to clarify the role of some tools, we also analyze the MJLS with finite state Markov chain and present illustrative examples.

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[1]The observability idea that after a number of observations that equals the system dimension, the initial state value can be precisely retrieved. This is inherently a nonstochastic idea.

An outline of the content of this paper is as follows. In section 2 we provide the bare essential of notations, state the model, and discuss the general ideas of the paper. Section 3 provides some preliminaries. Necessary and sufficient conditions for detectability are treated in section 4, and some sufficient conditions are presented in section 5. The finite MJLS is analyzed in section 6, and the control problem is studied in section 7. Some illustrative examples are exhibited in section 8. Finally, section 9 presents some conclusions.

2. Problem formulation and general ideas. Let $\mathbb{R}^n$ represent the usual linear space of all n-dimensional vectors and $\mathcal{R}^{r,n}$ (respectively, $\mathcal{R}^n$) the normed linear space formed by all $r \times n$ real matrices (respectively, $n \times n$). For $V \in \mathcal{R}^{n,r}$, V' denotes the transpose of V. $\sigma^+(V)$ and $\sigma^-(V)$ stand, respectively, for the largest and smallest singular value of V and $\|V\| = \sigma^+(V)$. For $V, W \in \mathcal{R}^n$, $V > W$ ($V \geq W$) indicates that $V - W$ is positive definite (semidefinite).

Let $\mathcal{H}_\infty^{r,n}$ denote the linear space formed by sequences of matrices $H = \{H_i \in \mathcal{R}^{r,n}; i \in \mathcal{Z}\}$ such that $\sup_{i \in \mathcal{Z}} \|H_i\| < \infty$; also, $\mathcal{H}_\infty^n \equiv \mathcal{H}_\infty^{n,n}$ and $\|H\|_\infty = \sup_{i \in \mathcal{Z}} \|H_i\|$. For $H, V \in \mathcal{H}_\infty^n$, $H \geq V$ indicates that $H_i \geq V_i$ for each $i \in \mathcal{Z}$; similarly, for $H \in \mathcal{H}_\infty^{r,n}$ and $V \in \mathcal{H}_\infty^{n,s}$, the "product" HV indicates the element of $\mathcal{H}_\infty^{r,s}$ formed by the sequence $\{H_i V_i, i \in \mathcal{Z}\}$, and equivalent understanding should apply to any basic mathematical operation involving elements of $\mathcal{H}_\infty^{r,n}$. In what follows, capital letters denote elements of $\mathcal{H}_\infty^{r,n}$, and capital letters with an index denote elements of $\mathcal{R}^{r,n}$.

The system we deal with is the discrete-time MJLS with infinite countably Markov chain, defined in a fixed stochastic basis $(\Omega, \mathfrak{F}, (\mathfrak{F}_k), \mathcal{P})$ by

$$(1) \qquad \Psi : \begin{cases} x(k+1) = A_{\theta(k)} x(k) + B_{\theta(k)} u(k), & k \geq 0, \\ y(k) = C_{\theta(k)} x(k) + D_{\theta(k)} u(k), & x(0) = x, \ \theta(0) = \theta, \end{cases}$$

where y is the output process and u is the input, an $(\mathfrak{F}_k)$-adapted process. The mode θ is the state of an underlying discrete-time Markov chain $\Theta = \{\theta(k); k \geq 0\}$ taking values in $\mathcal{Z} = \{1, 2, \dots\}$ and having a stationary transition probability matrix $\mathbb{P} = [p_{ij}]$, $i, j \in \mathcal{Z}$. The state of the system is the compound variable $(x(k), \theta(k))$. The matrices A_i belong to the sequence of matrices $A \in \mathcal{H}_\infty^n$, and similarly for $B \in \mathcal{H}_\infty^{n,r}$, $C \in \mathcal{H}_\infty^{q,n}$, and $D \in \mathcal{H}_\infty^{q,r}$. In addition, without loss of generality, we also assume that $C'D = 0$.

In this paper we deal with detectability for systems described by (1). The departure point is the following concept of detectability that follows from property (iii) of section 1. We emphasize that the specific notion of convergence is not relevant; the essence of the concept is the relation among convergence of state, input and output, and a particular sense of convergence is adopted later in connection with the choice of the cost functional.

DEFINITION 1 (detectability). *The system Ψ is detectable if the state converges provided that the output and the input converge.*

With the detectability concept above at hands, which trivially embraces property (iii) in the introduction, the issues pursued here are primarily summarized as follows:

(I) Relate the concept with the autonomous version of the system, aiming at mimicking item (i) mentioned in the introduction.

(II) Show that it retrieves property (ii) mentioned in the introduction.

(III) Investigate the extent to which the above concept is related to the weak detectability concept in [1] and [2] for MJLS, and the usual concept for deterministic linear systems.

We consider a cost functional that is an ℓ_2-measurement of the output (the expected accumulated energy in the output process path),

$$\mathcal{Y}_u(x,\theta) = E_{x,\theta} \left\{ \sum_{k=0}^{\infty} |y(k)|^2 \right\}, \tag{2}$$

defined for an admissible control u whenever $x(0) = x$ and $\theta(0) = \theta$. We also denote for the autonomous system obtained from Ψ with $u \equiv 0$,

$$\mathcal{Y}_0(x,\theta) = \mathcal{Y}_{u \equiv 0}(x,\theta). \tag{3}$$

In agreement with (2), we adopt the corresponding ℓ_2-convergence notion for each Ψ-processes, namely, we say that the output y converges whenever $\mathcal{Y}_u(\cdot,\cdot) < \infty$; similar notion holds for u and x.

Our approach starts from a novel point of view, which hinges on the following steps. We first locate an invariant linear subspace for the autonomous system, in the sense that the trajectories remain almost surely confined to it. Then we indicate the role that the invariant space plays in the convergence of an arbitrary state trajectory, showing that the existence of an invariant space for which the autonomous system is stable, together with the convergence to this set of an arbitrary trajectory, is equivalent to convergence to the origin of such a trajectory (see section 4 and Theorem 12).

Note that the announced result reduces to a tautology if the invariant space is taken to be the origin, and to make the above result suitable to deal with (I), we seek the largest of such an invariant space. It turns out to be the linear subspace $\mathcal{F} = \{(x,\theta) : \mathcal{Y}_0(x,\theta) < \infty\}$, and in Theorem 18 we state that detectability according to Definition 1 is equivalent to requiring that

(A1) the autonomous system is stable in $\mathcal{F}$,

(A2) the state x converges to $\mathcal{F}$ provided that both y and u converge.

Notice that condition (A1) accounts for the autonomous version of system Ψ only, and it is consistent with the notion of detectability for finite dimensional linear deterministic systems. Together with condition (A2) for system Ψ (not only the autonomous version), they build the essentials to complete the aforementioned mechanism yielding (iii). Due to (A2), a complete counterpart for property (i) is not viable in the present setup (see Example 3 in connection), and any attempt to enlarge $\mathcal{F}$ is worthless, as we show in Lemma 17.

Section 5 addresses (II), where we show that detectability according to Definition 1 generalizes uniform observability as in [1, 3, 5, 12], which, by its turn, generalizes previous observability concepts for MJLS, like those in [11]. We also show that an earlier ℓ_2-detectability concept in [7] is stricter than detectability. Moreover, we introduce a notion of uniform observability in the invariant space $\mathcal{F}$ that serves as a sufficient condition for (A2). See Proposition 28 for a summary.

Regarding (III), in section 6 we show that $\mathcal{F}^\perp$ is uniformly observable in the finite Markov chain case, which renders condition (A2) always true. Thus, we have that detectability is equivalent to (A1) in the finite case, allowing us to show that the weak detectability in [2] and the usual detectability concept in the deterministic linear case are necessary and sufficient conditions (in their particular contexts) for detectability according to Definition 1 (see Remark 3). The fact that (A2) holds true for the case in which the Markov chain is finite explains why no such condition appears in those simpler scenarios. By contrast, (A2) may fail in the infinite Markov chain case, as illustrated in Example 2.

Another important feature of the setting and results here is that, unlike previous ones, the focus is not constrained (i.e., is not *ad hoc*) to the optimal jump-linear-quadratic (JLQ) control and/or controls in linear feedback form, where detectability appears as a dual notion to stabilizability. It covers any $(\mathfrak{F}_k)$-adapted converging control that induces a finite cost $\mathcal{Y}_u$ for each initial state, assuring that it is stabilizing, and clearly encompassing the optimal solution (see Remark 1). In particular for the JLQ control, we show that the solution to the associated infinite coupled algebraic Riccati equation is unique (see section 7).

3. Preliminaries. In this section we introduce some basic machinaries, which will allow us to devise our approach toward detectability for (1). We consider the autonomous version of (1), which will be essential to relate detectability with stability and convergence of the state trajectory (see (A.1) and (A.2) in section 2). We define the various essential elements such as invariant space, some notions of convergence, some useful spaces, operators, and some preliminary results.

We consider the autonomous version of system (1):

$$\Psi_0 : \begin{cases} x_0(k+1) = A_{\theta(k)}x_0(k), & k \geq 0, \\ y_0(k) = C_{\theta(k)}x_0(k), & x_0(0) = x, \theta(0) = \theta. \end{cases}$$

Sometimes we refer to the autonomous system by the pair $(A, \mathbb{P})$ or by the triplet $(A, C, \mathbb{P})$. In addition, in what follows, for each $i \in \mathcal{Z}$, let $\mathcal{S}_i \subset \mathbb{R}^n$ stand for a vector subspace and let $\mathcal{S} = \{\mathcal{S}_i, i \in \mathcal{Z}\}$.

DEFINITION 2 (Ψ_0-invariant space). *Consider the autonomous system Ψ_0. We say that $\mathcal{S}$ is an invariant space if $x_0(k) \in \mathcal{S}_{\theta(k)}$ implies that $x_0(t) \in \mathcal{S}_{\theta(t)}$ almost surely (a.s.) for each $t \geq k$.*

DEFINITION 3 (projections onto $\mathcal{S}^\perp$). *For each $i \in \mathcal{Z}$, let $P_i \in \mathcal{R}^n$ denote the orthogonal projection onto $\mathcal{S}_i^\perp$. Clearly, $P = \{P_i, i \in \mathcal{Z}\} \in \mathcal{H}_\infty^n$.*

DEFINITION 4 (Ψ_0-convergence). *We say that $x(\cdot)$ converges (in the ℓ_2 sense) to the Ψ_0-invariant space $\mathcal{S}$ if*

$$\sum_{k=0}^{\infty} E_{x,\theta}\{|P_{\theta(k)}x(k)|^2\} < \infty.$$

We say that $x(\cdot)$ converges if it converges to the trivial Ψ_0-invariant space $\mathcal{S} = 0$.

DEFINITION 5 (ℓ_2-stability). *Consider the autonomous system Ψ_0. We say that $(A, \mathbb{P})$ is ℓ_2-stable in the invariant space $\mathcal{S}$ if $x_0(\cdot)$ converges for each initial condition $\theta \in \mathcal{Z}$ and $x \in \mathcal{S}_\theta$. We say that $(A, \mathbb{P})$ is ℓ_2-stable if it is ℓ_2-stable in $\mathcal{S}$ with $\mathcal{S}_i = \mathbb{R}^n$, $i \in \mathcal{Z}$*

Notice that $x(\cdot)$ converges if and only if $\sum_{k=0}^{\infty} E\{|x(k)|^2\} < \infty$, since $P = I$ whenever $\mathcal{S}$ is trivial. Also, ℓ_2-stability of $(A, \mathbb{P})$ is equivalent to convergence of $x_0(\cdot)$ for each initial condition $\theta \in \mathcal{Z}$ and $x \in \mathbb{R}^n$.

We will need the following property related with the concept of ℓ_2-stability in $\mathcal{S}$ and the projections P.

