THE EXTREMAL SOLUTION OF A BOUNDARY REACTION PROBLEM

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Abstract. We consider

$$\Delta u = 0 \text{ in } \Omega, \qquad \frac{\partial u}{\partial \nu} = \lambda f(u) \text{ on } \Gamma_1, \qquad u = 0 \text{ on } \Gamma_2$$

where $\lambda > 0$, $f(u) = e^u$ or $f(u) = (1+u)^p$ and Γ_1, Γ_2 is a partition of $\partial\Omega$ and $\Omega \subset \mathbb{R}^N$. We determine sharp conditions on the dimension N and $p > 1$ such that the extremal solution is bounded, where the extremal solution refers to the one associated to the largest λ for which a solution exists.

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1. Introduction

We study the semilinear boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = \lambda f(u) & \text{on } \Gamma_1 \\ u = 0 & \text{on } \Gamma_2 \end{cases} \tag{1}$$

where $\lambda > 0$ is a parameter, $f(u)$ is a nonlinear smooth function of u, $\Omega \subset \mathbb{R}^N$ is a smooth, bounded domain and Γ_1, Γ_2 is a partition of $\partial\Omega$ into surfaces separated by a smooth interface. We will assume that

$$f \text{ is smooth, nondecreasing, convex and } f(0) > 0, \tag{2}$$

$$\liminf_{t\to+\infty} \frac{f'(t)t}{f(t)} > 1. \tag{3}$$

Assumption (3) is not essential, but it simplifies some of the arguments and holds for the examples $f(u) = e^u$, $f(u) = (1+u)^p$, $p > 1$. In some related works the following weaker condition is usually assumed:

$$\lim_{u\to\infty} \frac{f(u)}{u} = \infty. \tag{4}$$

We say that u is a weak solution of (1) if $u \in W^{1,1}(\Omega)$, $f(u) \in L^1(\Gamma_1)$ and

$$\int_\Omega u(-\Delta\varphi) = \int_{\Gamma_1} \lambda f(u)\varphi \qquad \text{for all } \varphi \in C^2(\bar{\Omega}) \text{ such that } \varphi\Big|_{\Gamma_2} \equiv 0 \text{ and } \frac{\partial\varphi}{\partial\nu}\Big|_{\Gamma_1} \equiv 0.$$

Problem (1) shares many properties with the following generalization of the so-called Gelfand's problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases} \tag{5}$$

which has been widely considered [3, 4, 10, 11, 21, 22]. In particular, the following result can be proved as in [3] (see Section 1.1 in [12] for further details).

Proposition 1.1. *Assume that f satisfies* (2) *and* (3)*. Then there exists $\lambda^* \in (0,\infty)$ such that*

- (1) *has a smooth solution for* $0 \le \lambda < \lambda^*$*,*
- (1) *has a weak solution for* $\lambda = \lambda^*$*,*
- (1) *has no solution for* $\lambda > \lambda^*$ *(even in the weak sense) .*

Moreover, for $0 \le \lambda < \lambda^*$ there exists a minimal solution u_λ which is bounded, positive and stable, in the sense that the linearized operator at u_λ is positive, i.e.

$$\inf_{\varphi\in C^1(\overline{\Omega}),\varphi=0 \text{ on } \Gamma_2} \frac{\int_\Omega |\nabla\varphi|^2\,dx - \lambda\int_{\Gamma_1} f'(u_\lambda)\varphi^2\,ds}{\int_{\Gamma_1}\varphi^2\,ds} > 0. \tag{6}$$

The monotone limit $u^* := \lim_{\lambda\nearrow\lambda^*} u_\lambda$ is a weak solution for $\lambda = \lambda^*$ and satisfies

$$\lambda^* \int_{\Gamma_1} f'(u^*)\varphi^2 \le \int_\Omega |\nabla\varphi|^2\,dx, \quad \forall\varphi \in C^1(\overline{\Omega}), \varphi = 0 \text{ on } \Gamma_2. \tag{7}$$

We call u^* the extremal solution of (1).

Remark 1.2. *Under assumption* (3) *we have $u^* \in H^1(\Omega)$. The proof is analogous to the argument for* (5) *in* [4]*, so we skip it.*

Proposition 1.1 suggests the following natural question : is u^* a bounded solution?

