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An extension of Palumbo's method of solution for the Grad–Shafranov equation

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We present an extension of Palumbo's method for solving Grad Shafranov equation for ideal axisymmetric magnetohydrodynamic equilibria. The method relies essentially on an expansion in even powers of radius for the square of the gradient of the magnetic stream function and a suitable coordinate transformation that allows reducing the problem of mapping magnetic surface to quadratures. The resulting exact equilibria have magnetic surface contours different from isodynamic ones, allowing modeling flattened or D-shaped plasma tori. © *2009 American Institute of Physics*. [DOI: 10.1063/1.3208041]

I. INTRODUCTION

Since the beginning of magnetic fusion research, the obtainment of exact axisymmetric toroidal ideal magnetohydrodynamic (MHD) equilibria, in terms of simple analytical solutions, has been the subject of several studies. The problem is generally formulated in cylindrical coordinates (r, φ, and z) with axisymmetry and an elliptic second order partial differential equation for the poloidal magnetic stream function ψ widely known as Grad–Shafranov (GS) equation, is obtained.[1,2] Solutions of the GS equation depend explicitly on two arbitrary hypotheses on the functional dependence with ψ of plasma pressure p and poloidal current I. In spite of the fact that in principle infinite solutions are possible [one for any reasonable choice for $p(\psi)$ and $I(\psi)$], only few, corresponding to linear cases of the GS equation, are actually known in terms of simple analytical expressions.[1,3–7] To overcome such limitation, the fusion community has developed several numerical algorithms to solve the GS equation for given boundary conditions and assigned $p(\psi)$ and $I(\psi)$, that are routinely used for project designs, MHD stability, and transport calculations. Anyway, the knowledge of exact solutions of the GS equation is of relevance not only to validate numerical code outputs but also as starting models for MHD studies. As an alternative to directly solve the GS equation, Palumbo considered a special class of isodynamic equilibria (for which the modulus of the magnetic field is constant on magnetic surfaces),[8] reducing the problem to essentially two coupled ordinary differential equations that, once solved numerically, allows the construction of closed magnetic surfaces in terms of elliptic integrals. Bishop and Taylor extended Palumbo's derivation a little further,[9] showing that the same magnetic surfaces correspond to a degenerate class of ideal MHD equilibria for which infinite choices are possible for $p(\psi)$ and $I(\psi)$ and an additional constant term can be added to the square of $I(\psi)$. Recently, some papers related to Palumbo's approach have been published,[10–14] in connection with equilibrium properties and also as generalizations to helical or flowing equilibria.

Since the originality of Palumbo's treatment is really appealing, here we want to consider an extension of it that allows obtaining a broader class of equilibria for which the shape of magnetic surfaces has some flexibility. Once the isodynamic condition is relaxed, magnetic surfaces will not be uniquely determined and $p(\psi)$ and $I(\psi)$ will be eventually defined. The main assumptions rely on the use of r^2 and ψ as independent coordinates and on an expansion of the square of the gradient of ψ in terms of powers of r^2 (truncated at r^4) in which the coefficients depends on ψ.

In Sec. II the derivation of all relevant equations will be presented, following essentially the procedure shown in Ref. 11. In Sec. III a special class of solutions will be considered, some plots of magnetic surfaces corresponding to flattened and D-shaped equilibria will be shown, followed by some plots of associated safety factor profiles. Section IV is devoted to conclusions.

II. SOLUTION PROCEDURE

Assuming axisymmetry, in cylindrical coordinates r, φ, and z, ideal static MHD equilibria can be described in terms of a magnetic stream function $\psi(r,z)$, in such a way that the magnetic field can be written as $\vec{B}=\nabla\psi\times\hat{e}_\varphi/r+I(\psi)\hat{e}_\varphi/r$, where I is a surface function representing a poloidal current stream function. Since the plasma pressure must also be a surface function, the following GS equation (in standard units) arises:[1,2]

$$\Delta^*\psi=-r^2\frac{dp}{d\psi}-I\frac{dI}{d\psi}, \tag{1}$$

where $\Delta^*\equiv r(\partial/\partial r)(1/r)(\partial/\partial r)+(\partial^2/\partial z^2)$.

