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On the variational cohomology of *g*-invariant foliations

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Let S be an integrable Pfaffian system. If it is invariant under a transversally free infinitesimal action of a finite dimensional real Lie algebra g, we show that the "vertical" variational cohomology of S is equal to the Lie algebra cohomology of g with values in the space of the "horizontal" cohomology in a maximum dimension. This result, besides giving an effective algorithm for the computation of the variational cohomology of an invariant Pfaffian system, provides a method for detecting obstructions to the existence of infinitesimal actions leaving a given system invariant. © *2003 American Institute of Physics.* [DOI: 10.1063/1.1607513]

I. INTRODUCTION

We study here a problem that arises naturally in connection with the integration of differential systems invariant under finite or infinitesimal group actions, the theory of such systems, as conceived by Sophus Lie and later brought to its full light by Élie Cartan, being discussed in Ref. 7. Let $\mathcal{D}$ be such a differential system (viewed as a sub-manifold of some Jet or Grassmannian bundle) invariant under the action of a finite dimensional real Lie algebra g of infinitesimal contact transformations and let us further assume that $\mathcal{D}$ is integrable and of finite type (otherwise we are led into the realm of infinite Lie pseudogroups) and that the infinitesimal action of g operates transitively in a direction transverse to each solution (Ref. 7, Sec. 13). Then the integration of $\mathcal{D}$ can be reduced to the integration of a finite family of integrable Pfaffian systems that are invariant under the actions of Abelian or simple algebras, these infinitesimal actions being transversally free (Ref. 7, Sec. 4).

The integration of differential equations, in the sense of devising methods that will eventually lead to explicit solutions or at least that will contribute to simplify and reduce the integration problem (e.g., reduce the order of the equations), was a major theme in the second half of the last century, as witnessed by Lie's own writings (Refs. 8, 9, 11, 13). It is therefore not surprising that Lie's ultimate concern should have been precisely the search of such methods. Using the structure of continuous groups, he could easily say which, among the many integration methods known at that time, were the *best* (the sharpest, in the sense that they involved the least number of operations and the lowest orders for these operations) and would claim, with reason, that his were the best (Refs. 8, 10, 11). Inasmuch, he showed in Ref. 12 that "*la méthode du dernier multiplicateur de Jacobi*" was the best on the grounds that the infinite continuous group of all volume preserving transformations is simple.

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Élie Cartan abandoned this pursuit, at least as a priority, since, as he claimed, in this endeavor we most often fail rather than succeed. He was the first to point out that we should actually study the *structure* underlying an integration problem and, in understanding this structure, he created integration methods, for invariant differential systems, inconceivable at Lie's time (Refs. 2–5). For such systems, this underlying structure has a very precise meaning and is a direct consequence of the structure of g and of the prescribed invariant infinitesimal action (Refs. 1, 7, Secs. 5–6).

If the differential system $\mathcal{D}$ is invariant under the action of a contact Lie algebra g then its associated Pfaffian system S, obtained by restricting all the contact 1-forms to the underlying manifold $\mathcal{R}$ of $\mathcal{D}$, is integrable and invariant by the restriction of g. Conversely, if S is invariant by the infinitesimal action Φ of a Lie algebra g' then we can extend each vector field $\Phi(v)$, $v \in g'$, to an infinitesimal contact transformation so as to obtain a contact algebra g that leaves $\mathcal{D}$ invariant and such that $\Phi(g') = g|\mathcal{R}$. In general, the actions of g and g' are not transversally free but, as evidenced in Ref. 7, Sec. 5, one can in many cases reach this appropriate setting essentially via restriction and prolongation operations. It therefore becomes relevant, in view of applying the Lie and Cartan theory, to know whether a given integrable Pfaffian system S admits a transversally free invariant infinitesimal action of a given Lie algebra g or, more generally, of some Lie algebra g. Showing the existence of such infinitesimal actions is a rather delicate problem that has to be analyzed in each specific case since there does not seem to exist any general method. On the other hand, showing non-existence can be achieved by displaying some obstructions via cohomological methods and this is actually our main concern in this paper. One last word is due. Whereas the structure of a differential system is a global concept, the integration of such a system can, in a first approach, be viewed as a local problem. Since any integrable Pfaffian system admits locally many automorphisms, in fact, they form an infinite Lie pseudogroup of order one, there exist, in a neighborhood of each point, many transversally free infinitesimal actions leaving the system invariant and the Lie and Cartan theory can always be applied.

The Euler–Lagrange (variational) complex associated to an integrable Pfaffian system S is finite. As is usual, we call *horizontal* that part of the complex preceding the Euler operator E and *vertical* that part subsequent to this operator. The horizontal part is a finite augmentation of E and the vertical part a finite resolution. We show, in Sec. V (Theorem 1), that if S is invariant under a transversally free infinitesimal action of the Lie algebra g then the above finite resolution is equivalent, in positive dimensions, to the Lie algebra complex of g taking values in the horizontal cohomology of maximum dimension. In particular, the resulting cohomology spaces are equal whereupon any discrepancy between the two cohomologies will put in evidence an obstruction to the existence of such an infinitesimal action. The above equivalence also provides an effective method for the computation of the vertical variational cohomology of an invariant Pfaffian system.