In the context of (5), no complete answer has been given yet. For the case $f(u) = e^u$, that is the original Gelfand problem, it was shown by Joseph and Lundgren [21] that if Ω is a ball, then u^* is bounded if and only if $N < 10$. Crandall and Rabinowitz [11] showed that if $f(u) = e^u$ and $N < 10$ then for *any* smooth and bounded domain, u^* is bounded. Brezis and Vázquez [4] provided a different proof of the unboundedness of u^* in the case $\Omega = B_1$ and $N \ge 10$: they established in particular that a singular energy solution which is stable must be the extremal one. In applying this criterion in dimension $N \ge 10$ they use Hardy's inequality valid for $N \ge 3$:

$$\frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2} \le \int_{\mathbb{R}^N} |\nabla\varphi|^2, \qquad \forall\varphi \in C_0^\infty(\mathbb{R}^N) \tag{8}$$

Other explicit nonlinearities, for instance $f(u) = (1+u)^p$ with $p > 1$, are considered in these references, but in the general case, little is known. In this direction, we mention the result of Nedev [23], which asserts that for any function f satisfying (2) and (4), and any smooth bounded domain in $\mathbb{R}^N$, $N \le 3$, u^* is bounded. This result has been extended by

Cabré to the case $N = 4$ and Ω strictly convex [5]. Finally, Cabré and Capella [6] showed that if Ω is a ball and $N \leq 9$ then for any nonlinearity f satisfying (2),(4) the extremal solution is bounded.

Proving that u^* is unbounded seems to be much more difficult. Besides the radial case Dávila and Dupaigne [14] have shown that in domains that are small perturbations of a ball and for the nonlinearities e^u and $(1+u)^p$ the extremal solution is singular for large dimensions ($N \geq 11$ and $N > 2 + \frac{4p}{p-1} + 4\sqrt{\frac{p}{p-1}}$ respectively).

Returning to (1), we are interested in determining whether the extremal solution u^* is bounded or singular in the cases $f(u) = e^u$ and $f(u) = (1+u)^p$, $p > 1$.

Theorem 1.3. *Let $f(u) = e^u$. In any dimension $N \geq 10$ there exists a domain $\Omega \subset \mathbb{R}^N$ and a partition in smooth sets Γ_1, Γ_2 of $\partial\Omega$ such that $u^* \notin L^\infty(\Omega)$.*

The proof is an adaptation of the argument of Brezis and Vázquez [4], using a stability criterion. In our case the singular solution has the form $u_0(x) = -\log|x|$ for $x \in \partial\mathbb{R}^N_+$ and its linearized stability in dimension $N \geq 10$ is obtained thanks to :

$$\int_{\mathbb{R}^N_+} |\nabla\varphi|^2 \geq H_N \int_{\partial\mathbb{R}^N_+} \frac{\varphi^2}{|x|}, \quad \forall \varphi \in C_0^\infty(\overline{\mathbb{R}^N_+}), \tag{9}$$

which holds for $N \geq 3$ and where the best constant

$$H_N := \inf\left\{ \frac{\int_{\mathbb{R}^N_+} |\nabla\varphi|^2}{\int_{\partial\mathbb{R}^N_+} \frac{\varphi^2}{|x|}} \ : \ \varphi \in H^1(\mathbb{R}^N_+), \varphi\Big|_{\partial\mathbb{R}^N_+} \not\equiv 0 \right\} \tag{10}$$

is given by

$$H_N = 2\frac{\Gamma(\frac{N}{4})^2}{\Gamma(\frac{N-2}{4})^2} \qquad \forall N \geq 3, \tag{11}$$

where Γ is the Gamma function. Inequality (9) is known as Kato's inequality and a proof of it was given by Herbst [20].

We will give here a different proof of this result which offers a sharper version, analogous to improvements of (8) found by Brezis and Vázquez [4] or Vázquez and Zuazua [24] (see also [2, 4, 13, 19, 24] for other improved versions of the Hardy inequality (8)) :

Theorem 1.4. *Let $B = B_1(0)$ be the unit ball in $\mathbb{R}^N$, $N \geq 3$. Then for any $1 \leq q < 2$ there exists $c = c(N,q) > 0$ such that*

$$\int_{\mathbb{R}^N_+ \cap B} |\nabla\varphi|^2 \geq H_N \int_{\partial\mathbb{R}^N_+ \cap B} \frac{\varphi^2}{|x|} + c\|\varphi\|^2_{W^{1,q}(\mathbb{R}^N_+ \cap B)}, \quad \forall \varphi \in C_0^\infty(\overline{\mathbb{R}^N_+} \cap B),$$

As a converse to Theorem 1.3 we prove :

Theorem 1.5. *Let $f(u) = e^u$, $N \leq 9$ and suppose $\Omega \subset \mathbb{R}^N_+$ is open, bounded and satisfies:*

- $\partial\Omega = \Gamma_1 \cup \Gamma_2$, *where* $\Gamma_1 \subset \partial\mathbb{R}^N_+$ *and* $\Gamma_2 \subset \mathbb{R}^N_+$
- Ω *is symmetric with respect to the hyperplanes* $x_1 = 0, \ldots, x_{N-1} = 0$, *and*
- Ω *is convex with respect to all directions* $x_1, \ldots, x_{N-1}$.