Defining R_0 as the radius corresponding to a circular magnetic axis associated with $\psi=0$, ψ_0 as a typical value for the magnetic stream ψ, it is possible to write Eq. (1) in a dimensionless form,

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$$4x\frac{\partial^2\tilde{\psi}}{\partial x^2}+\frac{\partial^2\tilde{\psi}}{\partial y^2}=-x\frac{d\tilde{p}}{d\tilde{\psi}}-\tilde{I}\frac{d\tilde{I}}{d\tilde{\psi}}, \qquad (2)$$

where $\tilde{\psi}=\psi/\psi_0$, $x=r^2/R_0^2$, $y=z/R_0$, $\tilde{I}=IR_0/\psi_0$, and $\tilde{p}=pR_0^4/\psi_0^2$.

If we choose as independent variables x and $\tilde{\psi}$, it is possible to look for special solutions corresponding to $|R_0\nabla\psi|^2=\alpha(\tilde{\psi})+\beta(\tilde{\psi})x+\gamma(\tilde{\psi})x^2$; if $\gamma(\tilde{\psi})=0$ the case examined by Palumbo[8] and Bishop and Taylor[9] will be recovered.

Using the functions $u(x,\tilde{\psi})=\partial\tilde{\psi}/\partial x$ and $v(x,\tilde{\psi})=\partial\tilde{\psi}/\partial y$ it is possible to transform Eq. (2) as

$$4x\frac{\partial u}{\partial x}+4xu\frac{\partial u}{\partial\tilde{\psi}}+v\frac{\partial v}{\partial\tilde{\psi}}=-x\frac{d\tilde{p}}{d\tilde{\psi}}-\tilde{I}\frac{d\tilde{I}}{d\tilde{\psi}}. \qquad (3)$$

Since $|R_0\nabla\tilde{\psi}|^2=4xu^2+v^2$, by differentiating it with respect to $\tilde{\psi}$, it is possible to obtain the relation

$$8xu\frac{\partial u}{\partial\tilde{\psi}}+2v\frac{\partial v}{\partial\tilde{\psi}}=\alpha'+\beta'x+\gamma'x^2,$$

where $'\equiv d/d\tilde{\psi}$ has been introduced. Comparison with Eq. (3) yields

$$4x\frac{\partial u}{\partial x}=-\left(\tilde{I}\tilde{I}'+\frac{\alpha'}{2}\right)-x\left(\tilde{p}'+\frac{\beta'}{2}\right)-x^2\frac{\gamma'}{2},$$

which can be integrated giving

$$u(x,\tilde{\psi})=\frac{\ln x}{4}M+\frac{x}{4}L-\frac{x^2}{16}\gamma'+\delta, \qquad (4)$$

where $M(\tilde{\psi})\equiv-(\tilde{I}\tilde{I}'+\alpha'/2)$, $L(\tilde{\psi})\equiv-(\tilde{p}'+\beta'/2)$ and $\delta(\tilde{\psi})$ is an integrating factor. From $d\tilde{\psi}=(\partial\tilde{\psi}/\partial x)dx+(\partial\tilde{\psi}/\partial y)dy=udx+vdy$, it follows that $dy=(1/v)d\tilde{\psi}-(u/v)dx$, and if we require it to be an exact differential, the following condition must be fulfilled:[11]

$$-2v^2\left[\frac{\partial}{\partial x}\left(\frac{1}{v}\right)+\frac{\partial}{\partial\tilde{\psi}}\left(\frac{u}{v}\right)\right]=\frac{\partial}{\partial x}v^2+u\frac{\partial}{\partial\tilde{\psi}}v^2-2v^2\frac{\partial}{\partial\tilde{\psi}}u=0. \qquad (5)$$