LEMMA 6. *Assume that $(A, \mathbb{P})$ is ℓ_2-stable in $\mathcal{S}$. Then, $(A - AP, \mathbb{P})$ is ℓ_2-stable.*

Proof. Consider the following version of system Ψ:

$$(4) \qquad x_P(k+1) = (A_{\theta(k)} - A_{\theta(k)}P_{\theta(k)})x_P(k), \quad x_P(0) = x, \ \theta(0) = \theta.$$

Let us employ the trajectory of system Ψ_0, $x_0(k) = A_{\theta(k-1)} \cdots A_{\theta(0)}x_0$, with initial condition x_0 being the projection of x into $\mathcal{S}_\theta$, i.e., $x_0 = (I - P_\theta)x$. Since $\mathcal{S}$ is a invariant space, we have that $x_0(k) \in \mathcal{S}_{\theta(k)}$, $k \geq 0$.

We start by showing inductively that $x_P(\cdot)$ evolves as an open-loop trajectory according to $x_P(k) = x_0(k)$, for $k \geq 1$, for all $x \in \mathbb{R}^n$. For $k = 1$ we have that

$$x_P(1) = (A_\theta - A_\theta P_\theta)x = A_\theta(I - P_\theta)x = x_0(1).$$

From the induction assumption, we have that $x_P(k) = x_0(k)$ for $k \geq 1$, and recalling that $x_0(k) \in \mathcal{S}_{\theta(k)}$ we evaluate

$$x_P(k + 1) = (A_{\theta(k)} - A_{\theta(k)}P_{\theta(k)})x_0(k) = A_{\theta(k)}x_0(k) = x_0(k + 1)$$

and the induction is completed. Due to the facts that (i) $(A, \mathbb{P})$ is ℓ_2-stable in $\mathcal{S}$, (ii) $x_P(1) = x_0(1) \in \mathcal{S}_{\theta(1)}$ a.s., and (iii) $x_P(k)$, $k \geq 1$ evolves as a trajectory of the autonomous system for any $x_P(0) = x$, $\theta(0) = \theta$, we have from the definition of ℓ_2-stability in $\mathcal{S}$ that $x_P(\cdot)$ converges and, thus, $(A - AP, \mathbb{P})$ is ℓ_2-stable. $\square$

In what follows, we introduce a certain space $\mathcal{H}_F^n$, an element $X(k)$ related with the second moment of the state, an operator $\mathcal{L}$ related with the evolution of $X(k)$, and some associated results which will be useful to present the results of the paper in a concise manner.

Let $\mathcal{H}_1^n$ denote the linear space formed by sequences of matrices $H = \{H_i = H_i' \geq 0; i \in \mathcal{Z}\}$ such that $\sum_{i \in \mathcal{Z}} tr\{H_i\} < \infty$. Let $\mathcal{H}_F^n \subset \mathcal{H}_1^n$ denote the closed cone formed by sequences of symmetric positive semidefinite matrices $H = \{H_i = H_i' \geq 0; i \in \mathcal{Z}\}$. For $H, V \in \mathcal{H}_F^n$ we define the inner product

$$\langle H, V \rangle = \sum_{i \in \mathcal{Z}} tr\{H_i'V_i\}$$

and the Frobenius norm

$$\|H\|_F = \langle H, I \rangle. \tag{5}$$

Recall from the definition of the Ψ_0-invariant subspace $\mathcal{S}$ that $\mathcal{S}_i = \{x : P_i x = 0\}$. In connection, we define the spaces $\bar{\mathcal{S}} = \{H \in \mathcal{H}_F^n : PHP' = 0\} \subset \mathcal{H}_F^n$ and $\bar{\mathcal{S}}^\perp = \{H \in \mathcal{H}_F^n : H - PHP' = 0\}$. PHP' is the orthogonal projection of H onto $\bar{\mathcal{S}}^\perp$; indeed, P inherits from P_i the property that $P^2 = P$, and it is easy to check that $\langle PHP', H - PHP' \rangle = \langle H, PHP' - P^2HP^{2\prime} \rangle = 0$.

DEFINITION 7 (convergence in $\mathcal{H}_F^n$). *We refer to convergence of sequences in $\mathcal{H}_F^n$ in the ℓ_1 sense: we say that a sequence $H(\cdot) \in \mathcal{H}_F^n$ converges to the space $\bar{\mathcal{S}}$ whenever $\sum_{k=0}^\infty \|PH(k)P'\|_F < \infty$; we say that $H(\cdot)$ converges if it converges to the trivial space $\bar{\mathcal{S}} = 0$.*

We define $X(\cdot) \in \mathcal{H}_F^n$ and $U(\cdot) \in \mathcal{H}_F^r$ as

$$X_i(k) = E\{x(k)x(k)'1_{\{\theta(k)=i\}}\}$$
$$U_i(k) = E\{u(k)u(k)'1_{\{\theta(k)=i\}}\} \; \forall i \in \mathcal{Z}, k \geq 0, \tag{6}$$

where $1_{\{.\}}$ is the Dirac indicator function. We write $X_0(\cdot)$ when we refer to the autonomous system. We define $\mathcal{Y}_u^{t,T}$ similarly to the functional $\mathcal{Y}$ in (2) as follows:

$$\mathcal{Y}_u^{t,T}(x, \theta) = E_{x,\theta}\left\{\sum_{k=t}^{t+T-1} |y(k)|^2\right\} = \sum_{k=t}^{t+T-1} \left(\langle X(k), C'C \rangle + \langle U(k), D'D \rangle\right) \tag{7}$$

whenever $x(0) = x, \theta(0) = \theta$; for simplicity we write $\mathcal{Y}_u^{t=0,T}(x, \theta) = \mathcal{Y}_u^T(x, \theta)$ and also $\mathcal{Y}_{u\equiv0}^T(x, \theta) = \mathcal{Y}_0^T(x, \theta)$.

Using the notation above we can write $E_{x,\theta}\{|x(k)|^2\} = \|X(k)\|_F$ and this provides a connection between convergence in the ℓ_1 sense of $X(\cdot) \in \mathcal{H}_F^n$ with the ℓ_2 convergence of $x(\cdot)$. A further connection is presented in the next lemma; the proof is presented in Appendix 9.

LEMMA 8. $x(\cdot)$ converges to $\mathcal{S}$ if and only if $X(\cdot)$ converges to $\bar{\mathcal{S}}$.

Now, let us define for $V \in \mathcal{H}_\infty^{n,r}$ the linear operator $\mathcal{L}_V : \mathcal{H}_F^r \to \mathcal{H}_F^n$

$$(8) \qquad \mathcal{L}_{Vi}(H) = \sum_{j \in \mathcal{Z}} p_{ji} V_j H_j V_j'.$$

It is shown in [7] that the limit in (8) is well defined. We denote $\mathcal{L}^0(H) = H$, and for $k \geq 1$, we can define $\mathcal{L}^k(H)$ recursively by $\mathcal{L}^k(H) = \mathcal{L}(\mathcal{L}^{k-1}(H))$. Also, $r_\sigma(\mathcal{L})$ denotes the spectral radius of $\mathcal{L}$. Operator $\mathcal{L}$ is related to system Ψ as follows; the result is adapted from [7].

PROPOSITION 9. The following assertions hold:
(i) $X_0(k+1) = \mathcal{L}_A(X_0(k))$, $k \geq 0$;
(ii) $(A, \mathbb{P})$ is ℓ_2-stable if and only if $r_\sigma(\mathcal{L}_A) < 1$.

For the nonautonomous system Ψ, the evolution of X is still related to the operator $\mathcal{L}$, as follows. See Appendix 9 for the proof.

LEMMA 10. Let $\alpha \neq 0$. Then,

$$X(k+1) \leq (1 + \alpha^2)\mathcal{L}_A(X(k)) + (1 + 1/\alpha^2)\mathcal{L}_B(U(k)), \quad k \geq 0.$$

The following basic properties concerning the operator $\mathcal{L}$, which are easy to check by inspection, will be useful.

PROPOSITION 11. The following properties hold, for $V, W \in \mathcal{H}_\infty^n$ and $H, Y \in \mathcal{H}_F^n$:
(i) $\mathcal{L}_{VW}(H) = \mathcal{L}_V(WHW')$;
(ii) $\mathcal{L}_{V+W}(H) \geq (1 - \alpha^2)\mathcal{L}_V(H) + (1 - 1/\alpha^2)\mathcal{L}_W(H) \ \forall \alpha \neq 0$;
(iii) $\mathcal{L}_{V+W}(H) \leq (1 + \alpha^2)\mathcal{L}_V(H) + (1 + 1/\alpha^2)\mathcal{L}_W(H) \ \forall \alpha \neq 0$;
(iv) $\mathcal{L}_V(H) \geq \mathcal{L}_V(Y)$ whenever $H \geq Y$;
(v) $\|\mathcal{L}_V(H)\|_F \leq \|V\|_\infty^2 \|H\|_F$.

We finish the section with the following facts that we believe are worth mentioning. $\bar{\mathcal{S}}$ inherits from $\mathcal{S}$ the property that it is a Ψ_0-invariant subspace, that is, $PX_0(k)P' = 0$, $k \geq 0$, implies that $PX_0(t)P' = 0$, $t \geq k$. The notion of convergence in $\mathcal{H}_F^n$ is usual, in the sense that a sequence $H(\cdot) \in \mathcal{H}_F^n$ converges to the space $\bar{\mathcal{S}}$ if and only if $\sum_{k=0}^\infty \inf_{V \in \bar{\mathcal{S}}} \|H(k) - V\|_F < \infty$. Actually, the proof follows immediately from the fact that for each $H(k)$ there exists $V \in \bar{\mathcal{S}}$ for which $\|H(k) - V\|_F = \|PH(k)P'\|_F$ (indeed, $V = H(k) - PH(k)P'$).

4. A necessary and sufficient condition for detectability. We show in section 4.1 that a general state trajectory $x(\cdot)$ converges if and only if there exists an invariant space $\mathcal{S}$ for which: (i) $(A, \mathbb{P})$ is ℓ_2-stable in $\mathcal{S}$ and (ii) $x(\cdot)$ converges to $\mathcal{S}$. In section 4.2 we introduce the Ψ_0-invariant space $\mathcal{F}$ and we show the appropriateness of $\mathcal{F}$ to formulate the equivalence between detectability and conditions (A1) and (A2).

4.1. Conditions for state convergence. In this section we examine the state convergence of system Ψ and its interplay with the Ψ_0-invariant subspace $\mathcal{S}$. The first requirement is the existence of a Ψ_0-invariant space $\mathcal{S}$ in which $(A, \mathbb{P})$ is ℓ_2-stable, namely, the condition (A1) holds.

Notice that (A1) does not impose any condition on the system Ψ (or, even on Ψ_0) in the subspaces orthogonal to $\mathcal{S}$. From this perspective, one can infer that (A1) can be employed only to establish the convergence of Ψ_0-trajectories that remain a.s. in $\mathcal{S}$ or, at most, of Ψ_0-trajectories that converge to $\mathcal{S}$. Surprisingly, the combination of (A1) with convergence to $\mathcal{S}$ of Ψ-trajectories guarantee the convergence of any trajectory of Ψ with converging inputs, as the next theorem shows. Clearly, if $\mathcal{S}$ is trivial, Theorem 12 becomes a tautology.

THEOREM 12. *Consider system Ψ and assume that the input converges. The state $x(\cdot)$ converges if and only if there exists an invariant space $\mathcal{S}$ such that the following conditions hold:*

(i) $(A, \mathbb{P})$ *is ℓ_2-stable in $\mathcal{S}$;*

(ii) $x(\cdot)$ *converges to $\mathcal{S}$.*

Proof. (Necessity.) Since $x(\cdot)$ converges to the origin, $\mathcal{S} = 0$ trivially satisfies (i) and (ii).