Throughout the years, several authors have given distinct though essentially equivalent formulations to the variational complex. We adopt here the approach described in Ref. 6 since it emphasizes the relationship of this complex with the algebra of generalized symmetries. Inasmuch as the usual de Rham complex on a manifold M is the differential complex associated to the algebra of all the vector fields on M, the horizontal part of the variational complex is a de Rham complex associated to the algebra of all trivial symmetries (total derivatives) and the vertical part is a de Rham complex associated to the algebra of all generalized symmetries. Our first task, in this paper, consists in writing down explicitly the complex we shall be dealing with, namely the restriction of the general complex defined in Ref. 6 to an integrable Pfaffian system. This, unfortunately, is a rather long and boring task hence we only state, in Sec. II, a well known lemma that provides all the necessary technical information relevant to the restriction procedure and thereafter construct directly, in Secs. III and IV, the desired complex. It turns out that the trivial symmetries become simply the vector fields annihilated by S and the generalized symmetries become the equivalence classes of the infinitesimal automorphisms of S modulo the trivial symmetries. There is of course nothing new about this restricted complex, just a different make-up. Invariant systems are examined in Sec. V and some examples are discussed in Sec. VI.

For simplicity, we assume that all the data are C^∞ smooth though, in each specific case, C^k

smoothness for some k will suffice. We also assume that all the manifolds are connected and second countable though not necessarily orientable and that all the objects such as functions, vector fields, differential forms, etc. are globally defined on these manifolds unless stated otherwise (e.g., local coordinates, local generators of Pfaffian systems and distributions).

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II. PFAFFIAN SYSTEMS AND THEIR PROLONGATIONS

Let S be a Pfaffian system, for the time being not necessarily integrable, and $\Sigma = S^{\perp}$ the corresponding distribution, both defined on the manifold M (Ref. 7, Sec. 2). In terms of partial differential equations, it seems preferable to view the distribution Σ (or the Pfaffian system S) as a section of the Grassmannian bundle $\mathbf{G}_1^p M$ of linear contact elements of dimension $p = \operatorname{rank}\Sigma$ or, still better, as the submanifold $\mathcal{R}$ of $\mathbf{G}_1^p M$, an image of this section (Ref. 6, p. 614). Since the dimension p will remain unchanged throughout the present discussion, we abbreviate $\mathbf{G}_k^p M$ by $\mathbf{G}_k$, where k is the order of the contact elements under consideration. Let $\mathcal{R}_1 \subset \mathbf{G}_2$ denote the first prolongation of $\mathcal{R}$.

Lemma 1: The distribution Σ is integrable (involutive) if and only if $\mathcal{R}_1$ projects onto $\mathcal{R}$. A proof of this result can be found in Ref. 7, Sec. 13. This proof tells us, in particular, that a fiber $(\mathcal{R}_1)_X$, $X \in \mathcal{R}$, is either empty or else contains a single element, say Y. The first order linear holonomic contact element $\Sigma_X^{(1)}$ associated to Y is the unique holonomic element tangent to $\mathcal{R}$ at the point X. Since it does not make much sense to define the variational complex for other that *formally integrable* equations (or at least equations that, after prolongation, become formally integrable at large enough orders), we see that in the present situation it becomes natural to assume that Σ is integrable. This does not mean, however, that variational complexes cannot be associated to nonintegrable Pfaffian systems. In this latter context, we ought to specify or determine the dimension $q \leqslant p$ of linear contact elements for which a *sufficient number* of integral contact elements do exist (e.g., Pfaffian systems that are *in involution*, in the sense of Cartan, at dimension q) and consider the variational complex in the realm of the bundles $\mathbf{G}_k^q M$. We shall nevertheless restrict our attention to integrable systems.

When S (or Σ) is integrable, then $\mathcal{R}_1$ is the set of all second order contact elements determined by the p-dimensional integral manifolds of S and the assignement $Y \in \mathcal{R}_1 \mapsto \Sigma_X^{(1)} \subset T_X \mathcal{R}$, $X = \rho_{1,2}(Y)$, is an integrable distribution $\Sigma^{(1)}$, defined on the manifold $\mathcal{R}$, equivalent to Σ via the diffeomorphism $\beta_1 = \rho_{0,1} : \mathcal{R} \to M$. In general, the k-th prolongation $\mathcal{R}_k \subset \mathbf{G}_{k+1}$ is the set of all $(k+1)$-st order contact elements determined by the p-dimensional integral manifolds of S. Furthermore, the assignment $Y \in \mathcal{R}_k \mapsto \Sigma_X^{(k)} \subset T_X \mathcal{R}_{k-1}$, $X = \rho_{k,k+1} Y$, where $\Sigma_X^{(k)}$ is the linear holonomic contact element at order k associated to Y, is an integrable distribution $\Sigma^{(k)}$, defined on the manifold $\mathcal{R}_{k-1}$, equal to the annihilator of the restriction, to $\mathcal{R}_{k-1}$, of the canonical contact structure S_k of $\mathbf{G}_k$ ($[\Sigma^{(k)}]^{\perp} = \iota^* S_k$, $\iota : \mathcal{R}_{k-1} \hookrightarrow \mathbf{G}_k$; cf. Ref. 6, Sec. 2). For any pair of integers $h \leqslant k$, the distributions $\Sigma^{(k)}$ and $\Sigma^{(h)}$ are equivalent via the diffeomorphism $\rho_{h,k} : \mathcal{R}_{k-1} \to \mathcal{R}_{h-1}$ ($\mathcal{R}_0 = \mathcal{R}$, $\Sigma^{(0)} = \Sigma$).