Using $v^2=\alpha+\beta x+\gamma x^2-4xu^2$ and Eq. (4) in Eq. (5) allows to obtain a polynomial in powers of x and $\ln x$, with coefficients that are functions of $\tilde{\psi}$. Owing to the independency of x and $\tilde{\psi}$, Eq. (5) can be satisfied if all the coefficients vanish, this implies $M=0$ and also the following five ordinary differential equations for the coefficients of x^0, x, x^2, x^3, and x^4, respectively,

$$\beta+\alpha'\delta-4\delta^2-2\alpha\delta'=0, \qquad (6)$$

$$8\gamma-8\beta\delta'+4\delta\beta'-16L\delta+\alpha'L-2\alpha L'=0, \qquad (7)$$

$$\alpha'\gamma'-2\alpha\gamma''-40\gamma'\delta+32\gamma\delta'-4\beta'L+12L^2+8\beta L'=0, \qquad (8)$$

$$\beta'\gamma'-2\beta\gamma''-12\gamma'L+8\gamma L'=0, \qquad (9)$$

$$-9\gamma'^2+8\gamma\gamma''=0. \qquad (10)$$

The main difference with Bishop–Taylor (BT) case can now be appreciated, since the number of equations is equal to the number of unknowns, making the system determined. In BT case, the same procedure with $\gamma=0$, gives three equations for four unknowns, in such a way that L remains undetermined and it is possible to obtain a degenerate class of infinite equilibria corresponding to fixed shapes for magnetic surfaces.

Once Eqs. (6)–(10) are solved, in the case of equatorial symmetry it is possible to construct closed magnetic surfaces through

$$y(x,\tilde{\psi})=\pm\int_{x_1(\tilde{\psi})}^{x}dx'\frac{u(x',\tilde{\psi})}{v(x',\tilde{\psi})},$$

being $x_1(\tilde{\psi})$ a solution of $v^2(x,\tilde{\psi})=\alpha+\beta x+\gamma x^2-4x[(x/4)L-(x^2/16)\gamma'+\delta]^2=0$, corresponding to the inner intersection of the magnetic surface with the equatorial plane.

Equation (10) can be immediately integrated, yielding

$$\gamma(\tilde{\psi})=\gamma_0(1+\epsilon\tilde{\psi})^{-8}, \qquad (11)$$

where γ_0 and ϵ are constants. When $\gamma_0\neq0$ and $\epsilon\neq0$, v^2 becomes a quintic polynomial in x and it is possible to construct cases in which three real roots and two complex conjugate ones are allowed, starting from cases with $\epsilon=0$, for which v^2 is a cubic and three real roots exist. Since infinite choices are possible for γ_0 and ϵ, infinite consistent solutions for α, β, L, and δ may be obtained. Correspondingly, magnetic surface shapes are not fixed, and in this sense the class of equilibria is wider than that found by Bishop and Taylor.

III. A SPECIAL CLASS OF SOLUTIONS

In order to show concrete examples of solutions, we will restrict to the case $\epsilon=0$, showing that magnetic surface shapes strongly depend on the sign of γ_0. This greatly simplifies Eqs. (6)–(10) since from Eq. (9) $L=L_0$ must be a constant too. In order to numerically solve the remaining equations, it is necessary to establish starting conditions at the magnetic axis. Requiring that $u=v=0$ and $\partial(v^2)/\partial x=0$ at $\psi=0$ and $x=1$ gives the following conditions:

$$\alpha(0)\equiv\alpha_0=\gamma_0,$$

$$\beta(0)\equiv\beta_0=-2\gamma_0,$$

$$\delta(0)\equiv\delta_0=-L_0/4.$$

Equation (8) can be integrated yielding

$$\beta(\tilde{\psi})=3L_0\tilde{\psi}+8\frac{\gamma_0}{L_0}\delta(\tilde{\psi}), \qquad (12)$$

while Eqs. (6) and (7) can then be rewritten as

$$3L_0\tilde{\psi}+8\frac{\gamma_0}{L_0}\delta-4\delta^2+\alpha'\delta-2\alpha\delta'=0, \qquad (6')$$