(Sufficiency.) We show that $x(\cdot)$ converges provided (i) and (ii) hold and $u(\cdot)$ converges. Recall from Lemma 6 that condition (i) provides that $(A - AP, \mathbb{P})$ is ℓ_2-stable for P the projection in $\mathcal{S}$ and from Proposition 9 (ii) we have that $r_\sigma(\mathcal{L}_{A-AP}) < 1$. Let $\alpha \neq 0$ be such that $(1 + \alpha^2)^2 r_\sigma(\mathcal{L}_{A-AP}) < 1$. For ease of notation, we define the operators $\hat{\mathcal{L}}, \tilde{\mathcal{L}}, \bar{\mathcal{L}} : \mathcal{H}_F^n \to \mathcal{H}_F^n$ as

$$\hat{\mathcal{L}}(H) = (1 + \alpha^2)^2 \mathcal{L}_{A-AP}(H),$$
$$\tilde{\mathcal{L}}(H) = (1 + \alpha^2)(1 + 1/\alpha^2)\mathcal{L}_{AP}(H), \text{ and}$$
$$\bar{\mathcal{L}}(H) = (1 + 1/\alpha^2)\mathcal{L}_B(H)$$

for $H \in \mathcal{H}_F^n$. We also define the series $Z(\cdot)$ with $Z(k) \in \mathcal{H}_F^n$, $k \geq 0$, by

$$\begin{cases} Z(k + 1) = \hat{\mathcal{L}}(Z(k)) + \tilde{\mathcal{L}}(X(k)) + \bar{\mathcal{L}}(U(k)), \ k \geq 0, \\ Z(0) = X(0). \end{cases}$$

Noticing that

$$Z(m) = \hat{\mathcal{L}}^m(X(0)) + \sum_{k=0}^{m-1} \hat{\mathcal{L}}^k \Big(\tilde{\mathcal{L}}(X(-k + m - 1)) + \bar{\mathcal{L}}(U(-k + m - 1)) \Big),$$

we write

$$(9) \quad \begin{aligned} \sum_{m=0}^{\infty} \langle Z(m), I \rangle &= \sum_{m=0}^{\infty} \langle \hat{\mathcal{L}}^m(X(0)), I \rangle \\ &+ \sum_{m=0}^{\infty} \sum_{k=0}^{m-1} \Big\langle \hat{\mathcal{L}}^k \Big(\tilde{\mathcal{L}}(X(-k + m - 1)) + \bar{\mathcal{L}}(U(-k + m - 1)) \Big), I \Big\rangle. \end{aligned}$$

For the second term in the right-hand side of (9) we evaluate

$$(10) \quad \sum_{m=0}^{\infty} \sum_{k=0}^{m-1} \left\langle \hat{\mathcal{L}}^k \Big(\tilde{\mathcal{L}}(X(m-k-1)) + \bar{\mathcal{L}}(U(m-k-1)) \Big), I \right\rangle$$

$$= \sum_{k=0}^{\infty} \sum_{m=k+1}^{\infty} \langle \hat{\mathcal{L}}^k \Big(\tilde{\mathcal{L}}(X(m-k-1)) + \bar{\mathcal{L}}(U(m-k-1)) \Big), I \rangle$$

$$= \sum_{k=0}^{\infty} \left\langle \hat{\mathcal{L}}^k \left(\sum_{m=0}^{\infty} \tilde{\mathcal{L}}(X(m)) + \bar{\mathcal{L}}(U(m)) \right), I \right\rangle$$

$$= \sum_{k=0}^{\infty} \langle \hat{\mathcal{L}}^k(\Upsilon), I \rangle,$$

where we set

$$\Upsilon = \sum_{m=0}^{\infty} \tilde{\mathcal{L}}(X(m)) + \hat{\mathcal{L}}(U(m)) = (1+\alpha^2)(1+1/\alpha^2) \sum_{m=0}^{\infty} \mathcal{L}_{AP}(X(m))$$

$$+ (1+1/\alpha^2) \sum_{m=0}^{\infty} \mathcal{L}_B(U(m)).$$

We need to show that Υ is well defined, i.e., that $\Upsilon \in \mathcal{H}_F^n$; the result is presented in the next lemma.

LEMMA 13. $\|\sum_{k=0}^{\infty} \mathcal{L}_B(U(k))\|_F < \infty$ *if* $u(\cdot)$ *converges.*
$\|\sum_{k=0}^{\infty} \mathcal{L}_{AP}(X(k))\|_F < \infty$ *if condition* (ii) *in Theorem* 12 *holds.*
Proof. From Proposition 11 (v), we obtain

$$(11) \quad \left\| \sum_{k=0}^{\infty} \mathcal{L}_B(U(k)) \right\|_F \le \sum_{k=0}^{\infty} \|\mathcal{L}_B(U(k))\|_F \le \|B\|_\infty^2 \sum_{k=0}^{\infty} \|U(k)\|_F < \infty.$$

For the second assertion, we employ Proposition 11 (i), (v), to evaluate

$$(12) \quad \left\| \sum_{k=0}^{\infty} \mathcal{L}_{AP}(X(k)) \right\|_F \le \sum_{k=0}^{\infty} \|\mathcal{L}_{AP}(X(k))\|_F = \sum_{k=0}^{\infty} \|\mathcal{L}_A(PX(k)P')\|_F$$

$$\le \|A\|_\infty \sum_{k=0}^{\infty} \|PX(k)P'\|_F.$$

Since from the assumption $x(\cdot)$ converges to $\mathcal{S}$, we have that $\sum_{k=0}^{\infty} \|PX(k)P'\|_F < \infty$ from Lemma 8 and the result follows. $\square$

Proof of sufficiency of Theorem 12 continued. Lemma 13 provides, under the assumptions of the theorem, that Υ is well defined, and from (9) and (10) we obtain

$$(13) \quad \sum_{k=0}^{\infty} \|Z(k)\|_F = \sum_{k=0}^{\infty} \langle \hat{\mathcal{L}}^k(X(0) + \Upsilon), I \rangle,$$

and recalling that $r_\sigma(\hat{\mathcal{L}}) < 1$, we have that

$$(14) \quad \sum_{k=0}^{\infty} \|Z(k)\|_F < \infty.$$

Now we show by induction that

$$(15) \qquad\qquad X(k) \leq Z(k) \ \forall k \geq 0.$$

For $k = 0$ we defined $Z(0) = X(0)$; assuming $X(k) \leq Z(k)$ one can check employing Lemma 10 and Proposition 11 (iii), (iv), that

$$
\begin{aligned}
X(k+1) &\leq (1+\alpha^2)\mathcal{L}_{A+AP-AP}(X(k)) + (1+1/\alpha^2)\mathcal{L}_B(U(k)) \\
&\leq (1+\alpha^2)^2\mathcal{L}_{A-AP}(X(k)) + (1+\alpha^2)(1+1/\alpha^2)\mathcal{L}_{AP}(X(k)) \\
&\qquad + (1+1/\alpha^2)\mathcal{L}_B(U(k)) \\
&= \hat{\mathcal{L}}(X(k)) + \tilde{\mathcal{L}}(X(k)) + \bar{\mathcal{L}}(U(k)) \\
&\leq \hat{\mathcal{L}}(Z(k)) + \tilde{\mathcal{L}}(X(k)) + \bar{\mathcal{L}}(U(k)) = Z(k+1)
\end{aligned}
$$

and the induction is complete. From (15) we obtain that for each $k \geq 0$, $Z(k) - X(k) \geq 0$ in such a manner that $Z(k) - X(k) \in \mathcal{H}_F^n$ and $\langle Z(k) - X(k), I \rangle \geq 0$. This leads to $\|Z(k)\|_F \geq \|X(k)\|_F$ and from (14) we obtain

$$\sum_{k=0}^{\infty} \|X(k)\|_F \leq \sum_{k=0}^{\infty} \|Z(k)\|_F < \infty,$$

and Lemma 8 with trivial $\mathcal{S} = 0$ provides that $x(\cdot)$ converges. $\square$

4.2. The main result. The first result of this section follows in a straightforward manner from Theorem 12 and the definition of detectability. We omit the proof.

LEMMA 14. *System Ψ is detectable if and only if there exists an invariant space $\mathcal{S}$ such that the following conditions hold:*

(i) *$(A, \mathbb{P})$ is ℓ_2-stable in $\mathcal{S}$;*

(ii) *$x(\cdot)$ converges to $\mathcal{S}$ provided that $y(\cdot)$ and $u(\cdot)$ converge.*

Notice that, for $\mathcal{S}$ trivial, Lemma 14 becomes a tautology; indeed, item (i) holds trivially and item (ii) reduces to the definition of detectability. The larger the invariant space $\mathcal{S}$ is, the more significant the result will be. Along this line, in this section we introduce the set $\mathcal{F} = \{\mathcal{F}_i, i \in \mathcal{Z}\}$ as

$$(16) \qquad\qquad \mathcal{F}_i = \{x \in \mathbb{R}^n : \mathcal{Y}_0(x, i) < \infty\} \quad \forall i \in \mathcal{Z}$$

and we show that $\mathcal{F}$ is the largest of such Ψ_0-invariant space.

The first step is to show that $\mathcal{F}$ is indeed a Ψ_0-invariant space. We need the following preliminary result, adapted from [7].

PROPOSITION 15. *For each $t \geq 0$, there exists $H \in \mathcal{H}_F^n$ such that $\mathcal{Y}_0^t(x, i) = x'H_i x$.*

LEMMA 16. *$\mathcal{F}$ is a Ψ_0-invariant space.*

Proof. ($\mathcal{F}_i$ is a vector subspace.) For $x_1, x_2 \in \mathcal{F}_i$ and $\alpha, \beta \in \mathbb{R}$, from Proposition 15 it is simple to check that

$$
\begin{aligned}
\mathcal{Y}_0^T(\alpha x_1 + \beta x_2, i) &= (\alpha x_1 + \beta x_2)'H_i(\alpha x_1 + \beta x_2) \\
&\leq 2\alpha^2 x_1'H_i x_1 + 2\beta^2 x_2'H_i x_2 \\
&= 2\alpha^2 \mathcal{Y}_0^T(x_1, i) + 2\beta^2 \mathcal{Y}_0^T(x_1, i) \quad \forall T \geq 0.
\end{aligned}
$$

Taking limits, we obtain

$$\mathcal{Y}_0(\alpha x_1 + \beta x_2, i) \leq 2\alpha^2 \mathcal{Y}_0(x_2, i) + 2\beta^2 \mathcal{Y}_0(x_2, i) < \infty$$

in such a manner that $\alpha x_1 + \beta x_2 \in \mathcal{F}_i$.

($\mathcal{F}$ is invariant.) We set $x_0(k) \in \mathcal{F}_{\theta(k)}$ and assume without loss that $k = 0$. Let us deny the assertion and assume that $x_0(s) \notin \mathcal{F}_{\theta(s)}$ for some $s > 0$, with probability (w.p.) $\epsilon > 0$. In this situation, it is simple to check that

$$(17) \qquad \forall \gamma > 0, \exists t_\gamma > 0 : \sum_{k=s}^{s+t_\gamma} \|y_0(k)\|^2 \geq \gamma \quad \text{w.p. } \epsilon.$$

Employing the Tchebychev inequality, we evaluate

$$(18) \qquad E\left\{ \sum_{k=s}^{s+t_\gamma} \|y_0(k)\|^2 \right\} \geq \gamma P\left\{ \sum_{k=s}^{s+t_\gamma} \|y_0(k)\|^2 > \gamma \right\} = \gamma\epsilon,$$

and we conclude that $\mathcal{Y}_0(x(0), \theta(0)) \geq \gamma\epsilon$ for all $\gamma > 0$, which is a contradiction in view of the fact that $x_0(0) \in \mathcal{F}_{\theta(0)}$. $\quad\square$

Next we show that $\mathcal{F}$ is the largest Ψ_0-invariant space that possibly meets the condition (i) in Lemma 14.