III. THE HORIZONTAL OPERATOR

We now construct directly the so called *horizontal* part of the variational complex associated to an integrable Pfaffian system S, namely that part preceding the Euler operator. We denote by $\mathcal{B}$ the algebra of all the (globally defined) vector fields η tangent to the distribution $\Sigma = S^{\perp}$ ($\eta \in \Gamma(\Sigma)$) and by $\mathcal{F}$ the ring of all the (globally defined) C^{∞} functions on the underlying manifold. The dual space $\mathcal{H} = \mathcal{B}^*$, with respect to the $\mathcal{F}$-module structure, is equal to the set of global sections of the dual bundle $\Sigma^* \simeq T^*M/S$ and, correspondingly, $\wedge \mathcal{H} \simeq \Gamma(\wedge \Sigma^*)$. The differential,

$$d_H : \wedge^s \mathcal{H} \to \wedge^{s+1} \mathcal{H},$$

is defined by the usual formula:

$$d_H\mu(\eta_1,\dots,\eta_{s+1})=\sum_i(-1)^{i+1}\theta(\eta_i)\,\mu(\eta_1,\dots,\widehat{\eta_i},\dots,\eta_{s+1})$$

$$+\sum_{i<j}(-1)^{i+j}\mu([\eta_i,\eta_j],\eta_1,\dots,\widehat{\eta_i},\dots,\widehat{\eta_j},\dots,\eta_{s+1}), \qquad (1)$$

where the η_i are vector fields tangent to Σ, $\theta(\eta_i)$ is the usual Lie derivative and $[\eta_i,\eta_j]$ the usual Lie bracket. Let $(U;x^1,\dots,x^p,y^1,\dots,y^q)$ be a foliated chart of Σ for which the integral manifolds, in U, are given by the equations $y^\lambda=c^\lambda$. Then an element of $\wedge\mathcal{H}$ has the local expression

$$\mu=\sum a_{i_1\cdots i_s}dx^{i_1}\wedge\cdots\wedge dx^{i_s},$$

the coefficients $a_{i_1\cdots i_s}$ being C^∞ functions on U, and

$$d_H\mu=\sum(da_{i_1\cdots i_s}|\Sigma)\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_s}=\sum\frac{\partial a_{i_1\cdots i_s}}{\partial x^i}dx^i\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_s}, \qquad (2)$$

where $da_{i_1\cdots i_s}|\Sigma$ (resp., dx^i) stands for the restriction of this differential to the integral manifolds of Σ.

We now extend the differential (1) by adding, in the cochains, a term that corresponds in Ref. 6 to the module $\mathcal{C}$ of all the contact 1-forms. Here we consider $\mathcal{C}=\Gamma(S)$ to be the module of all the global sections of S, take the cochain space $\Phi^{r,s}=(\wedge^r\mathcal{C})\otimes(\wedge^s\mathcal{H})$ and consider its elements as horizontal forms with values in $\wedge^r\mathcal{C}$. The extended differential,

$$d_H:(\wedge^r\mathcal{C})\otimes(\wedge^s\mathcal{H})\to(\wedge^r\mathcal{C})\otimes(\wedge^{s+1}\mathcal{H}),$$

is then defined by

$$d_H(\omega\otimes\mu)(\eta_1,\dots,\eta_{s+1})=\sum_i(-1)^{i+1}\theta(\eta_i)[\mu(\eta_1,\dots,\widehat{\eta_i},\dots,\eta_{s+1})\omega]$$

$$+\sum_{i<j}(-1)^{i+j}\mu([\eta_i,\eta_j],\eta_1,\dots,\widehat{\eta_i},\dots,\widehat{\eta_j},\dots,\eta_{s+1})\omega, \qquad (3)$$

where $\eta_i\in\Gamma(\Sigma)$ and $\theta(\eta_i)$ is the Lie derivative. The second term on the right hand side belongs of course to $\wedge^r\mathcal{C}$, the same being true for the first term since S is integrable and consequently $\theta(\eta_i)\mathcal{C}\subset\mathcal{C}$. Let

$$(U;x^1,\dots,x^p,y^1,\dots,y^q)$$

be a foliated chart for the distribution Σ in which the integral manifolds are given by the slices $y^\lambda=c^\lambda$. Then a typical element of $\Phi^{r,s}$ is locally a sum of terms,

$$\mu=a\,dy^{j_1}\wedge\cdots\wedge dy^{j_r}\otimes dx^{i_1}\wedge\cdots\wedge dx^{i_s}$$

and

$$d_H\mu=\sum_i\frac{\partial a}{\partial x^i}dy^{j_1}\wedge\cdots\wedge dy^{j_r}\otimes dx^i\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_s}, \qquad (4)$$

where dx^i stands for the restriction $dx^i|\Sigma$. The last formula as well as the formula (2), though helpful in theoretical considerations, is most often useless in practice since it requires the local integration of Σ. We can nevertheless remedy this situation as follows: We consider any coordinate system $(U;x^i,y^j)$ with the sole requirement that the family $\{dx^i|\Sigma\}$ be free at every point of U, thus providing a field of coframes for $\Sigma^*|U$. Next, we consider the local basis $\{\eta_i\}$ of Σ defined by $\langle \eta_i, dx^j\rangle = \delta_i^j$. Since Σ is integrable and since each η_i projects onto $\partial/\partial x^i$, it follows that $[\eta_i, \eta_j]=0$. A typical element of $\Phi^{r,s}$ can now be written locally as a sum of terms,

$$\mu = a\, \omega^{j_1}\wedge\cdots\wedge\omega^{j_r}\otimes dx^{i_1}\wedge\cdots\wedge dx^{i_s},$$

where dx^i stands for $dx^i|\Sigma$ and $\{\omega^\lambda = dy^\lambda - \Sigma_i Y_i^\lambda\, dx^i\}$ is a local basis of S. The formula (3) then reduces to

$$d_H\mu = \sum_i\, [\theta(\eta_i)(a\, \omega^{j_1}\wedge\cdots\wedge\omega^{j_r})]\otimes dx^i\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_s}, \tag{5}$$

and a similar formula can replace (2), the derivatives $\partial a\ldots/\partial x^i$ being then replaced by $\theta(\eta_i)(a\ldots)$.