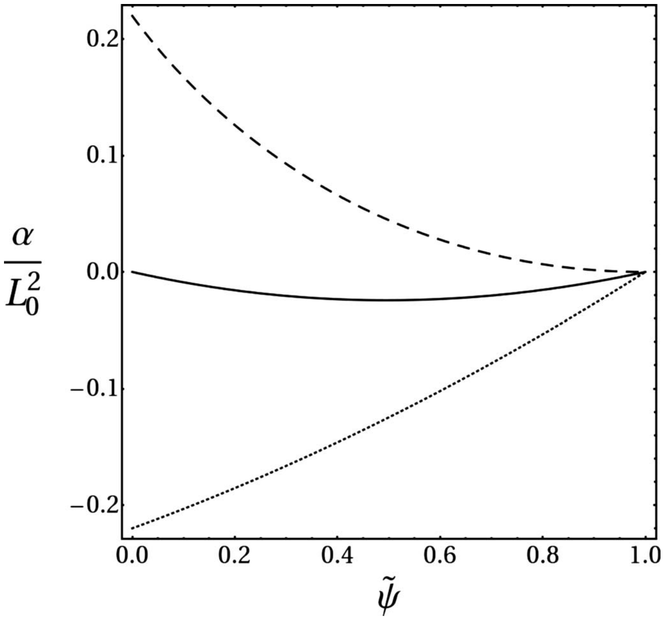

FIG. 1. α/L_0^2 as function of $\tilde{\psi}$ (normalized to unity), for constant γ_0/L_0^2, obtained from Eqs. (6′) and (7′). The bold, dashed, and dotted curves correspond to $\gamma_0/L_0^2=0$, $\gamma_0/L_0^2=0.22$, and $\gamma_0/L_0^2=-0.22$, respectively.

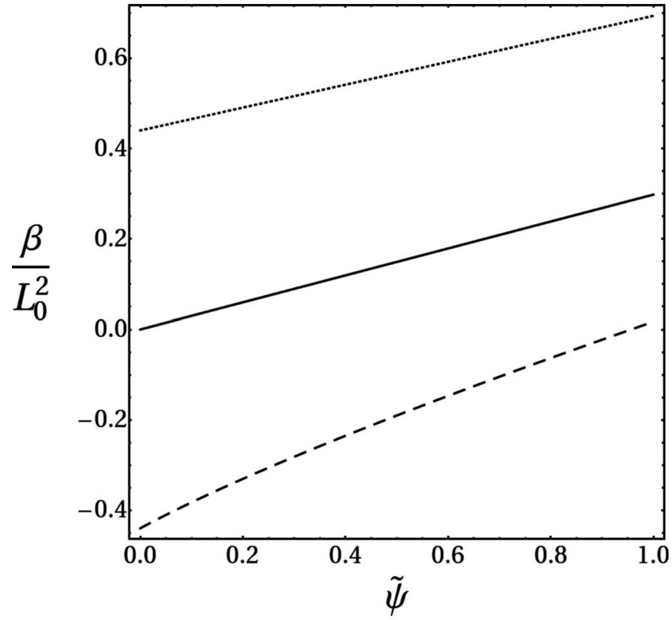

FIG. 2. β/L_0^2 as function of $\tilde{\psi}$ (normalized to unity), for constant γ_0/L_0^2, obtained from Eq. (12). The bold, dashed, and dotted curves correspond to $\gamma_0/L_0^2=0$, $\gamma_0/L_0^2=0.22$, and $\gamma_0/L_0^2=-0.22$, respectively.

$$4L_0\delta-8\gamma_0+32\frac{\gamma_0}{L_0}\delta\delta'+24\delta' L_0\tilde{\psi}-\alpha' L_0=0. \qquad (7')$$

First derivatives of α and δ at the magnetic axis ($\psi=0$ and $x=1$) are given by

$$\alpha'(0)=-L_0-8\frac{\gamma_0}{L_0}-\frac{6\gamma_0}{L_0(1-4\gamma_0/L_0^2)},$$

$$\delta'(0)=\frac{3}{4(1-4\gamma_0/L_0^2)},$$

and in order to avoid singularities it is necessary to exclude $4\gamma_0/L_0^2=1$.