LEMMA 17. *If $\mathcal{S}$ is such that $(A, \mathbb{P})$ is ℓ_2-stable in $\mathcal{S}$, then $\mathcal{S} \subset \mathcal{F}$.*

Proof. Let us deny the assertion of the lemma and assume that there exists $i \in \mathcal{Z}$ for which $\mathcal{F}_i \subset \mathcal{S}_i$ strictly. We have that there exists $x \in \mathcal{S}_i$ with $x \notin \mathcal{F}_i$ and from the definition of $\mathcal{F}$ we conclude that $\mathcal{Y}_0(x, i) = \infty$, which provides that the associated output does not converge. Then, $(A, \mathbb{P})$ is not ℓ_2-stable in $\mathcal{S}$. $\quad\square$

Lemmas 14 and 17 allow us to derive the main result of the paper.

THEOREM 18. *System Ψ is detectable if and only if the following conditions hold:*
(A1) *$(A, \mathbb{P})$ is ℓ_2-stable in $\mathcal{F}$;*
(A2) *$x(\cdot)$ converges to $\mathcal{F}$ provided $y(\cdot)$ and $u(\cdot)$ converge.*

Proof. (Sufficiency.) (A1) and (A2) satisfy the conditions for detectability in Lemma 14.

(Necessity.) Since $(A, C, \mathbb{P})$ is detectable, from Lemma 14 we have that there exists $\mathcal{S}$ for which $(A, \mathbb{P})$ is ℓ_2-stable in $\mathcal{S}$ and Lemma 17 provides that $\mathcal{S} \subset \mathcal{F}$. Lemma 14 also yields that $x(\cdot)$ converges to $\mathcal{S}$ provided $y(\cdot)$ and $u(\cdot)$ converges; this fact together with the fact that $\mathcal{S} \subset \mathcal{F}$ lead immediately to (A2).

Now, notice from the concept of detectability that, in particular for the autonomous system Ψ_0, $x_0(\cdot)$ converges whenever the corresponding output $y(\cdot)$ converges or, equivalently, whenever $x(0) \in \mathcal{F}_{\theta(0)}$. This means that $(A, \mathbb{P})$ is ℓ_2-stable in $\mathcal{F}$ and (A1) holds. $\quad\square$

5. Sufficient conditions for (A1) and (A2). In this section we deal with other detectability and observability concepts that appear in the literature of MJLS and we present the role that they play as sufficient conditions (expressed entirely in terms of the autonomous version of the system) for (A1) and (A2), and therefore for the detectability concept here.

Initially, we introduce a concept of uniform observability related to the Ψ_0-invariant space $\mathcal{S}$. From (7), recall that we set $\mathcal{Y}_0^T(x, \theta) = E\{\sum_{k=0}^{T-1} |y_0(k)|^2\}$, where $y_0(\cdot)$ denotes the output trajectory of the autonomous system Ψ_0 with $x_0(0) = x$.

DEFINITION 19 (uniform observability w.r.t. $\mathcal{S}$). *Consider the autonomous system Ψ_0. We say that $(A, C, \mathbb{P})$ is uniformly observable with respect to (w.r.t.) $\mathcal{S}$ if*

there exists $T, \epsilon > 0$ such that $\mathcal{Y}_0^T(x, \theta) \geq \epsilon \|x\|^2$ whenever $x \in \mathcal{S}_\theta^\perp$. We say that $(A, C, \mathbb{P})$ is uniformly observable if it is uniformly observable w.r.t. 0.

A particular case of this concept with trivial $\mathcal{S}$ appears in [1, 3, 5, 12], and it generalizes previous observability concepts for MJLS, like the ones appearing in [11].

In Lemma 22 in what follows, we show that uniform observability is a sufficient condition for the state convergence to $\mathcal{S}$ and, in particular, for (A2) to hold when $\mathcal{S} \subset \mathcal{F}$. For the proof, we need the next two lemmas; their proofs are presented in Appendix 9. Recall that P_i denotes the orthogonal projection onto $\mathcal{S}_i^\perp$, $i \in \mathcal{Z}$.

LEMMA 20. *If $(A, C, \mathbb{P})$ is uniformly observable w.r.t. $\mathcal{S}$, then there exist $T, \epsilon > 0$ such that $\mathcal{Y}_0^T(x, \theta) \geq \epsilon |P_\theta x|^2$ for each $x \in \mathbb{R}^n$ and $\theta \in \mathcal{Z}$.*

LEMMA 21. *There exist $\delta_1, \delta_2 > 0$ for which*

$$\mathcal{Y}_u^{t,T}(x, \theta) \geq \delta_1 E\{\mathcal{Y}_0^T(x(t), \theta(t))\} - \delta_2 \sum_{k=t}^{T+t-1} E\{|u(k)|^2\} \; \forall x \in \mathbb{R}^n \text{ and } \theta \in \mathcal{Z}.$$

LEMMA 22. *If $(A, C, \mathbb{P})$ is uniformly observable w.r.t. $\mathcal{S}$, then $x(\cdot)$ converges to $\mathcal{S}$ provided that $y(\cdot)$ and $u(\cdot)$ converge. In addition, if $\mathcal{S} \subset \mathcal{F}$, then (A2) holds.*

Proof. Provided that $u(\cdot)$ and the output $y(\cdot)$ converge, i.e., $E[\sum_{k=0}^\infty \|u(k)\|^2] < \infty$ and $\mathcal{Y}_u(x, \theta) < \infty$, respectively, we show that $(A, C, \mathbb{P})$ uniformly observable w.r.t. $\mathcal{S}$ suffices for convergence of the state to $\mathcal{S}$, namely, $E[\sum_{k=0}^\infty \|P_{\theta(k)} x(k)\|^2] < \infty$. For each $t \geq 0$, we employ Lemmas 20 and 21 to evaluate

$$(19) \quad \mathcal{Y}_u(x, \theta) \geq \mathcal{Y}_u^{t,\infty}(x, \theta) = \sum_{k=0}^\infty \mathcal{Y}_u^{t+kT,T}(x, \theta)$$

$$\geq \sum_{k=0}^\infty \left(\delta_1 E\{\mathcal{Y}_0^T(x(t+kT), \theta(t+kT))\} - \delta_2 \sum_{\ell=t+kT}^{t+(k+1)T-1} E\{|u(k)|^2\} \right)$$

$$\geq \delta_1 \epsilon \sum_{k=0}^\infty E\{|P_{\theta(t+kT)} x(t+kT)|^2\} - \delta_2 \sum_{k=0}^\infty E\{|u(k)|^2\}.$$

Summing (19) for $t = 0, \dots, T-1$, we obtain

$$T \mathcal{Y}_u(x, \theta) \geq \delta_1 \epsilon \sum_{m=0}^\infty E\{|P_{\theta(m)} x(m)|^2\} - T \delta_2 \sum_{k=0}^\infty E\{|u(k)|^2\},$$

which leads to

$$\sum_{m=0}^\infty E\{|P_{\theta(m)} x(m)|^2\} \leq \frac{T}{\delta_1 \epsilon} \mathcal{Y}_u(x, \theta) + \frac{T \delta_2}{\delta_1 \epsilon} \sum_{k=0}^\infty E\{|u(k)|^2\} < \infty$$

and the first assertion is proven.

Now, from Definition 19 we obtain that if $(A, C, \mathbb{P})$ is uniformly observable w.r.t. $\mathcal{S} \subset \mathcal{F}$, then it is uniformly observable w.r.t. $\mathcal{F}$. The result then follows immediately from the first assertion. $\square$

Notice from Lemma 22 with trivial $\mathcal{S}$ that, if $(A, C, \mathbb{P})$ is uniformly observable, then $x(\cdot)$ converges to $\mathcal{S} = 0$, provided $y(\cdot)$ and $u(\cdot)$ converges, i.e., $(A, C, \mathbb{P})$ is detectable, which implies the following corollary.

COROLLARY 23. *If $(A, C, \mathbb{P})$ is uniformly observable, then Ψ is detectable.*

Next, we are concerned with an earlier ℓ_2-detectability sense; see [7] in a setting similar to the one of this paper, or [8] in the continuous time case, or [6] and [12] in the finite dimensional case.

DEFINITION 24 (ℓ_2-detectability). *Consider the autonomous system Ψ_0. We say that $(A, C, \mathbb{P})$ is ℓ_2-detectable if there exists $L \in \mathcal{H}_\infty^{q,n}$ for which $(A + LC, \mathbb{P})$ is ℓ_2-stable.*

LEMMA 25. *If $(A, C, \mathbb{P})$ is ℓ_2-detectable, then system Ψ is detectable.*

Proof. We assume that $(A, C, \mathbb{P})$ is ℓ_2-detectable and $y(\cdot)$ and $u(\cdot)$ converge, and we show that $x(\cdot)$ converges in a similar manner to the proof of sufficiency of Theorem 12. Here we only point out the differences. For $L \in \mathcal{H}_\infty^{q,n}$ as in the ℓ_2-detectability definition, $r_\sigma(\mathcal{L}_{A+LC}) < 1$, see Proposition 9 (ii) in connection. We chose $\alpha \neq 0$ in such a way that $(1 + \alpha^2)^2 r_\sigma(\mathcal{L}_{A+LC}) < 1$ holds. The operators $\hat{\mathcal{L}}, \tilde{\mathcal{L}}, \bar{\mathcal{L}} : \mathcal{H}_F^n \to \mathcal{H}_F^n$ are defined as

$$\hat{\mathcal{L}}(H) = (1 + \alpha^2)^2 \mathcal{L}_{A+LC}(H),$$
$$\tilde{\mathcal{L}}(H) = (1 + \alpha^2)(1 + 1/\alpha^2) \mathcal{L}_{LC}(H) \text{ and}$$
$$\bar{\mathcal{L}}(H) = (1 + 1/\alpha^2) \mathcal{L}_B(H) \quad \text{for } H \in \mathcal{H}_F^n.$$

In parallel with Lemma 13, we also need to show that $\sum_{k=0}^\infty \mathcal{L}_{LC}(X(k)) < \infty$. In fact, since $y(\cdot)$ converges for the autonomous system $(A+LC, \mathbb{P})$, we get that $\mathcal{Y}_0(x, \theta) < \infty$ and from (7) we evaluate

$$\infty > \mathcal{Y}_0(x, \theta) \geq \sum_{k=0}^\infty \langle X(k), C'C \rangle = \sum_{k=0}^\infty \|C'X(k)C\|_F$$

and employ Proposition 11 (i), (v), to obtain

$$(20) \quad \left\|\sum_{k=0}^\infty \mathcal{L}_{LC}(X(k))\right\|_F \leq \sum_{k=0}^\infty \|\mathcal{L}_{LC}(X(k))\|_F = \sum_{k=0}^\infty \|\mathcal{L}_L(CX(k)C')\|_F$$

$$\leq \|L\|_\infty \sum_{k=0}^\infty \|CX(k)C'\|_F < \infty. \quad \square$$

Remark 1. In [7] it is shown that ℓ_2-detectability (together with ℓ_2-stabilizability) ensures that the optimal linear state feedback control that arises in the JLQ problem is ℓ_2-stabilizing, considering an additional assumption on matrices D_i as in Lemma 35. Lemma 25 generalizes this result in the sense that $x(\cdot)$ converges provided that the output and input converge; here, neither optimality nor linear state feedback is required.