Let $\mathcal{I}^r = \mathcal{I}^r(S)$ denote the module of all the (globally defined) invariant forms ω of degree r with respect to the Pfaffian system S, namely those satisfying the following condition (Ref. 7, Sec. 4):

$$\theta(\eta)\omega = 0, \ \ \forall\ \eta\in\Gamma(S^\perp).$$

Then, $\mathcal{I}^0 = \mathcal{I}$ is the ring of all the (global) first integrals of S, $\mathcal{I}^r$ is a graded $\mathcal{I}$-sub-algebra of A and the formula (3) shows that the sequence

$$0\rightarrow \mathcal{I}^r \rightarrow \Phi^{r,0}\xrightarrow{d_H}\Phi^{r,1} \tag{6}$$

is exact. We next show that the sequence

$$0\rightarrow \mathcal{I}\rightarrow \Phi^{0,0}\xrightarrow{d_H}\Phi^{0,1}\xrightarrow{d_H}\cdots\xrightarrow{d_H}\Phi^{0,p}\rightarrow 0 \tag{7}$$

is locally exact. In fact, let $(U; x^i, y^\lambda)$ be a foliated chart for Σ. Then the formula (2) defines, for each fixed set of values $y^\lambda = c^\lambda$, the differential of the *de Rham* complex on the corresponding slice, whereupon results the local exactness of (7) since the usual homotopy operators can be written incorporating the parameters y^λ. Let us finally show that

$$\Phi^{r,p-1}\xrightarrow{d_H}\Phi^{r,p}\rightarrow 0 \tag{8}$$

is locally exact. A typical element of $\Phi^{r,p}$ is, locally, a sum,

$$\omega = \sum\ a_{j_1,\ldots,j_r}\, dy^{j_1}\wedge\cdots\wedge dy^{j_r}\otimes dx^1\wedge\cdots\wedge dx^p,$$

hence, upon integrating for example along x^1, we obtain the element

$$\Omega = \sum\ A_{j_1,\ldots,j_r}\, dy^{j_1}\wedge\cdots\wedge dy^{j_r}\otimes dx^2\wedge\cdots\wedge dx^p,\ \ \frac{\partial A_{[j]}}{\partial x^1} = a_{[j]},$$

such that $d_H\Omega = \omega$.

IV. THE VERTICAL OPERATOR

Let us next construct the so called *vertical* part of the Euler–Lagrange complex, namely that part subsequent to the Euler operator. We denote by $\mathcal{A}$ the algebra of all the infinitesimal automorphisms of S and by $\mathcal{B}$ the ideal of those vector fields that are tangent to $\Sigma = S^\perp$. The system

S being integrable, any vector field tangent to Σ is an infinitesimal automorphism. Based on the lemma 1, it can be shown that the algebra $\mathcal{S}(\mathcal{R})$ of generalized symmetries of the equation $\mathcal{R}$ associated to S ($\mathcal{S}(D)$ in the notations of Ref. 6, Sec. 9) identifies with $\mathcal{A}/\mathcal{B}$. The following remarks will be used later.

(a) $\mathcal{A}$ is a module over the ring $\mathcal{I}$, $\mathcal{B}$ is a module over $\mathcal{F}$, hence $\mathcal{S}$ is a module over $\mathcal{I}$.

(b) If $\xi \in \mathcal{A}$ is tangent to a leaf of S at a point x_0, then it is also tangent to this leaf at all of its points.

We now define $\Xi^r = \Phi^{r,p}/d_H \Phi^{r,p-1}$ ($p = \dim \Sigma$), denote by $\mathbf{q}_r : \Phi^{r,p} \rightarrow \Xi^r$ the quotient map and observe that Ξ^r is an $\mathcal{I}$-module since $d_H \mathcal{I} = 0$. An element $\omega \otimes \mu \in (\wedge^r \mathcal{C}) \otimes (\wedge^s \mathcal{H})$ can also be considered as an $\mathcal{I}$-multilinear form on $\mathcal{S}(\mathcal{R})$ with values in $\wedge^s \mathcal{H}$ by setting

$$(\omega \otimes \mu)([\xi_1], \ldots, [\xi_r]) = \omega(\xi_1, \ldots, \xi_r)\mu, \quad \xi_i \in \mathcal{A},$$

where each $[\xi_i]$ is the class of ξ_i *modulo* $\mathcal{B}$. In fact, when $\eta \in \mathcal{B}$, then $(\omega \otimes \mu)(\ldots, \eta, \ldots) = 0$ and consequently $(\omega \otimes \mu)([\xi_1], \ldots, [\xi_r])$ is well defined on $\mathcal{S}(\mathcal{R})$. Furthermore (Ref. 6, Sec. 11), since

$$[d_H(\omega \otimes \mu)](\xi_1, \ldots, \xi_r) = d_H[(\omega \otimes \mu)(\xi_1, \ldots, \xi_r)],$$

the form $d_H(\omega \otimes \mu)$, $\omega \otimes \mu \in (\wedge^r \mathcal{C}) \otimes (\wedge^{p-1} \mathcal{H})$, considered as a multilinear form on $\mathcal{S}(\mathcal{R})$, takes values that vanish under the projection $\mathbf{q}_0 : \Phi^{0,p} \rightarrow \Xi^0$. Hence, to any element $\sigma \in \Xi^r$, we can associate an $\mathcal{I}$-multilinear form $[\sigma]$, defined on $\mathcal{S}(\mathcal{R})$ and taking values in Ξ^0, as follows: We take $\Omega = \Sigma\, \omega_i \otimes \mu_i \in \Phi^{r,p}$ such that $\mathbf{q}_r(\Omega) = \sigma$ and set

$$[\sigma]([\xi_1], \ldots, [\xi_r]) = \mathbf{q}_0\, \Omega([\xi_1], \ldots, [\xi_r]).$$