Once Eqs. (6′) and (7′) are numerically solved, the corresponding magnetic surface can be worked out as

$$y(x,\tilde{\psi})=\pm\frac{x_0(\tilde{\psi})+4\delta(\tilde{\psi})/L_0}{\sqrt{x_2(\tilde{\psi})-x_0(\tilde{\psi})}}F(\varphi,m)\pm\sqrt{x_2(\tilde{\psi})-x_0(\tilde{\psi})}E(\varphi,m), \qquad (13)$$

where $x_0(\tilde{\psi})\leqslant x_1(\tilde{\psi})\leqslant x_2(\tilde{\psi})$ are three real roots of $v^2(x,\tilde{\psi})=0$ with $v^2(x_1<x<x_2,\tilde{\psi})>0$, while $F(\varphi,m)$ and $E(\varphi,m)$ are the incomplete elliptic integrals of the first and second kinds, respectively, with arguments given by

$$\varphi=\sin^{-1}\sqrt{\frac{x_2(\tilde{\psi})-x}{x_2(\tilde{\psi})-x_1(\tilde{\psi})}},\quad m=\sqrt{\frac{x_2(\tilde{\psi})-x_1(\tilde{\psi})}{x_2(\tilde{\psi})-x_0(\tilde{\psi})}}.$$

Closed magnetic surfaces can be obtained only if $\gamma_0/L_0^2<1/4$. For different illustrative values of γ_0/L_0^2, the quantities α/L_0^2, β/L_0^2, and $-4\delta/L_0$ (representing the square of the dimensionless radius of circles of tangency between magnetic surfaces and z=constant planes) are shown in Figs. 1–3, respectively, as functions of $\tilde{\psi}$ normalized to unity at the outmost closed magnetic surface. The case $\gamma_0/L_0^2=0$ corresponds to BT equilibria (see Fig. 4 for magnetic surface contours), $\gamma_0/L_0^2<0$ corresponds to cases in which magnetic surfaces are progressively flattened (see Fig. 5 corresponding to $\gamma_0/L_0^2=-0.22$), while $0<\gamma_0/L_0^2<1/4$ corresponds to cases in which magnetic surfaces are progressively more D-shaped (see Fig. 6 corresponding to $\gamma_0/L_0^2=0.22$). In all cases the outer closed surface touches the symmetry axis, however, any closed surface can be chosen as the plasma

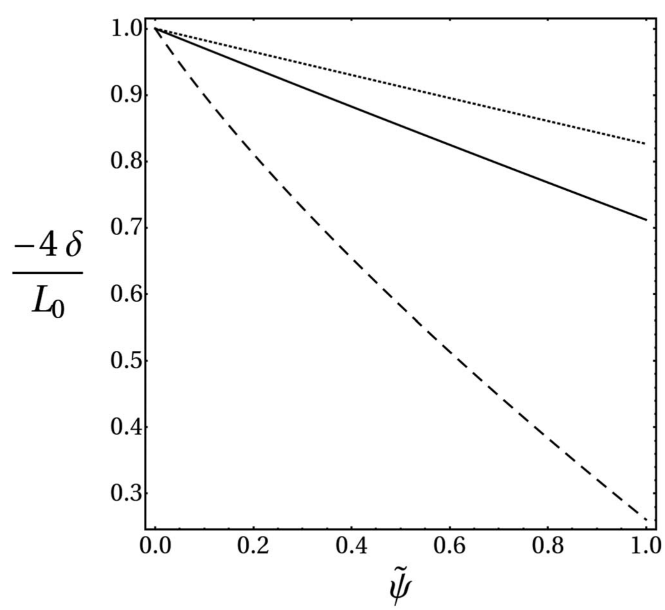

FIG. 3. $-4\delta/L_0$ as function of $\tilde{\psi}$ (normalized to unity), for constant γ_0/L_0^2, obtained from Eqs. (6′) and (7′). The bold, dashed, and dotted curves correspond to $\gamma_0/L_0^2=0$, $\gamma_0/L_0^2=0.22$, and $\gamma_0/L_0^2=-0.22$, respectively.