Then the extremal solution u^* of (1) *belongs to $L^\infty(\Omega)$.*

Remark 1.6. *In order to prove Theorem 1.5, one is at first tempted to imitate the classical argument of Crandall and Rabinowitz* [11]*: roughly speaking, one uses the stability inequality* (7) *and the equation* (1) *with test functions of the form $\varphi = e^{ju}$, $j \geq 1$. This does not lead to the optimal dimension $N = 9$ but applies to general domains (see Proposition*

1.7 below). We use instead test functions φ, which are not functions of u, but which have the expected behavior of e^{ju} near a singular point, assuming it exists.

Proposition 1.7. *Let $f(u) = e^u$ and assume $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain such that $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \subset \partial\mathbb{R}^N_+$ and $\Gamma_2 \subset \mathbb{R}^N_+$. Assume further that $N < 6$. Then the extremal solution u^* of* (1) *belongs to $L^\infty(\Omega)$.*

This raises the following question

Open Problem 1. *Does Theorem 1.5 hold in any smooth bounded domain $\Omega \subset \mathbb{R}^N_+$ such that $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \subset \partial\mathbb{R}^N_+$ and $\Gamma_2 \subset \mathbb{R}^N_+$?*

Next we look at (1) in the case $f(u) = (1+u)^p$, $p > 1$. Given $0 < \alpha < N-1$ define

$$w_\alpha(x) = \int_{\partial\mathbb{R}^N_+} K(x,y)|y|^{-\alpha}dy \quad \text{for } x \in \mathbb{R}^N_+, \tag{12}$$

where $K(x,y) = \frac{2x_N}{N\omega_N}|x-y|^{-N}$ is the Green's function for the Dirichlet problem in $\mathbb{R}^N_+$ (see e.g. [18]). Clearly, $w_\alpha > 0$ in $\mathbb{R}^N_+$. Moreover w_α is harmonic in $\mathbb{R}^N_+$ and w_α extends to a function belonging to $C^\infty(\overline{\mathbb{R}^N_+} \setminus \{0\})$ with

$$w_\alpha(x) = |x|^{-\alpha} \quad \text{for all } x \in \partial\mathbb{R}^N_+ \setminus \{0\}. \tag{13}$$

It is not difficult to verify that for some constant $C(N,\alpha)$ we have

$$\frac{\partial w_\alpha}{\partial \nu}(x) = C(N,\alpha)|x|^{-\alpha-1} \qquad \forall x \in \partial\mathbb{R}^N_+ \setminus \{0\}.$$

In Section 2 we shall show that

$$C(N,\alpha) = 2\frac{\Gamma(\frac{\alpha}{2}+\frac{1}{2})\Gamma(\frac{N-1}{2}-\frac{\alpha}{2})}{\Gamma(\frac{\alpha}{2})\Gamma(\frac{N-2}{2}-\frac{\alpha}{2})}. \tag{14}$$

A heuristic calculation shows that for (1) with nonlinearity $f(u) = (1+u)^p$, the expected behavior of a solution u which is singular at $0 \in \partial\Omega$ should be $u(x) \sim |x|^{\frac{1}{p-1}}$. The boundedness of u^* is then related to the value of $C(N, \frac{1}{p-1})$. Observe that $C(N, \frac{1}{p-1})$ is defined for $p > \frac{N}{N-1}$. In the sequel, when writing $C(N, \frac{1}{p-1})$ we will implicitly assume that this condition holds.

Theorem 1.8. *Consider* (1) *with $f(u) = (1+u)^p$. If $p\,C(N, \frac{1}{p-1}) \le H_N$ and $p \ge \frac{N}{N-2}$ there exists a domain Ω such that u^* is singular.*

Remark 1.9. *The condition $p\,C(N, \frac{1}{p-1}) \le H_N$ alone is not enough to guarantee that the extremal solution is singular for some domain. Actually this condition can hold for some values of p in the range $\frac{N}{N-1} < p < \frac{N}{N-2}$. In this case a singular solution exists in some domains, but it does not correspond to the extremal one. See Theorem 6.2 in* [4] *for a similar phenomenon.*

As a partial converse, we obtain

Theorem 1.10. *Consider* (1) *with $f(u) = (1+u)^p$. Assume $\Omega \subset \mathbb{R}^N_+$ is a bounded domain such that $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \subset \partial\mathbb{R}^N_+$ and $\Gamma_2 \subset \mathbb{R}^N_+$ and such that the following condition holds*

- *Ω is convex with respect to x' and*
- *$\Pi_N(\Omega) = \Gamma_1$, where Π_N is the projection on $\partial\mathbb{R}^N_+$.*

If $p\,C(N, \frac{1}{p-1}) > H_N$ or $1 < p < \frac{N}{N-2}$ then u^* is bounded.