The mapping $\sigma \mapsto [\sigma]$ being injective (Ref. 6, Sec. 11), we are led to consider the formula

$$\begin{aligned} d_V(\mathbf{q}_r(\omega \otimes \mu))([\xi_1], \ldots, [\xi_{r+1}]) \\ = \mathbf{q}_0 \left\{ \sum_i (-1)^{i+1} \theta([\xi_i]) [\, \omega([\xi_1], \ldots, \widehat{[\xi_i]}, \ldots, [\xi_{r+1}])\mu] \right\} \\ + \mathbf{q}_0 \left\{ \sum_{i<j} (-1)^{i+j} \omega([[\xi_i], [\xi_j]], [\xi_1], \ldots, \widehat{[\xi_i]}, \ldots, \widehat{[\xi_j]}, \ldots, [\xi_{r+1}])\mu \right\}, \end{aligned} \tag{9}$$

where $\omega \otimes \mu \in \Phi^{r,p}$ and $\xi_i \in \mathcal{A}$. The second term on the right hand side clearly belongs to Ξ^0. As for the first term, let us write

$$\mu_i = \omega([\xi_1], \ldots, \widehat{[\xi_i]}, \ldots, [\xi_{r+1}])\mu \in \Phi^{0,p},$$

and let us assume that some $\xi_j = \eta \in \mathcal{B}$. If $j \neq i$, then $\mu_i = 0$ and if $j = i$ then, since $d_H \mu_i = 0$,

$$\theta(\eta)\mu_i = i(\eta) d_H \mu_i + d_H i(\eta)\mu_i = d_H i(\eta)\mu_i$$

and consequently $\mathbf{q}_0\, \theta(\eta)\mu_i = 0$. In any case, (9) provides a well defined multilinear form on $\mathcal{S}(\mathcal{R})$ taking values in Ξ^0 and it can be shown (e.g., in coordinates) that the multilinear form $d_V(\mathbf{q}_r(\omega \otimes \mu))$ is the image of an element $\sigma \in \Xi^{r+1}$.

The Euler–Lagrange (variational) complex associated to the integrable Pfaffian system S is the finite sequence,

$$0 \rightarrow \mathcal{I} \rightarrow \Phi^{0,0} \xrightarrow{d_H} \Phi^{0,1} \xrightarrow{d_H} \cdots \xrightarrow{d_H} \Phi^{0,p-1} \xrightarrow{d_H} \Phi^{0,p} \xrightarrow{E} \Xi^1 \xrightarrow{d_V} \cdots \xrightarrow{d_V} \Xi^{q-1} \xrightarrow{d_V} \Xi^q \rightarrow 0,$$

where E, the *Euler* operator, is the composite,

$$\Phi^{0,p} \xrightarrow{\mathbf{q}_0} \Xi^0 \xrightarrow{d_V} \Xi^1 ,$$

and $q = \mathrm{rank}\, S$. This complex is locally exact and reduces, locally, to (7) since Ξ^r vanishes on account of (8).

V. INVARIANT PFAFFIAN SYSTEMS

Let S be an integrable Pfaffian system invariant under the infinitesimal action $\Phi: g \to \chi(M)$ of the finite dimensional real Lie algebra g on the manifold M (Ref. 7, Sec. 3). For every $v \in g$, $\Phi(v) \in \mathcal{A}$, hence the action induces a Lie algebra morphism $\Phi: g \to \mathcal{S}(\mathcal{R})$. If further we assume that the action Φ is transversally free (Ref. 7, Sec. 4), i.e., if

$$(i) \quad \dim g = \dim \Phi(g)_p , \quad \forall\ p \in M ,$$

$$(ii) \quad T_pM = \Sigma_p \oplus \Phi(g)_p , \quad \forall\ p \in M ,$$

then the above morphism is injective and the following result holds.

Lemma 2: The algebra $\mathcal{S}(\mathcal{R})$ is generated by $\Phi(g)$ over the ring $\mathcal{I}$ of first integrals of S, any $\mathbf{R}$-basis of $\Phi(g)$ is an $\mathcal{I}$-basis of $\mathcal{S}(\mathcal{R})$ and the $\mathcal{I}$-dual $\mathcal{S}(\mathcal{R})^*$ identifies with $\mathcal{I}^1$.