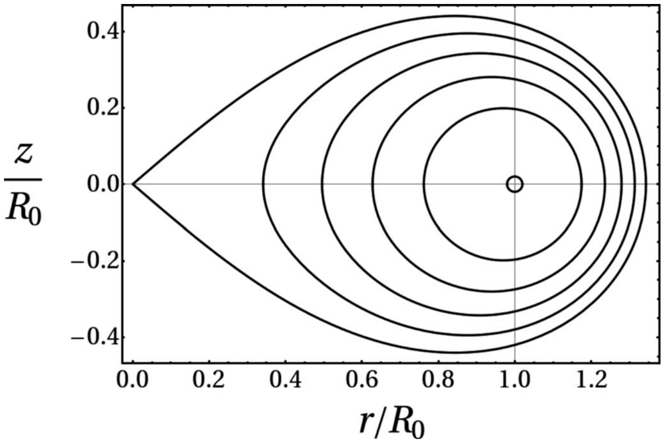

FIG. 4. Magnetic surface contours obtained from Eq. (13) for $\gamma_0/L_0^2=0$, corresponding to BT equilibria.

boundary. Dimensionless pressure profiles, vanishing at the outmost closed magnetic surface (where $\tilde{\psi}$ is normalized to unity), are shown in Fig. 7 in terms of $\tilde{p}(\tilde{\psi})/L_0^2=[\beta(1)-\beta(\tilde{\psi})]/2L_0^2-(1-\tilde{\psi})/L_0$ as function of dimensionless radius at the equatorial plane, for the previously chosen values of γ_0/L_0^2.

In all cases α vanishes at the outer closed magnetic surface and, since $\tilde{I}^2=k-\alpha$, with k=constant, when $\gamma_0/L_0^2<0$ the equilibria are paramagnetic for the toroidal magnetic field, while for $\gamma_0/L_0^2>0$ they are diamagnetic.

In the paper of Bishop and Taylor, the shown safety factor $q(\tilde{\psi})$ for isodynamic equilibria ($k=0$ and $\gamma_0=0$) must contain some mistake since it vanishes at the magnetic axis and also at the outmost closed magnetic surface, while it should be monotonically increasing with $\tilde{\psi}$, reaching the value of 0.5 (this is contained in Ref. 12, inverting the formula for rotational transform). Since the existence of extrema for the safety factor inside the plasma is symptomatic of possible violation of Mercier's necessary criterion for ideal MHD stability,[15] it may be of interest to correctly work out its expression for the class of equilibria here presented. In our notation,

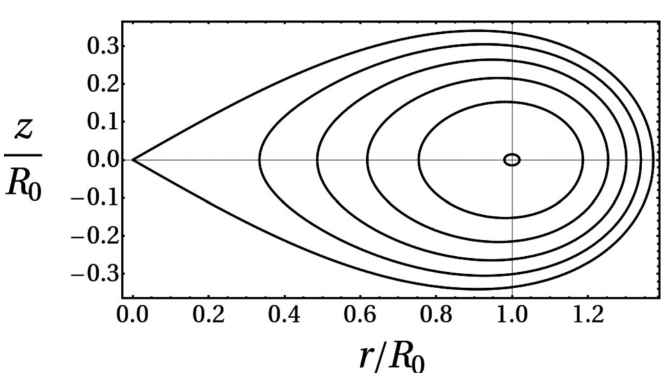

FIG. 5. Magnetic surface contours obtained from Eq. (13) for $\gamma_0/L_0^2=-0.22$. The surfaces become progressively flattened in the $z/R_0=y$ axis as γ_0/L_0^2 becomes more negative.