The next concept is named W_S-detectability, and it was introduced in [5] as an attempt to deal with detectability in the present context. It can be seen as a particularization of detectability with $u \equiv 0$.
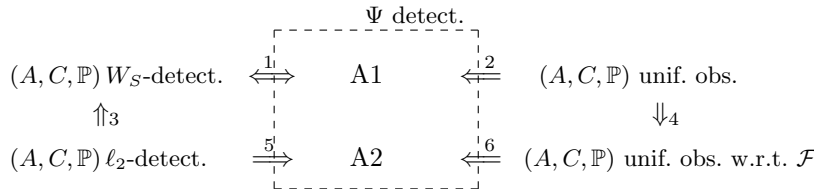
DEFINITION 26 (W_S-detectability). *Consider the autonomous system Ψ_0. We say that $(A, C, \mathbb{P})$ is W_S-detectable provided that $x_0(\cdot)$ converges whenever the output $y_0(\cdot)$ converges.*

It follows directly from the definitions that the concept is equivalent to ℓ_2-stability in $\mathcal{F}$.

PROPOSITION 27. *$(A, C, \mathbb{P})$ is W_S-detectable if and only if $(A, \mathbb{P})$ is ℓ_2-stable in $\mathcal{F}$ ((A1) holds).*

We finish with a summary of the main results of this section. For ease of reference, the relations are numbered. The relation (1) follows from Proposition 27, (2) follows from Corollary 23 and Theorem 18, (3) and (5) follows from Lemma 25, (4) follows immediately from definition, and (6) follows from Lemma 22.

PROPOSITION 28. *The following relations hold:*

$$
\begin{array}{ccccc}
& & \Psi \text{ detect.} & & \\
(A,C,\mathbb{P})\,W_S\text{-detect.} & \overset{1}{\Longleftrightarrow} & \text{A1} & \overset{2}{\Longleftarrow} & (A,C,\mathbb{P}) \text{ unif. obs.} \\
\Uparrow 3 & & & & \Downarrow 4 \\
(A,C,\mathbb{P})\,\ell_2\text{-detect.} & \overset{5}{\Longrightarrow} & \text{A2} & \overset{6}{\Longleftarrow} & (A,C,\mathbb{P}) \text{ unif. obs. w.r.t. } \mathcal{F}
\end{array}
$$

Remark 2. In principle the detectability concept depends on the nonautonomous system (made explicit by assumption (A2)). However, for systems that are ℓ_2-detectable, uniformly observable, or uniformly observable w.r.t. $\mathcal{F}$, (A2) always holds true, as indicated in Proposition 28. In section 6 we show that this is also the situation for MJLS with finite Markov state.

6. Finite MJLS. Recall from the main result of the paper, Theorem 18, that the system is detectable if and only if (A1) and (A2) hold. In this section, we show that (A2) is made redundant when the Markov state space is finite, $\mathcal{Z} = \{1, \ldots, N\}$. This leads to the main result of the section: (A1) is a necessary and sufficient condition for detectability, in parallel with detectability notions for linear deterministic systems and previous concepts for MJLS [2]. The result here also generalizes previous results in the literature, which require that the control is in the linear state feedback form.

We start showing that uniform observability w.r.t. $\mathcal{S}$ always holds with $\mathcal{S} = \mathcal{N}$, where the set $\mathcal{N} = \{\mathcal{N}_i, i \in \mathcal{Z}\}$ is defined as

$$
(21) \qquad \mathcal{N}_i = \{x \in \mathbb{R}^n : \mathcal{Y}_0(x, i) = 0\}, \quad i \in \mathcal{Z}.
$$

Notice by inspection of (16) and (21) that $x \in \mathcal{F}_\rangle$ whenever $x \in \mathcal{N}_\rangle$, thus yielding that $\mathcal{N} \subset \mathcal{F}$. One can also check that $\mathcal{N}$ is an invariant space, in a similar manner to the proof of Lemma 16. We state this property formally.

PROPOSITION 29. $\mathcal{N}$ *is an invariant space,* $\mathcal{N} \subset \mathcal{F}$.

The preliminary results of Proposition 30 and Lemma 31 in what follows will be needed. First, let us generalize the definition of the cost functional $\mathcal{Y}_0$, as follows. Suppose that the initial conditions (x, θ) are random variables with x a second order random variable. In this situation, we set $X(0) \in \mathcal{H}_F^n$ such that $X_i(0) = E\{xx'1_{\{\theta=i\}}\}$. Conversely, given any $X \in \mathcal{H}_F^n$, there exists a second order r.v. x and some distribution for θ in such a way that we can represent $X_i = E\{xx'1_{\{\theta=i\}}\}$. These considerations allow us to generalize the definition of $\mathcal{Y}_0$ by writing, for each $X \in \mathcal{H}_F^n$,

$$
(22) \qquad \mathcal{Y}_0^T(X) = \sum_{k=0}^{T-1} \langle X_0(k), C'C \rangle
$$

whenever $X(0) = X$. Notice that $\mathcal{Y}_0^T(x, \theta) = \mathcal{Y}_0^T(X)$ whenever X is defined as above with $X_\theta = xx'$ and $X_i = 0$, $i \neq \theta$.

The next preliminary result is adapted from [3, Prop. 1].

PROPOSITION 30. *Assume that* $\mathcal{Z} = \{1, \ldots, N\}$. *If* $\mathcal{Y}_0^{n^2 N}(X) = 0$, *then* $\mathcal{Y}_0^t(X) = 0$ *for all* $t \geq 0$.

LEMMA 31. *Let P be the projection onto $\mathcal{N}^\perp$. The following assertions hold:*

(i) $\mathcal{Y}_0(X) = 0$ *if and only if* $P'XP = 0$;

(ii) $(A, C, \mathbb{P})$ *is uniformly observable w.r.t.* $\mathcal{N}$ *if there exists* $\epsilon > 0$ *for which* $\mathcal{Y}_0^T(X) \geq \epsilon \|X\|_F$ *whenever* $X - PXP' = 0$.

For the proof of Lemma 31, see Appendix 9.

LEMMA 32. *Assume that* $\mathcal{Z} = \{1, \ldots, N\}$. *Then,* $(A, C, \mathbb{P})$ *is uniformly observable w.r.t.* $\mathcal{N}$.

Proof. Let P be the projection onto $\mathcal{N}^{\perp}$ and recall that $\bar{\mathcal{N}} = \{H \in \mathcal{H}_F^n : PHP' = 0\}$ and $\bar{\mathcal{N}}^{\perp} = \{H \in \mathcal{H}_F^n : H - PHP' = 0\}$. Let us show that there exist $\epsilon > 0$ such that $\mathcal{Y}_0^T(X) \geq \epsilon \|X\|_F$ whenever $X \in \bar{\mathcal{N}}^{\perp}$. Let us deny this assertion and assume that there exists a sequence $X_m \in \mathcal{N}_C$, $m = 1, 2, \ldots$, for which

$$(23) \qquad \mathcal{Y}_0^{n^2 N}(X_m) \leq m^{-1},$$

where

$$\mathcal{N}_C = \{X \in \bar{\mathcal{N}}^{\perp} : \|X\|_F = 1\} \subset \bar{\mathcal{N}}^{\perp}.$$

For the countably finite case, one can check that $\mathcal{N}_C$ is a compact set and this leads to the fact that there exists a subsequence X_{m_k} that converges to

$$(24) \qquad \bar{X} \in \mathcal{N}_C \subset \bar{\mathcal{N}}^{\perp}.$$

Moreover, it is not difficult to check that $\mathcal{Y}_0^{n^2 N}(\cdot)$ is continuous; this fact and (23) allow us to write that $\mathcal{Y}_0^{n^2 N}(\bar{X}) = \lim_{k \to \infty} \mathcal{Y}_0^{n^2 N}(X_{m_k}) \leq \lim_{k \to \infty} k^{-1} = 0$. Then, Proposition 30 yields that $\mathcal{Y}_0(\bar{X}) = 0$ and from Lemma 31 (i) we conclude that $\bar{X} \in \bar{\mathcal{N}}$, which is a contradiction, in view of (24). We have shown that there exist $\epsilon > 0$ such that $\mathcal{Y}_0^T(X) \geq \epsilon \|X\|_F$ whenever $X \in \bar{\mathcal{N}}^{\perp}$; Lemma 31 (ii) completes the proof. $\square$

The result of Lemma 32 cannot be extended to the countably infinite case, as we show in the following counterexample. In connection, note that the set $\mathcal{N}_C$ in the proof of Lemma 32 is no longer compact.

Example 1. Let $n = 1$, $p_{i\,i+1} = 1$, $A_i = 1$, $C_i = r^i$, $|r| < 1$. It is simple to check that $\mathcal{Y}_0(x, i) = r^i(1 - r)|x|^2$, in such a manner that for each $\gamma > 0$ there exists i such that $\mathcal{Y}_0(x, i) < \gamma|x|^2$, which implies that $(A, C, \mathbb{P})$ is not uniformly observable w.r.t. $\mathcal{N}$.

The next result follows from Lemmas 22 and 32 and the fact that $\mathcal{N} \subset \mathcal{F}$ (see Proposition 29); the proof is omitted.

LEMMA 33. *Assume that* $\mathcal{Z} = \{1, \ldots, N\}$. *Then,* (A2) *holds.*

The next result is immediate from Lemma 33 and Theorem 18. We state the result in terms of the triplet $(A, C, \mathbb{P})$ to emphasize that the detectability concept depends only on the autonomous version Ψ_0 of the system.

THEOREM 34. *Assume that* $\mathcal{Z} = \{1, \ldots, N\}$. $(A, C, \mathbb{P})$ *is detectable if and only if* (A1) *holds.*

Remark 3. The relation between detectability and other detectability concepts for finite scenarios is discussed here. The weak detectability concept for MJLS with finite Markov state was introduced in [2]. It requires that $x(\cdot)$ converges provided $\mathcal{Y}_0(x, \theta) = 0$. In [5] it was shown that this concept is equivalent to W_S-detectability when reduced to the finite case. Assuming $\mathcal{Z} = \{1, \ldots, N\}$, Proposition 27, Theorem 34, and the aforementioned facts provide the following relations:

$$(25) \qquad \text{weak detectability} \Leftrightarrow W_S\text{-detectability} \Leftrightarrow \text{A1} \Leftrightarrow \text{detectability}.$$

For (finite dimensional) linear deterministic systems, it was shown in [2] that the weak detectability concept retrieves the usual detectability concept. Then, from the relations in (25) we conclude that, for linear deterministic systems,

$$usual\ detectability\ concept \Leftrightarrow detectability.$$

7. Detectability and the jump linear quadratic problem. In this section we are concerned with the JLQ problem, which consists of obtaining the control $u(\cdot)$ that minimizes the cost functional $\mathcal{Y}_u(x, \theta)$. We also consider the related infinite coupled algebraic Riccati equations (ICARE).

We assume here with no loss of generality that the control is in linear state feedback form, $u(k) = G_{\theta(k)}x(k)$, $G \in \mathcal{H}_\infty^{r,n}$. Indeed, it is a well-known fact that the optimal control is in this form; see, e.g., [7]. In connection, we denote $\mathcal{Y}_G(\cdot) = \mathcal{Y}_u(\cdot)$ to emphasize the dependence on G.

Also a standard assumption in the JLQ problem, that $\inf_{i \in \mathcal{Z}} \sigma^-(D_i'D_i) = \xi > 0$, is in force here. In this situation, the convergence of the input and the output are directly connected and the condition in (A2) (e.g., in Theorem 18) related to the input is not essential; the following lemma formalizes the result.

LEMMA 35. *If $\inf_{i \in \mathcal{Z}} \sigma^-(D_i'D_i) = \xi > 0$ and $\mathcal{Y}_u(x, \theta) < \infty$, then $u(\cdot)$ converges.*

Proof. Employing (7) and the assumptions in the lemma, we evaluate $\infty > \mathcal{Y}_u(x, \theta) \geq \sum_{k=0}^\infty \langle U(k), D'D \rangle \geq \xi \sum_{k=0}^\infty \|U(k)\|_F$. $\square$

The next result establishes that a linear state feedback control is stabilizing whenever the associated cost is bounded.