Let us next recall some properties of the invariant forms associated to S. By definition (Ref. 7, Sec. 4), the exterior form ω is an invariant form of S if $\theta(\eta)\,\omega = 0$ for all $\eta \in \Gamma(S^\perp)$. It follows that $\theta(f\,\eta)\,\omega = 0$ for any function f, hence ω is an invariant if and only if $i(\eta)\omega = i(\eta)\,d\omega = 0$, for all $\eta \in \Gamma(S^\perp)$. Consequently, ω is invariant if and only if it can be expressed locally in terms of the first integrals of S and their differentials. When ω is invariant then so are the forms $f\,\omega$, $d\omega$ and $\theta(\xi)\,\omega$, where f is a first integral and ξ an infinitesimal automorphism of S, hence the set of all invariant forms is a differential algebra over the ring $\mathcal{I}$ invariant under the infinitesimal action Φ via the Lie derivative. Let $\{v_i\}$ be a basis of g. The linear forms $\omega^i \in \mathcal{C}$ defined by the conditions $\langle \Phi(v_i), \omega^j \rangle = \delta_i^j$ are a global basis of invariant forms of S, a so-called Cartan basis (Ref. 7, Secs. 6,8), and

$$d\omega^i = \sum_{j<k} c^i_{jk}\, \omega^j \wedge \omega^k ,$$

where $\{-c^i_{jk}\}$ is the set of structure constants of g with respect to the above basis. The real subspace $\Omega \subset \Gamma(T^*M)$ generated by forms ω^i only depends on Φ and acts as an $\mathbf{R}$-dual to the space $h = \Phi(g)$. Let us denote by $\mathcal{F}$ be the ring of C^∞ functions on M and by $\wedge\,\Omega$ the exterior algebra of Ω over the field $\mathbf{R}$. Since $\{\omega^i\}$ is a global basis of the Pfaffian system S, it follows that $\mathcal{C} \simeq \Omega \otimes_{\mathbf{R}} \mathcal{F}$ and, more generally, that

$$\Phi^{r,s} = (\wedge^r \mathcal{C}) \otimes_{\mathcal{F}} (\wedge^s \mathcal{H}) \simeq (\wedge^r \Omega) \otimes_{\mathbf{R}} (\wedge^s \mathcal{H}) \simeq (\wedge^r \mathcal{S}(\mathcal{R})^*) \otimes_{\mathcal{I}} (\wedge^s \mathcal{H}) . \tag{10}$$

Furthermore, since $d\omega(\xi_1, \xi_2) = -\omega([\,\xi_1, \xi_2\,])$, for any $\omega \in \Omega$ and $\xi_i \in \Phi(g)$, it also follows that the formula (9) reduces, whenever $\omega \in \wedge^r \Omega$ and $\xi_i \in \Phi(g) \subset \mathcal{S}(\mathcal{R})$, to the expression

$$[\, d\omega \otimes \mathbf{q}_0 \mu + (-1)^{\deg \omega}\, \omega \wedge d_V(\mathbf{q}_0 \mu)\,]\,(\xi_1, \ldots, \xi_{r+1}) , \tag{11}$$

where $d_V(\mathbf{q}_0 \mu)(\xi)$ is the Lie derivative $\mathbf{q}_0(\theta(\xi)\mu)$. We shall see later that the above formula still holds for any invariant form ω since $d_H \omega = 0$ implies $d_V \omega = d\omega$.

We next consider the elements $\sigma \in \Xi^r$ as i $\mathcal{I}$-multilinear (and skew-symmetric) forms $[\sigma]$ defined on $\mathcal{S}(\mathcal{R})$ and taking values in Ξ^0 (cf. Sec. IV). Each form $[\sigma]$ restricts to an $\mathcal{R}$-multilinear form $\tau = [\sigma]_{\mathbf{R}}$ defined on the real subspace $h \in \mathcal{S}(\mathcal{R})$ and conversely, on account of Lemma 2, each $\mathcal{R}$-multilinear form τ defined on h and taking values in $\sigma \in \Xi^0$ extends, by $\mathcal{I}$-multilinearity, to a form $[\sigma]$ defined on $\mathcal{S}(\mathcal{R})$ and such that $[\sigma]_{\mathbf{R}} = \tau$. Furthermore, since in the realm of real vector spaces the subspace $d_H(\Phi^{0,p-1})$ admits a complement in $\Phi^{0,p}$, the form

τ lifts to a real form $\overline{\tau}$ defined on h and taking values in $\Phi^{0,p} = \wedge^p \mathcal{H}$. Finally, using $\mathcal{F}$-multilinearity on h, $\overline{\tau}$ extends to an $\mathcal{F}$-multilinear form $\tilde{\tau}$ defined on $\chi(M)$ with values in $\wedge^p \mathcal{H}$ by requiring that $i(\eta)\tilde{\tau}=0$, for all $\eta \in \Gamma(\Sigma)$. We thus obtain an element $\tilde{\tau} \in \Phi^{r,p}$ such that $[\tilde{\tau}]_{\mathbf{R}} = \tau$ and consequently the assignment $\sigma \mapsto [\sigma]$ of Sec. IV becomes bijective when S is invariant under a transversally free infinitesimal action. Identifying g with h and thereafter Ω with g^*, the expression (11) or equivalently the formula (9) shows that the cochain complex defining the cohomology of g with values in Ξ^0 and relative to the representation $\rho(v)(\mathbf{q}_0\mu) = \mathbf{q}_0(\theta(\xi)\mu)$, $v \in g$, $\mu \in \Phi^{0,p}$ and $\xi = \Phi(v)$, is equal to

$$\Xi^0 \xrightarrow{d_V} \Xi^1 \xrightarrow{d_V} \Xi^2 \xrightarrow{d_V} \cdots . \tag{12}$$

Since $\mathbf{q}_0$ is surjective, we can rewrite the above complex by

$$\Phi^{0,p} \xrightarrow{E} \Xi^1 \xrightarrow{d_V} \Xi^2 \xrightarrow{d_V} \cdots , \tag{13}$$

without affecting the cohomology groups in positive dimensions, the latter being the vertical part of the Euler–Lagrange complex associated to S, namely the finite resolution of E.