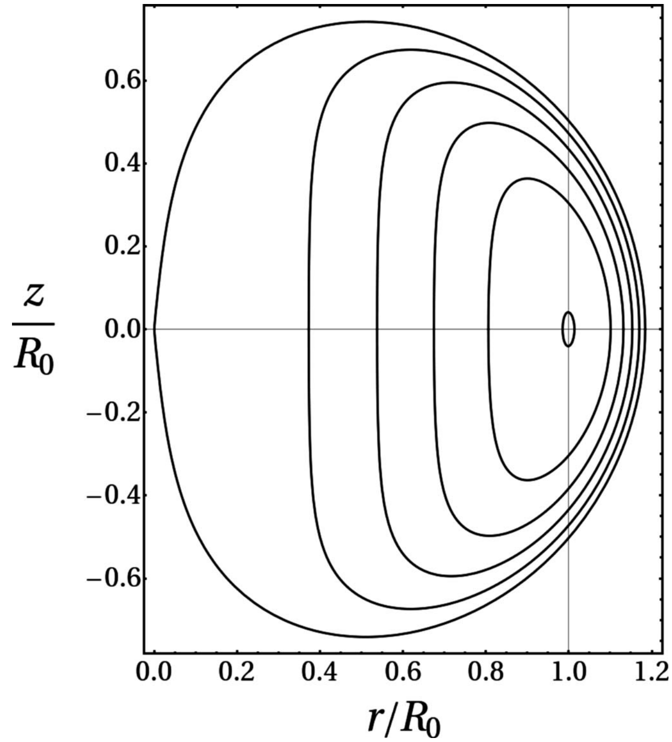

FIG. 6. Magnetic surface contours obtained from Eq. (13) for $\gamma_0/L_0^2=0.22$. This is a typical case in the interval $0<\gamma_0/L_0^2<1/4$, where the surfaces are progressively more D-shaped.

$$q(\tilde{\psi}) = \frac{1}{\pi}\int_{x_1(\tilde{\psi})}^{x_2(\tilde{\psi})} \frac{\tilde{I}(\tilde{\psi})}{x\partial\tilde{\psi}/\partial y}dx = \frac{\sqrt{k-\alpha(\tilde{\psi})}}{\pi}\int_{x_1(\tilde{\psi})}^{x_2(\tilde{\psi})} \frac{dx}{xv(x,\tilde{\psi})}.$$

In the case of $\gamma=\gamma_0$ the last integral can be rewritten in terms of a complete elliptic integral of the third kind, Π, as

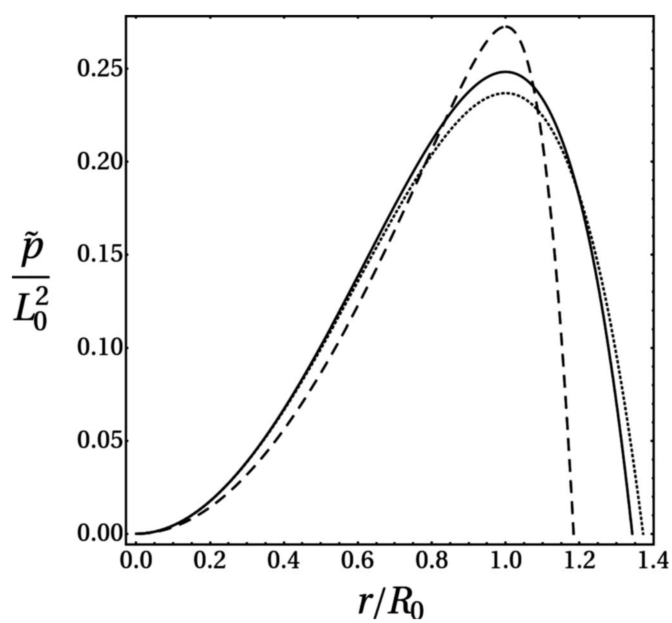

FIG. 7. Dimensionless plasma pressure $\tilde{p}/L_0^2$ as function of the normalized radius r/R_0, for constant γ_0/L_0^2. The bold, dashed, and dotted curves correspond to $\gamma_0/L_0^2=0$, $\gamma_0/L_0^2=0.22$, and $\gamma_0/L_0^2=-0.22$.