In the above, Ω is said to be convex with respect to x' if $(tx', x_N) + ((1-t)y', x_N) \in \Omega$ whenever $t \in [0,1]$, $x = (x', x_N) \in \Omega$ and $y = (y', x_N) \in \Omega$. Π_N is defined by $\Pi_N(x', x_N) = x'$ for all $x = (x', x_N) \in \mathbb{R}^N_+$.

Remark 1.11. *The interested reader can verify that Theorem 1.10 (and the same proof) hold if*

- Ω *is convex with respect to all directions* $x_1, \ldots, x_{N-1}$ *and*
- Ω *is symmetric with respect to the hyperplanes* $x_1 = 0, \ldots, x_{N-1} = 0$.

The organization of the paper is as follows. In Section 2 we derive formula (14) and we prove Theorem 1.4 in Section 3. In Section 4 we analyze the exponential case and give a proof of Theorems 1.3 and 1.5. The proofs of Theorems 1.10 and 1.8 are given in Section 5.

Throughout ω_N denotes the area of the unit ball in $\mathbb{R}^N$ and hence the area of the sphere S^{N-1} is $N\omega_N$.

2. Computation of $C(N, \alpha)$

We write $x = (x', x_N) \in \mathbb{R}^N_+$ with $x' \in \mathbb{R}^{N-1}$, $x_N > 0$. It follows from (12) and a simple change of variables that

$$w_\alpha(x', x_N) = w_\alpha(e(x'), x_N) \qquad \text{for all rotations } e \in \mathcal{O}(N-1)$$

and similarly

$$w_\alpha(Rx', Rx_N) = R^{-\alpha} w_\alpha(x', x_N). \tag{15}$$

Differentiating with respect to x_N yields

$$\frac{\partial w_\alpha}{\partial x_N}(Rx', Rx_N) = R^{-\alpha-1} \frac{\partial w_\alpha}{\partial x_N}(x', x_N).$$

Let $x \in \partial\mathbb{R}^N_+$, $x = (x', 0)$ and plug $R = \frac{1}{|x|} = \frac{1}{|x'|}$ in the previous formula to find

$$\frac{\partial w_\alpha}{\partial \nu}(x) = -\frac{\partial w_\alpha}{\partial x_N}(x', 0) = |x|^{-\alpha-1}\left(-\frac{\partial w_\alpha}{\partial x_N}\Big(\frac{x'}{|x'|}, 0\Big)\right).$$

Define

$$C(N, \alpha) = -\frac{\partial w_\alpha}{\partial x_N}\Big(\frac{x'}{|x'|}, 0\Big) \tag{16}$$

and observe that it is independent of $x' \in \mathbb{R}^{N-1}$.

Using (15) and the radial symmetry of w in the variables x', there exists a function $v : [0, \infty) \to \mathbb{R}$ such that

$$w_\alpha(x', x_N) = |x'|^{-\alpha} w_\alpha\Big(\frac{x'}{|x'|}, \frac{x_N}{|x'|}\Big) = |x'|^{-\alpha} v\Big(\frac{x_N}{|x'|}\Big). \tag{17}$$

Writing $r = |x'|$, $t = \frac{x_N}{|x'|}$, we have

$$r^{-\alpha} v(t) = w_\alpha(x', rt), \quad \forall x' \in \mathbb{R}^{N-1},\ |x'| = r.$$

The equation $\Delta w = 0$ is equivalent to

$$(1+t^2)v''(t) + (2\alpha + 4 - N)tv'(t) + \alpha(\alpha - N + 3)v(t) = 0, \qquad t > 0, \tag{18}$$

while (13) implies

$$v(0) = 1.$$

The initial condition for v' is related to (16)

$$v'(0) = -C(N, \alpha).$$

In addition to these initial conditions we remark that w_α is a smooth function in $\mathbb{R}^N_+$ and this together with (17) implies that

$$\lim_{t\to\infty} v(t)t^\alpha \text{ exists.} \tag{19}$$

Using the change of variables $z = it$ with i the imaginary unit and defining the new unknown $h(z) := v(-iz)$ equation (18) becomes

$$(1 - z^2)h''(z) - (2\alpha + 4 - N)zh'(z) - \alpha(\alpha - N + 3)h(z) = 0, \tag{20}$$

with initial conditions

$$\lim_{t>0,\, t\to 0} h(it) = 1, \qquad \lim_{t>0,\, t\to 0} h'(it) = iC(N, \alpha). \tag{21}$$