LEMMA 36. *Assume that $(A, C, \mathbb{P})$ is detectable. If $G \in \mathcal{H}_\infty^{r,n}$ is such that $\mathcal{Y}_G(x, \theta) < \infty$ $\forall x \in \mathbb{R}^n, \theta \in \mathcal{Z}$, then $(A + BG, \mathbb{P})$ is ℓ_2-stable.*

Proof. Consider the system Ψ in closed loop form with $u(k) = G_{\theta(k)}x(k)$,

$$(26) \qquad \begin{cases} x(k+1) = (A_{\theta(k)} + B_{\theta(k)}G_{\theta(k)})x(k), & k \geq 0, \\ y(k) = (C_{\theta(k)} + D_{\theta(k)}G_{\theta(k)})x(k). \end{cases}$$

For each initial condition $x \in \mathbb{R}^n$ and $\theta \in \mathcal{Z}$ we have from the lemma that $\mathcal{Y}_u(x, \theta) = \mathcal{Y}_G(x, \theta) < \infty$, which means that the associated output $y(\cdot)$ converges; moreover, Lemma 35 provides that $u(\cdot)$ converges. In this situation, detectability yields that $x(\cdot)$ converges, and we conclude that $(A + BG, \mathbb{P})$ is ℓ_2-stable. $\square$

In what follows, we consider the following ICARE in the unknown $R \in \mathcal{H}_F^n$ that arises in the JLQ problem (see, e.g., [7]):

$$(27) \qquad 0 = (A_i + B_iG_i)' \sum_{j \in \mathcal{Z}} p_{ij}R_j(A_i + B_iG_i) + C_i'C_i + G_i'D_i'D_iG_i,$$

$$(28) \qquad G_i = -\left(D_i'D_i + B_i' \sum_{j \in \mathcal{Z}} p_{ij}R_jB_i \right)^{-1} B_i' \sum_{j \in \mathcal{Z}} p_{ij}R_jA_i, \quad i \in \mathcal{Z}.$$

The following results are adapted from [7].

PROPOSITION 37. *Assume that $R \in \mathcal{H}_F^n$ satisfies the ICARE (27)–(28). The following assertions hold:*

(i) *$\mathcal{Y}_G(x, \theta) \leq x'R_\theta x$;*

(ii) *If $(A + BG, \mathbb{P})$ is ℓ_2-stable, then $R \in \mathcal{H}_\infty^n$ is the unique solution of the ICARE. Moreover, the solution of the JLQ problem is $u(k) = G_{\theta(k)}x(k)$, where G is given by (28).*

THEOREM 38. *Assume that $(A, C, \mathbb{P})$ is detectable according to Definition* 1. *Then, the ICARE has at most one solution. Moreover, if $R \in \mathcal{H}_F^n$ is the solution of the ICARE, then $(A + BG, \mathbb{P})$ is ℓ_2-stable with the optimal control* (28).

Proof. Let $R \in \mathcal{H}_F^n$ be a solution of the ICARE. From Proposition 37 (i) we have that $\mathcal{Y}_G(x, \theta) \le x'R_\theta x$, for each x, θ, and Lemma 36 provides that $(A + BG, \mathbb{P})$ is ℓ_2-stable. Hence, Proposition 37 (ii) yields that R is the unique solution of the ICARE and the optimal control is given by (28). $\quad\square$

Remark 4. The results in this section generalize previous result in [7] from the fact that detectability here generalizes the ℓ_2-detectability notion employed there; see Lemma 25.

8. Examples. We start this section with an example showing that (A2) does not necessarily hold for MJLS with infinite countably Markov chain. Then Example 3 shows that the detectability notion according to Definition 1 depends on the collections of matrices B and D, and thus it cannot be related to the autonomous version Ψ_0 only.

We also show, via Example 4, that the detectability concept here generalizes the earlier ℓ_2-detectability and uniform observability concepts, in the sense that the converse relations of Proposition 28 involving those concepts does not hold.

Example 2. This example illustrates that (A2) does not necessarily hold true for MJLS with infinite countably Markov chain. Indeed, we present a system for which the state trajectory does not converge to $\mathcal{F}$ under converging input and output.

Assume that $p_{i\,i+1} = 1$, $i \in \mathcal{Z}$, in such a manner that $\theta(k) = k + i$ a.s. whenever $\theta(0) = i$. Let $n = 1$, $A_i = B_i = 1$, $D_i = 0$, $i \in \mathcal{Z}$. As regards to $C \in \mathcal{H}_F^1$, we set $C_1 = 0$ and $C_i = (i-1)^{-1/2}$, $i \ge 2$, in order to get that $C_{\theta(k)} = (k+i-1)^{-1/2}$, $k \ge 1$.

It is simple to check for the autonomous system that $\mathcal{Y}_0(x, \theta) = \sum_{k=0}^{\infty} x^2/(k+i-1)$, which converges if and only if $x = 0$, thus leading to

$$\mathcal{F} = 0.$$

Now, for simplicity, we consider fixed initial conditions $x = 1$ and $\theta = 1$. Consider the control given by $u(0) = 0$ and $u(k) = (k+1)^{-1/2} - k^{-1/2}$, $k \ge 1$. We get that $x(k) = k^{-1/2}$, $k \ge 1$ is the corresponding trajectory. It is a simple matter to check that (see [16, Chap. 2.6])

$$(29) \qquad E_{x,\theta} \left\{ \sum_{k=0}^{\infty} |u(k)|^2 \right\} = \sum_{k=1}^{\infty} \frac{(k^{1/2} - (k+1)^{1/2})^2}{k(k+1)} \le \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1,$$

and we have that the input converges. As regards to the output, we first evaluate

$$E \left\{ \sum_{k=0}^{\infty} x(k)'C_{\theta(k)}'C_{\theta(k)}x(k) | x = \theta = 1 \right\} = \sum_{k=1}^{\infty} \frac{1}{k^{4 \times 1/2}} = \sum_{k=1}^{\infty} \frac{1}{k^2} \le \sum_{k=0}^{\infty} \frac{1}{2^k} \le 2,$$

where, in the last inequality, we employed the evaluation in [16, Chap. 3.1]). Together with (29), they provide that

$$\mathcal{Y}(1, 1) \le 3,$$

which means that the output converges. However, we can also write that

$$E_{x,\theta} \left\{ \sum_{k=0}^{\infty} |x(k)|^2 \right\} = \sum_{k=0}^{\infty} \frac{1}{k} = \infty,$$

and the state does not converge to the trivial $\mathcal{F}$. Thus, (A2) does not hold and the system is not detectable. It is also interesting to mention that from Proposition 29 we have that $\mathcal{N} = 0$ and, thus, the example also illustrates that the state trajectory does not converge to the nonobserved space, despite the fact that both input and output converge.

Example 3. In the system of Example 2, let $B_i = 0$, $i \in \mathcal{Z}$. In this case, $\mathcal{Y}(x, i) = \mathcal{Y}_0(x, i) = \sum_{k=0}^{\infty} x^2/(k + i - 1)$, and the output converges if and only if $x(0) = x = 0$. Then, the system is trivially detectable in the sense of Definition 1. On the other hand, recall that the system in Example 2 is not detectable. This makes clear the dependence of the detectability concept on $B \in \mathcal{H}^{n,r}$ and $D \in \mathcal{H}^{q,r}$ and not only on the parameters of the autonomous system, which shows that the class of systems studied here share with general nonlinear systems the characteristic that observability and detectability in general depends on features of the input class.

Example 4. Consider the following version of the JLQ problem of section 7:

$$(30) \qquad \min_u \mathcal{Y}_u^{k_0}(x, \theta), \quad \text{where } \mathcal{Y}_u^{k_0}(x, \theta) = E\left\{\sum_{k=0}^{k_0} |y(k)|^2\right\},$$

y is the output of system Ψ defined in (1) and k_0 is a $\mathfrak{F}_k$-stopping time defined as the time that the Markov chain $\Gamma = \{\gamma(k), k \geq 0\}$ taking values on the set $\{n, f\}$ first enters the state f, i.e., $k_0 = \inf\{k : \gamma(k) = f\}$. We assume that the transition probabilities are given by $q_{nn} \geq 0$, $0 \leq q_{nf} \leq \nu < 1$ and $q_{ff} = 1$ (f is a cemetery state).

A possible physical interpretation is that the cemetery state f represents a critical failure of the system, which forces the system to stop for maintenance at time k_0; n and f stand for *normal* and *failure,* respectively.

We start showing that the problem (30) can be cast as the JLQ problem for the MJLS defined as

(31)

$$\bar{\Psi} : \begin{cases} \bar{\theta} \text{ takes values on } \bar{\mathcal{Z}} = \{(f), (1, n), (2, n), \dots\}; \\ \bar{\mathbb{P}} : \bar{p}_{ff} = q_{ff} = 1, \ \bar{p}_{(i,n)(j,n)} = (1 - q_{nf})p_{ij}, \ \bar{p}_{(i,n)f} = q_{nf}; \\ \text{system matrices: } \bar{A}_f = 0, \ \bar{A}_{(i,n)} = A_i, \ i \neq f, \text{ and similarly for } \bar{B}, \bar{C}, \bar{D}; \\ \text{initial conditions: } \begin{array}{l} x(0) = x; \\ \bar{\theta}(0) = f \text{ if } \gamma(0) = f; \ \bar{\theta}(0) = (\theta(0), n) \text{ otherwise.} \end{array} \end{cases}$$

LEMMA 39. *The problem* (30) *is equivalent to the JLQ problem for system* $\bar{\Psi}$.

Proof. Let us define the $\mathfrak{F}_k$-stopping time $t_0 = \inf\{k : \bar{\theta}(k) = f\}$. It is a simple matter to check that the random variables t_0 and k_0 have the same statistics and similarly for the variables of system Ψ and $\bar{\Psi}$ for $k \leq t_0$ (e.g., $y(k)$ and $\bar{y}(k)$ are statistically identical for $k \leq t_0$). Let $\bar{Y}_u(\cdot)$ represent the cost functional in (2) associated with system $\bar{\Psi}$. Then, we can write

$$(32) \quad \begin{aligned} \bar{Y}_u(x, \theta) &= \sum_{k=0}^{\infty} E\{|\bar{y}(k)|^2 1_{\{k \leq t_0\}}\} + \sum_{k=0}^{\infty} E\{|\bar{y}(k)|^2 1_{\{k > t_0\}}\} \\ &= \sum_{k=0}^{\infty} E\{|\bar{y}(k)|^2 1_{\{k \leq t_0\}}\} = \sum_{k=0}^{\infty} E\{|y(k)|^2 1_{\{k \leq k_0\}}\} = \mathcal{Y}_u^{k_0}(x, \theta). \quad \square \end{aligned}$$

The next lemma establishes that system $(\bar{A}, \bar{C}, \bar{\mathbb{P}})$ is a counterexample for the converse of Corollary 23; namely, it shows that the uniformly observable systems

form a strictly subset of the set of detectable systems. The proof is presented in Appendix 9.

LEMMA 40. *If $(A, C, \mathbb{P})$ is uniformly observable, then system $\bar{\Psi}$ is detectable. $(\bar{A}, \bar{C}, \bar{\mathbb{P}})$ is not uniformly observable.*

Example 5. Let us consider systems that present Markov chains with distinct communicating classes $\mathcal{Z}_j = \{i_{n_{j-1}}, \ldots, i_{n_j}\}$, $n_0 = 1$, $j = 1, \ldots, N$, for which $P\{\theta(k+1) \in \mathcal{Z}_j | \theta(k) \in \mathcal{Z}_i\} = 0$ for all $i \neq j$. Let us denote such a system by Ψ_c; we also denote $A^j = (A_i)$, $i \in \mathcal{Z}_j$ and similarly for C^j and $\mathbb{P}^j$. We refer to the system associated with a class $\mathcal{Z}_j$ as a subsystem $(A^j, C^j, \mathbb{P}^j)$.