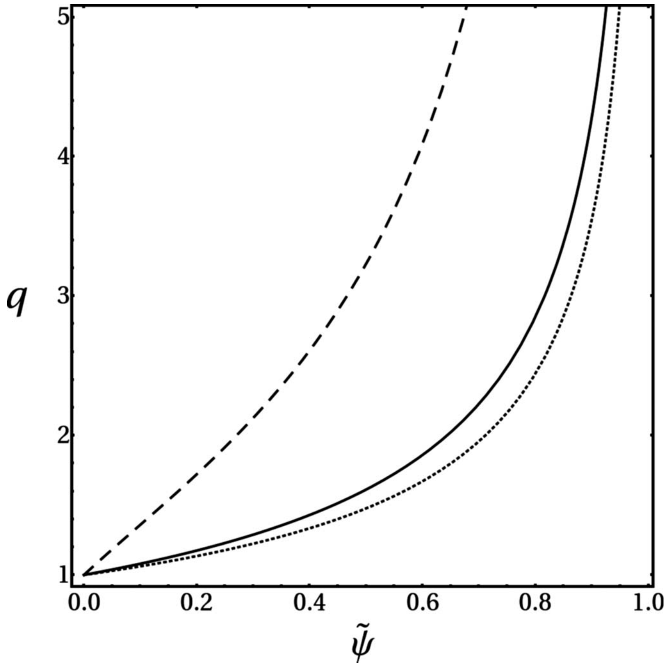

FIG. 8. Safety factor profiles q for constant γ_0/L_0^2, obtained from Eq. (14), as function of $\tilde{\psi}$ (normalized to unity). The bold, dashed, and dotted curves correspond to $\gamma_0/L_0^2=0$, $\gamma_0/L_0^2=0.22$, and $\gamma_0/L_0^2=-0.22$, respectively. For comparison, in all cases $q(0)=1$ has been chosen.

$$q(\tilde{\psi}) = \frac{2\sqrt{\dfrac{k-\alpha(\tilde{\psi})}{L_0^2}}}{\pi x_2\sqrt{x_2(\tilde{\psi})-x_0(\tilde{\psi})}} \times \Pi\left[\frac{x_2(\tilde{\psi})-x_1(\tilde{\psi})}{x_2(\tilde{\psi})}, \sqrt{\frac{x_2(\tilde{\psi})-x_1(\tilde{\psi})}{x_2(\tilde{\psi})-x_0(\tilde{\psi})}}\right]. \qquad (14)$$

In Fig. 8 $q(\tilde{\psi})$ is shown for the three typical cases already considered and $k/L_0^2=\gamma_0/L_0^2+1-x_0(0)$ chosen in such a way that q is unity at the magnetic axis in all cases. As it can be seen, $dq/d\tilde{\psi}$ is always larger in the case of D-shaped surfaces, which should indicate better stability properties.

IV. CONCLUSIONS

A class of exact solutions of GS equation for ideal axisymmetric MHD equilibria, which extends previous solutions obtained by Palumbo and Bishop and Taylor, has been presented. Depending on the sign of one parameter that enters the problem, flattened or D-shaped plasma tori can be described. The obtained solutions may be of some interest as starting model for MHD stability and transport studies, as well as to validate GS equilibrium code outputs, since they corresponds to cases in which $dp/d\psi$ and $IdI/d\psi$ are slightly nonlinear in ψ (while all known analytical solutions correspond to linear versions of GS equation) and their natural boundaries have some flexibility.

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