Theorem 1: *Let S be an integrable Pfaffian system invariant under a transversally free infinitesimal action of the Lie algebra g. Then the finite resolution of the Euler operator E is equal, in positive dimensions, to the cochain complex of the Lie algebra g with values in Ξ^0.*

Since $E = d_V \circ \mathbf{q}_0$, we also infer that the space of cocycles in $\Phi^{0,p}$ (i.e., ker E) is equal to the inverse image, by $\mathbf{q}_0$, of the 0-dimensional cohomology of g, namely the inverse image of the subspace of the g-invariant elements of Ξ^0 $(\rho(v)(\mathbf{q}_0\mu) = 0,\ \forall v \in g)$. Given an integrable Pfaffian system S, we can now confront its variational cohomology with the cohomology of g taking values in Ξ^0 and eventually detect obstructions to the existence of a transversally free infinitesimal action of g leaving S invariant. When $\dim_{\mathbf{R}} \Xi^0 < \infty$ (which is very seldom the case), we usually have more information on the cohomology of g. For instance, if g is semi-simple then its cohomology vanishes in dimensions one and two (Whitehead's lemmas) and consequently the same must hold for the variational cohomology.

VI. EXAMPLES

Throughout this section, we replace integrable Pfaffian systems by the corresponding integral foliations. Though all the foliations are naive, the resulting homological calculations are not always so. For the sake of not being too omissive on these calculations, we outline a few in the last example.

A. Example 1: The torus

Let $\mathcal{F}$ be the the foliation on the 2-dimensional torus T whose leaves are the cosets of a 1-dimensional sub-group H. Then $\mathcal{F}$ is invariant under the infinitesimal action generated by any element of the Lie algebra of T $(\equiv \mathbf{R}^2)$ this action being transversally free as soon as this element does not belong to the Lie algebra h of H. The variational cohomology at Ξ^1 is equal to $\mathbf{R}$ and, using Green's formula, one shows that the cohomology class of an element $[\omega] \in \Xi^1$ identifies with the real number $\int_T \omega$. When the slope of a generating element of h is rational, the calculations are very simple and both spaces Ξ^0 and Ξ^1 identify with the set of all global first integrals of $\mathcal{F}$. When this slope is irrational, the global first integrals of $\mathcal{F}$ reduce to the constants and it becomes more involved to describe the spaces Ξ^0 and Ξ^1 and to calculate the cohomology. This is an example where the advantage of the Lie algebra cohomology calculations becomes apparent.

B. Example 2: The Möbius strip

Let $\mathcal{F}$ be the foliation on the Möbius strip $\mathcal{M}$ whose leaves are the "double" circles, except for the central circle (under the usual identification $(1,y) \equiv (-1,-y)$, $\mathcal{F}$ is the foliation induced by the segments parallel to the x-axis). Both spaces Ξ^0 and Ξ^1 identify again with the set of all global first integrals of $\mathcal{F}$ which in turn identifies with the set of all the even functions defined on the interval $]-1,1[$. The variational cohomology at Ξ^1 vanishes and, whatever the representation $\rho: g=\mathbf{R} \to \mathrm{Der}\ \Xi^0$, the cohomology of g in dimension one is nontrivial (the derivative of an even function usually ceases to be even). This disagreement shows that $\mathcal{F}$ cannot be invariant by any transversally free infinitesimal action, a fact that is geometrically obvious since such an action would provide an orientation to $\mathcal{M}$.

C. Example 3: Foliation with a compact attractor

We consider, on the infinite cylinder $C=\mathbf{R}\times S^1$ with coordinates (t,θ), the foliation $\mathcal{F}$ obtained by integrating the vector field,

$$\eta = t\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}.$$

The nature of the spaces Ξ^0 and Ξ^1 is rather involved but a straightforward calculation shows that the cohomology at Ξ^1 is null. On the other hand, whatever the representation of the Lie algebra $g=\mathbf{R}$ into $\mathrm{Der}\ \Xi^0$, the Lie algebra cohomology in dimension one cannot vanish. Consequently, the foliation $\mathcal{F}$ does not admit any 1-dimensional transversally free infinitesimal action that leaves it invariant. This fact is also geometrically obvious since the local 1-parameter group (ϕ_u) generated by any such infinitesimal action would be defined, for small u, on a whole neighborhood of the limit circle $\{0\}\times S^1$ and would transform this circle into open compact subsets of the neighboring leaves, this being of course excluded.

An entirely similar situation arises in the double solid torus (two solid tori glued by their boundaries) upon taking the Reeb foliation inside each of the tori. The common boundary torus is the unique compact leaf.