The following result is adapted from [5].

PROPOSITION 41. *Consider system Ψ_c. Assumption* (A1) *holds for $(A, C, \mathbb{P})$ if and only if* (A1) *holds for each subsystem $(A^j, C^j, \mathbb{P}^j)$, $j = 1, \ldots, N$.*

Let us construct in this example a system Ψ_c composed of one uniformly observable subsystem plus one finite dimensional subsystem, composed by the two classes $\mathcal{Z}_1 = \{1, 2\}$ and $\mathcal{Z}_2 = \{3, 4, \ldots\}$, and probability matrix

$$
\mathbb{P} = \begin{bmatrix}
p_{11} & (1 - p_{11}) & 0 & \ldots & \\
(1 - p_{22}) & p_{22} & 0 & \ldots & \\
0 & 0 & p_{33} & p_{34} & \ldots \\
\vdots & \vdots & p_{43} & p_{44} & \ldots \\
& & \vdots & \vdots &
\end{bmatrix},
$$

and data $A_1 = a_1$, $A_2 = a_2$, $C_1 = 1$, $C_2 = 0$, $p_{11} > 0$, and $p_{22}a_2^2 > 1$. We assume that $(A^2, C^2, \mathbb{P}^2)$ is uniformly observable.

It is shown in [1] that $(A^1, C^1, \mathbb{P}^1)$ is uniformly observable. Since uniform observability implies (A1) (see Proposition 28), we have that (A1) holds for both $(A^1, C^1, \mathbb{P}^1)$ and $(A^2, C^2, \mathbb{P}^2)$ and Proposition 41 yields that (A1) holds for the overall system Ψ_c.

Uniform observability also implies uniform observability w.r.t. $\mathcal{F}$ for each subsystem (see Proposition 28), and it is simple to check that the overall system is uniformly observable w.r.t. $\mathcal{F}$. In this situation, Proposition 28 yields that (A2) holds for the overall system.

Then, Theorem 18 implies that system Ψ_c is detectable. However, for this simple example, it was shown in [5] that the overall system is not ℓ_2-detectable. This implies that the converse of Lemma 25 does not hold, and we conclude that detectability here generalizes ℓ_2-detectability.

9. Conclusions. This paper deals with detectability for discrete-time Markov jump linear systems with countably infinite Markov state. Beginning with Definition 1, which expresses an idea that at same time is purposeful and captures the abstract notion of detectability, we show that it can be written down in terms of conditions (A1) and (A2). Condition (A1) alone refers to the autonomous systems and its behavior within the invariant space $\mathcal{F}$. It is reminiscent of detectability concepts related with finite dimensional linear systems. Condition (A2) refers to the complete system Ψ and its behavior within set $\mathcal{F}^\perp$. It comes as an essential condition, connected to the fact that the observed part of the autonomous system, represented by $\mathcal{F}^\perp$, may not be uniformly observable, contrary to the finite dimensional case. Example 2 shows that (A2) may fail in the infinite Markov state case. This clarifies that, unlike the finite dimensional contexts, the detectability notion yielding property (i) (stated in section 1) cannot be expressed in terms of the parameters of the autonomous version Ψ_0;

thus, (iii) cannot be completely reproduced. Exceptions are pointed out in Remark 2.

Regarding the issues (I)–(III), note that (I) is accomplished in the sense that we show that $\mathcal{F}$ is the largest Ψ_0 invariant space for which (A1) and (A2) together possibly hold. Along this line, a remarkable feature is that (A2) is weaker than the natural extension of the finite dimensional case, that the trajectories converge to the nonobservable space $\mathcal{N} \subset \mathcal{F}$. In addition, (II) and (III) are accomplished by showing that the notion of detectability generalizes previous notions of ℓ_2-detectability and uniform observability, as well as detectability notions for the finite Markov state case and the usual detectability concept for linear deterministic systems, in their respective scenarios. Moreover, these relations provide a generalization for earlier results concerning stability of trajectories with associated finite cost, in the sense that here we are not constrained to linear feedback form nor optimal control; see Remark 1). A particularization of the results for the JLQ optimal control problem, which was the initial motivation for this work, provides that the JLQ control is stabilizing and the solution to the associated ICARE is unique.

Finally, although the analysis here concludes a circle of ideas toward detectability of MJLS, which has began in [1, 3], we believe that the approach via invariant subspaces proposed here may be useful elsewhere, in contexts such as nonlinear systems or other infinite dimensional systems.

Appendix: Proof of Lemmas 8 and 10.

Proof of Lemma 8. It is simple to check that

$$
\begin{aligned}
E[|P_{\theta(k)}x(k)|^2] &= tr\{E[P_{\theta(k)}x(k)x(k)'P'_{\theta(k)}]\} \\
&= \sum_{i \in \mathcal{Z}} tr\{P_i E[x(k)x(k)'1_{\{\theta(k)=i\}}]P'_i\} \\
&= \sum_{i \in \mathcal{Z}} tr\{P_i X_i(k)P'_i\} = \|PX(k)P'\|_F,
\end{aligned}
$$

which provides that $\sum_{k=0}^{\infty} E[|P_{\theta(k)}x(k)|^2] < \infty$ if and only if $\sum_{k=0}^{\infty} \|PX(k)P'\|_F < \infty$. $\square$

Proof of Lemma 10. For any scalar $\alpha \neq 0$, we have that

$$
\begin{aligned}
X_i(k+1) &= E\{[A_{\theta(k)}x(k) + B_{\theta(k)}u(k)][A_{\theta(k)}x(k) + B_{\theta(k)}u(k)]'1_{\{\theta(k+1)=i\}}\} \\
&\leq E\{[(1+\alpha^2)(A_{\theta(k)}x(k)x(k)'A'_{\theta(k)}) \\
&\quad\quad + (1+1/\alpha^2)(B_{\theta(k)}u(k)u(k)'B'_{\theta(k)})]1_{\{\theta(k+1)=i\}}\} \\
&= \sum_{j \in \mathcal{Z}} E\{[(1+\alpha^2)(A_j x(k)x(k)'A'_j) \\
&\quad\quad + (1+1/\alpha^2)(B_j u(k)u(k)'B'_j)]1_{\{\theta(k+1)=i,\theta(k)=j\}}\} \\
&= (1+\alpha^2) \sum_{j \in \mathcal{Z}} p_{ji} E\{A_j x(k)x(k)'A'_j 1_{\{\theta(k)=j\}}\} \\
&\quad\quad + (1+1/\alpha^2) \sum_{j \in \mathcal{Z}} p_{ji} E\{B_j u(k)u(k)'B'_j 1_{\{\theta(k)=j\}}\} \\
&= (1+\alpha^2)\mathcal{L}_A(X(k)) + (1+1/\alpha^2)\mathcal{L}_B(U(k)), \quad k \geq 0. \quad \square
\end{aligned}
$$

Appendix: Proofs of Lemmas 20 and 21.

Proof of Lemma 20. Since $P_i x$ is the projection of x onto $\mathcal{S}_i^\perp$, we have that $P_i x \in \mathcal{S}_i^\perp$ and from the hypothesis of the lemma we have that there exists $T, \epsilon > 0$ such that $\mathcal{Y}_0^T(P_i x, \theta) > \epsilon |P_i x|^2$. Employing this fact and Proposition 15 we evaluate, for $\alpha > 0$,

$$
\begin{aligned}
\mathcal{Y}_0^T(x, i) = x'Hx &= (x - P_i x + P_i x)'H(x - P_i x + P_i x) \\
&\geq (1 - \alpha^2)(x - P_i x)'H(x - P_i x) + (1 - 1/\alpha^2)(P_i x)'H(P_i x) \\
&\geq (1 - 1/\alpha^2)(P_i x)'H(P_i x) = (1 - 1/\alpha^2)\mathcal{Y}_0^T(P_i x, i) \\
&> (1 - 1/\alpha^2)\epsilon |P_i x|^2. \qquad \square
\end{aligned}
$$

The next results are needed for the proof of Lemma 21.

LEMMA 42. $E_{x,\theta}\{\mathcal{Y}_0^T(x(t), \theta(t))\} = \mathcal{Y}_0^T(X(t))$.

Proof. Using (7) we can write that

$$
\mathcal{Y}_0^{t,T}(x, \theta) = \sum_{k=t}^{t+T-1} \langle X(k), C'C \rangle = E_{x,\theta}\{\mathcal{Y}_0^T(x(t), \theta(t))\}.
$$

However,

$$
\sum_{k=t}^{t+T-1} \langle X(k), C'C \rangle = \sum_{\ell=0}^{T-1} \langle \tilde{X}(\ell), C'C \rangle = \mathcal{Y}_0^T(\tilde{X}(0)),
$$

where $\tilde{X}_i(\ell) = X_i(t+\ell) = E_{x,\theta}\{x(t+\ell)x(t+\ell)'1_{\{\theta(t+\ell)=i\}}\}$ for $\ell = 0, \ldots, T-1$, which shows the result. $\square$

LEMMA 43. *Let* $V \in \mathcal{H}_\infty^n$. *The following inequality holds:*

$$
(33) \qquad \langle \mathcal{L}_V(X(k)), C'C \rangle \geq (1 - \alpha^2)\langle \mathcal{L}_V(\mathcal{L}_A(X(k-1))), C'C \rangle - \kappa \|U(k-1)\|_F
$$

for some $0 < \alpha < 1$ *and* $\kappa > 0$.

Proof. From Lemma 10 we evaluate

$$
(34) \qquad
\begin{aligned}
\langle \mathcal{L}_V(X(k)), C'C \rangle &\geq (1 - \alpha^2)\langle \mathcal{L}_V(\mathcal{L}_A(X(k-1))), C'C \rangle \\
&\quad + (1 - 1/\alpha^2)\langle \mathcal{L}_V(\mathcal{L}_B(U(k-1)), C'C \rangle
\end{aligned}
$$

and for the second term on the right-hand side of (34) we employ Proposition 11 (v) to obtain, for $0 < \alpha < 1$,

$$
(35) \qquad
\begin{aligned}
(1 - 1/\alpha^2)&\langle \mathcal{L}_V(\mathcal{L}_B(U(k-1))), C'C \rangle \\
&\geq (1 - 1/\alpha^2)\|C\|_\infty^2 \|V\|_\infty^2 \|B\|_\infty^2 \langle U(k-1), I \rangle \\
&= -\kappa \langle U(k-1), I \rangle,
\end{aligned}
$$

where $\kappa = -(1 - 1/\alpha^2)\|C\|_\infty^2 \|V\|_\infty^2 \|B\|_\infty^2 > 0$. The result follows immediately from (34) and (35). $\square$

Proof of Lemma 21. From (33) with $V = I$, we get that

$$
(36) \qquad \langle X(m), C'C \rangle \geq (1 - \alpha^2)\langle \mathcal{L}_A(X(m-1)), C'C \rangle - \kappa \|U(m-1)\|_F.
$$

For the first term on the right-hand side of (36), we employ (33) with $V = A$, and we repeat this step recursively for $m = k - 1, \ldots, t + 1$, to obtain

$$
\langle X(k), C'C \rangle \geq (1 - \alpha^2)^{k-t} \langle \mathcal{L}_A^{k-t}(X(t)), C'C \rangle
$$
$$
(37) \hspace{3cm} - \kappa \sum_{\ell=t}^{k-1} (1 - \alpha^2)^{-\ell+k-1} \|U(\ell)\|_F.
$$

Noticing that $(1 - \alpha^2)^{k-t} \geq (1 - \alpha^2)^T$ for $k - t \leq T$ and $-(1 - \alpha^2)^k \geq -1$ for all $k \geq 0$, we get that

$$
(38) \hspace{1cm} \langle X(k), C'C \rangle \geq (1 - \alpha^2)^T \langle \mathcal{L}_A^{k-t}(X(t)), C'C \rangle - \kappa \sum_{\ell=t}^{k-1} \|U(\ell)\|_F
$$

for $t \leq k \leq T + t - 1$. Then, from (7) and (38) we evaluate

$$
\mathcal{Y}_u^{t,T}(x, \theta) = \sum_{k=t}^{T+t-1} \langle X(k), C'C \rangle + \langle U(k), D'D \rangle
$$
$$
\geq (1 - \alpha^2)^T \sum_{k=t}^{T+t-1} \langle \mathcal{L}_A^{k-t}(X(t)), C'C \rangle - \kappa \sum_{k=t}^{T+t-1} \sum_{\ell=t}^{k-1} \|U(\ell)\|_F
$$
$$
\geq (1 - \alpha^2)^T \sum_{k=0}^{T-1} \langle \mathcal{L}_A^k(X(t)), C'C \rangle - \kappa T \sum_{k=t}^{T+t-1} \|U(k)\|_F
$$

or, equivalently,

$$
\mathcal{Y}_u^{t,T}(x, \theta) \geq \delta_1 \mathcal{Y}_0^T(X(t)) - \delta_2 \sum_{k=t}^{T+t-1} \|U(k)\|,
$$

where $\delta_1 = (1 - \alpha^2)^T$ and $\delta_2 = \kappa T$. Lemma 42 completes the proof. $\square$

Appendix: Proof of Lemma 31. For the proof of Lemma 31, we write X in the following form [14, Them. 7.5.2],

$$
(39) \hspace{3cm} X_i = v_i^1 v_i^{1\prime} + \cdots + v_i^{r_i} v_i^{r_i\prime},
$$

where $v_i^m \in \mathbb{R}^n, m = 1, \ldots, r$, and $r_i = \text{rank}(X_i) \leq n$, and we write the trajectory $X(k)$ as a linear combination of trajectories $X^{i,m}(k)$ associated with initial condition v_i^m, as follows.

Let $x^{i,m}(0) = v_i^m \in \mathcal{N}(\mathcal{O}_i)$. Let $x^{i,m}(k) \in \mathbb{R}^n, m = 1, \ldots, r_i$, be given by the difference equation $x^{i,m}(k + 1) = A_{\theta(k)} x^{i,m}(k), \theta(0) = i$. Let $X^{i,m}(k) \in \mathcal{H}_F^{n0}$ be the second moment matrix $X_j^{i,m}(k) = E\{x^{i,m}(k) x^{i,m}(k)' 1_{\{\theta(k)=j\}}\}, j \in \mathcal{Z}$. Notice that $X_i^{i,m}(0) = v_i^m v_i^{m\prime}$ and $X_j^{i,m}(0) = 0$ for $j \neq i$, and we can write $X_i = \sum_{j \in \mathcal{Z}} \sum_{m=1}^{r_j} X_i^{j,m}(0)$. Then, from the linearity of the operator $\mathcal{L}$ we have that, provided $X(0) = X$,

$$
X_{0,i}(k) = \sum_{j \in \mathcal{Z}} \sum_{m=1}^{r_j} X_i^{j,m}(k)
$$

which leads to

(40)
$$\mathcal{Y}_0(X) = \sum_{k=0}^{\infty}\langle X_0(k), C'C\rangle = \sum_{k=0}^{\infty}\left\langle \sum_{j\in\mathcal{Z}}\sum_{m=1}^{r_j} X^{j,m}(k), C'C\right\rangle$$
$$= \sum_{j\in\mathcal{Z}}\sum_{m=1}^{r_j}\sum_{k=0}^{\infty}\langle X^{j,m}(k), C'C\rangle = \sum_{j\in\mathcal{Z}}\sum_{m=1}^{r_j}\mathcal{Y}_0(v_j^m, j). \qquad \Box$$

Proof of Lemma 31. (i) Notice that (39) provides that

(41)
$$PXP' = 0 \Leftrightarrow P_i v_i^m v_i^{m\prime} P_i' = 0 \quad \forall i, m$$
$$\Leftrightarrow v_i^m \in \mathcal{N}_i \quad \forall i, m \Leftrightarrow \mathcal{Y}_0(v_i^m, i) = 0 \quad \forall i, m.$$

From (40), $\mathcal{Y}_0(X) = 0$ is equivalent to $\mathcal{Y}_0(v_i^m, i) = 0$, for each i and m, and, from (41), this is equivalent to $PXP' = 0$.

(ii) We shall show that $\mathcal{Y}_0(x, i) \geq \epsilon|x|^2$ whenever $x \in \mathcal{N}_i^\perp$. Let $X \in \mathcal{H}_F^n$ be defined as $X_i = xx'$ and $X_j = 0$, $j \neq i$. Since $x \in \mathcal{N}_i^\perp$, we have that $P_i x = x$ and it is simple to check that $PXP' = X$ and $X \in \bar{\mathcal{N}}^\perp$. Then, from the assumption in the lemma, we obtain $\mathcal{Y}_0(X) \geq \epsilon\|X\|_F$ and it follows that $\mathcal{Y}_0(x, i) = \mathcal{Y}_0(X) \geq \epsilon\|X\|_F = \epsilon|x|^2$. $\qquad \Box$

Appendix: Proof of Lemma 40.

Proof of Lemma 40. First we show that (A2) holds for $(\bar{A}, \bar{C}, \bar{\mathbb{P}})$. Notice that the requirements in Assumption (A2) hold trivially whenever $\bar{\theta}(0) = f$; indeed, $\bar{x}(k) = 0$, $k > 0$, a.s..

Now consider $\bar{\theta}(0) \neq f$. From the uniform observability of $(A, C, \mathbb{P})$ we have that there exists $T, \epsilon > 0$ for which $\sum_{k=0}^{T-1} E\{|y(k)|^2 | T < k_0\} > \epsilon|x_0|^2$. Then, we define the stopping time $t_0 = \inf\{k : \bar{\theta}(k) = f\}$ and similarly to (32) we evaluate

$$\mathcal{Y}_0^T(x, \theta) \geq \sum_{k=0}^{T-1} E\{|\bar{y}(k)|^2 1_{\{t_0 \geq T\}}\} = \sum_{k=0}^{T-1} E\{|y(k)|^2 1_{\{k_0 \geq T\}}\}$$
$$= P\{k_0 \geq T\}\sum_{k=0}^{T-1} E\{|y(k)|^2 | k_0 \geq T\} \geq P\{k_0 \geq T\}\epsilon|x|^2.$$

Since $q_{\theta f} \leq \nu < 1$, we have that $P\{k_0 \geq T\} > 0$ whenever $\theta(0) = \theta \neq f$ ($\gamma(0) \neq f$). This allows us to write

(42)
$$\mathcal{Y}_0^T(x, \theta) \geq \epsilon_2|x|^2, \quad \theta \neq f,$$

where $\epsilon_2 = \epsilon P\{k_0 \geq T | \theta \neq f\} > 0$.

Now we show that (A1) holds for $(\bar{A}, \bar{C}, \bar{\mathbb{P}})$, that is, we show that $\mathcal{Y}_0(x, \theta) = \infty$ provided that $\sum_{k=0}^{\infty} E\{|\bar{x}(k)|^2\} = \infty$. Note that we can assume that $\theta(0) \neq f$, otherwise we have that $\sum_{k=0}^{\infty} E\{|\bar{x}(k)|^2\} = |x|^2 < \infty$. We evaluate, for any $\ell \geq 0$,

$$\mathcal{Y}_0(x, \theta) \geq \sum_{k=\ell}^{\infty} E\{|\bar{y}(k)|^2\} = \sum_{m=0}^{\infty}\sum_{k=mT+\ell}^{(m+1)T+\ell-1} E\{|\bar{y}(k)|^2\}$$
$$\geq \sum_{m=0}^{\infty}\sum_{k=mT+\ell}^{(m+1)T+\ell-1} E\{|\bar{y}(k)|^2 1_{\{t_0 > mT+\ell\}}\}$$
$$= \sum_{m=0}^{\infty} P(t_0 > mT+\ell)\sum_{k=mT+\ell}^{(m+1)T+\ell-1} E\{|\bar{y}(k)|^2 | t_0 > mT+\ell\},$$

and since $t_0 > mT + \ell$ implies in particular that $\theta(mT+\ell) \neq f$, we get from (42) that

$$
\mathcal{Y}_0(x,\theta) \geq \sum_{m=0}^{\infty} P(t_0 > mT+\ell)\epsilon_2 E\big\{|\bar{x}(mT+\ell)|^2\,\big|\,t_0 > mT+\ell\big\}
$$

(43)

$$
= \epsilon_2 \sum_{m=0}^{\infty} E\{|\bar{x}(mT+\ell)|^2 1_{\{t_0 > mT+\ell\}}\}.
$$

Summing up (43) for $\ell = 0,\ldots,T-1$, we obtain

$$
(44) \qquad T\mathcal{Y}_0(x,\theta) \geq \epsilon_2 \sum_{k=0}^{\infty} E\{|\bar{x}(k)|^2 1_{\{t_0 > k\}}\}.
$$

Now, recalling that $\theta(0) \neq f$, we evaluate

$$
E\{|\bar{x}(k)|^2 1_{\{t_0>k\}}\} = P\{t_0 > k\}E\big\{|\bar{x}(k)|^2\,\big|\,t_0 > k\big\}
$$

$$
= (1-q_{nf})^k E\big\{|\bar{x}(k)|^2\,\big|\,t_0 > k\big\} = (1-q_{nf})^k E\big\{|\bar{x}(k)|^2\,\big|\,t_0 = k\big\}
$$

(45)
$$
= \frac{(1-q_{nf})}{q_{nf}}(1-q_{nf})^{k-1}q_{nf}E\big\{|\bar{x}(k)|^2\,\big|\,t_0 = k\big\}
$$

$$
= \frac{(1-q_{nf})}{q_{nf}}P\{t_0 = k\}E\big\{|\bar{x}(k)|^2\,\big|\,t_0 = k\big\} = \frac{(1-q_{nf})}{q_{nf}}E\{|\bar{x}(k)|^2 1_{\{t_0=k\}}\}
$$

$$
= \epsilon_3 E\{|\bar{x}(k)|^2 1_{\{t_0=k\}}\},
$$

where $\epsilon_3 = (1-q_{nf})/q_{nf}$. Finally, (44), (45), and the fact that $\bar{x}(k) = 0$ a.s. for each $k > t_0$ lead to

$$
T\mathcal{Y}_0(x,\theta) \geq \epsilon_2 \sum_{k=0}^{\infty} E\{|\bar{x}(k)|^2 1_{\{t_0>k\}}\}
$$

$$
= \epsilon_2 \frac{\epsilon_3}{1+\epsilon_3}\frac{1+\epsilon_3}{\epsilon_3}\sum_{k=0}^{\infty} E\{|\bar{x}(k)|^2 1_{\{t_0>k\}}\}
$$

$$
= \epsilon_2 \frac{\epsilon_3}{1+\epsilon_3}\left(\sum_{k=0}^{\infty} E\{|\bar{x}(k)|^2 1_{\{t_0>k\}}\} + \sum_{k=0}^{\infty} E\{|\bar{x}(k)|^2 1_{\{t_0=k\}}\}\right)
$$

$$
= \epsilon_2 \frac{\epsilon_3}{1+\epsilon_3}\sum_{k=0}^{\infty} E\{|\bar{x}(k)|^2\} = \infty.
$$

We have shown that (A1) and (A2) holds for $(\bar{A}, \bar{C}, \bar{\mathbb{P}})$; Theorem 18 provides the first assertion in the lemma. For the second assertion, note that $\mathcal{Y}_0(x,\theta) = 0$ whenever $\theta = f$. ☐

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