D. Example 4: Spheres and rays

On the space $M=\mathbf{R}^{p+1}-0$, let $\mathcal{F}_1$ be the foliation whose leaves are the spheres centered at the origin and $\mathcal{F}_2$ the foliation whose leaves are the rays issued from the origin. We first consider the *sphere* foliation $\mathcal{F}_1$ and calculate Ξ^0, one possible argument being as follows: Each element $\mu \in \Phi^{0,q}$ identifies canonically with a differentiable 1-parameter family $(\overline{\mu}_\rho)$ of differentiable q-forms defined on the unit sphere S^p and, under this identification, $d_H\mu \in \Phi^{0,q+1}$ also identifies with $(d\overline{\mu}_\rho)$. We next take a differentiable 1-parameter family $(\overline{\mu}_t)$, $t>0$, of p-forms on S^p. Then, upon choosing a fixed volume form Ω on S^p (e.g., the volume form associated to the induced Euclidean metric), we can determine, by integration, a differentiable function $\varphi: \mathbf{R}_+ \to \mathbf{R}$ such that $\overline{\mu}_t - \varphi(t)\,\Omega$ is, for each t, a coboundary. Restating the Lemma 4.2 (p. 123) of Ref. 14 in its stronger version (as is proved in the subsequent two pages), we can use it to establish a stronger 1-parameter version of the Lemma 4.2 (p. 126) and prove in the aforementioned context that there exists a differentiable 1-parameter family $(\overline{\eta}_t)$ of $(p-1)$-forms defined on S^p such that $\overline{\mu}_t - \varphi(t)\,\Omega = d\,\overline{\eta}_t$. Returning to $\Phi^{0,p}$ and taking the form $\tilde{\Omega} = r^*\Omega$, $r: X \mapsto (1/\|X\|)\,X$, defined on M, we conclude that each $\mu \in \Phi^{0,p}$ determines a differentiable function φ such that $\mu - \varphi\,\tilde{\Omega} = d_H\eta$, where $\eta \in \Phi^{0,p-1}$, and consequently that Ξ^0 is equal to the set of all the real-valued differentiable functions defined on $\mathbf{R}_+$ i.e., to the set of all the global first integrals of $\mathcal{F}_1$.

Let us now calculate Ξ^1. Observing that $d\rho$ is a global generator of the Pfaffian system that annihilates $\mathcal{F}_1$, any element $\eta_1 \in \Phi^{1,q}$ writes $\eta_1 = \eta \wedge d\rho$, with $\eta \in \Phi^{0,q}$, and $d_H\eta_1 = (d_H\eta)\wedge d\rho$ since $d_H(d\rho)=0$. Consequently, the element $\mu_1 = \mu\wedge d\rho \in \Phi^{1,p}$ is equal to

$d_H\eta_1$, with $\eta_1\in\Phi^{1,p-1}$, if and only if $\mu=d_H\eta$ hence the present calculation reduces to the previous one and Ξ^1 is again equal to the set of all the global first integrals of $\mathcal{F}_1$.

It now becomes easy to show that the variational cohomology at Ξ^1 is null. The foliation $\mathcal{F}_1$ is of course invariant under many 1-dimensional transversally free infinitesimal actions and the vanishing of the Lie algebra cohomology in dimension one is easily be checked.

We next take the *radial* foliation $\mathcal{F}_2$. Here we can proceed locally, on open sets saturated by rays, and integration along these rays will show that $\Xi^q=0$, $0\leqslant q\leqslant p$. The variational as well as the Lie algebra cohomologies vanish, their comparison not revealing the following geometrical facts.

(a) When p is even, there cannot exist a transversally free infinitesimal action leaving the radial foliation $\mathcal{F}_2$ invariant. In fact, since the tangent spaces to $\mathcal{F}_1$ and $\mathcal{F}_2$ are complementary, any such infinitesimal action would project onto the spheres producing an infinitesimal action operating tangentially to the spheres and, in restriction to these spheres, would be free. However, even dimensional spheres do not admit nowhere vanishing vector fields.

(b) When p is odd, such transversally free infinitesimal actions do exist only for $p=1,3$. Their nonexistence for $p=7$ is essentially a consequence of the fact that S^7 is not a Lie group manifold and, for all the other values of p, that the corresponding spheres are not parallelizable.

We can enhance the variational cohomology by adding nontrivial cocycles to the space M. For example, let us take for the manifold M the portion of $\mathbf{R}^{p+1}-0$ in between the spheres $S^p(1)$ and $S^p(2)$ and identify these two spheres by the radial map. Then $\mathcal{F}_1$ induces a foliation $\overline{\mathcal{F}}_1$ in spheres, $\mathcal{F}_2$ a foliation $\overline{\mathcal{F}}_2$ in circles (in fact, $M\simeq S^p\times S^1$) and one shows, for the foliation $\overline{\mathcal{F}}_2$, that Ξ^0 is equal to the set of all the differentiable functions defined on the sphere $S^p(1)$ or, equivalently, to the set of all the global first integrals of $\overline{\mathcal{F}}_2$. Furthermore, Ξ^r is equal to the product of $\binom{p}{r}$ copies of Ξ^0 and $\Xi^p=\Xi^0$. As for the variational cohomology, we can again apply the 1-parameter version of the Lemmas 4.2 and conclude that it vanishes at Ξ^{r+1} whenever $r+1<p$ and that it is equal to $\mathbf{R}$ at Ξ^p. Stokes' formula will then show that the cohomology class of an element $[\omega]\in\Xi^p$ identifies with the real number $\int_M\omega$.

Returning to the geometric facts described earlier, we can retrace (a) by looking at the variational cohomology. In fact, since any vector field on an even dimensional sphere has a singularity, whatever the representation ρ of a Lie algebra g into $Der\,\Xi^0\simeq\chi(S^p)$, the corresponding Lie algebra cohomology cannot vanish in dimension one. As for the property (b), it requires a deeper analysis that seems to be out of reach in the present context. Nevertheless, it can be shown that transversally free Abelian infinitesimal actions leaving $\overline{\mathcal{F}}_2$ invariant cannot exist since the corresponding Lie algebra cohomologies with values in Ξ^0 would vanish in dimension p thus contradicting the variational cohomology.

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