On the other hand (19) implies

$$\lim_{t\in\mathbb{R},\, t\to\infty} h(it)t^\alpha \text{ exists.} \tag{22}$$

The substitution

$$g(z) = (1 - z^2)^{\frac{\alpha}{2}+\frac{1}{2}-\frac{N}{4}} h(z) \tag{23}$$

transforms equation (20) into

$$(1 - z^2)g''(z) - 2zg'(z) + (\nu(\nu + 1) - \frac{\mu^2}{1 - z^2})g(z) = 0, \tag{24}$$

with

$$\mu = \alpha + \frac{2 - N}{2}, \quad \nu = \frac{N - 4}{2}. \tag{25}$$

The general solution to (24) is well known. Indeed, equation (24) belongs to the class of Legendre's equations. Following [1], two linearly independent solutions of (24) are given by the Legendre functions $P^\mu_\nu(z)$, $Q^\mu_\nu(z)$, which are defined in $\mathbb{C} \setminus \{-1, 1\}$ and analytic in $\mathbb{C} \setminus (-\infty, 1]$ (see [1, Formulas 8.1.2 – 8.1.6]). Moreover the limits of $P^\mu_\nu(z)$, $Q^\mu_\nu(z)$ on both sides of $(-1, 1)$ exist and we shall use the notation

$$\begin{aligned} P^\mu_\nu(x + i0) &= \lim_{z\to x,\mathrm{Re}(z)>0} P^\mu_\nu(z), \quad -1 < x < 1, \\ P^\mu_\nu(x - i0) &= \lim_{z\to x,\mathrm{Re}(z)<0} P^\mu_\nu(z), \quad -1 < x < 1, \end{aligned} \tag{26}$$

and a similar notation for Q^μ_ν.

The solution g of (24) is therefore given by

$$g(z) = c_1 P^\mu_\nu(z) + c_2 Q^\mu_\nu(z),$$

for appropriate constants c_1, c_2. These constants are determined by the initial conditions (21), which imply:

$$c_1 P^\mu_\nu(0 + i0) + c_2 Q^\mu_\nu(0 + i0) = 1, \tag{27}$$

$$c_1 \frac{d}{dz} P^\mu_\nu(0 + i0) + c_2 \frac{d}{dz} Q^\mu_\nu(0 + i0) = iC(N, \alpha). \tag{28}$$

In order to evaluate $C(N, \alpha)$, we use also condition (22), which is equivalent to

$$\lim_{t\to\infty,\, t\in\mathbb{R}} (c_1 P^\mu_\nu(it) + c_2 Q^\mu_\nu(it))t^{\frac{N}{2}-1} \text{ exists.} \tag{29}$$

But according to [1, Formulas 8.1.3, 8.1.5]

$$P_\nu^\mu(z) \sim z^\nu \text{ as } |z| \to \infty$$
$$Q_\nu^\mu(z) \sim z^{-\nu-1} \text{ as } |z| \to \infty$$

This and (23),(29) imply that $c_1 = 0$ and we obtain from (27),(28)

$$C(N,\alpha) = -i\frac{\frac{d}{dz}Q_\nu^\mu(0+i0)}{Q_\nu^\mu(0+i0)} \tag{30}$$

From the properties and formulas in [1] the following values can be deduced:

$$Q_\nu^\mu(0+i0) = -i2^{\mu-1}\pi^{\frac{1}{2}}e^{i\mu\pi-i\nu\frac{\pi}{2}}\frac{\Gamma(\frac{\nu}{2}+\frac{\mu}{2}+\frac{1}{2})}{\Gamma(\frac{\nu}{2}-\frac{\mu}{2}+1)} \tag{31}$$

$$\frac{d}{dz}Q_\nu^\mu(0+i0) = 2^\mu\pi^{\frac{1}{2}}e^{i\mu\pi-i\nu\frac{\pi}{2}}\frac{\Gamma(\frac{\nu}{2}+\frac{\mu}{2}+1)}{\Gamma(\frac{\nu}{2}-\frac{\mu}{2}+\frac{1}{2})} \tag{32}$$

The relations (30),(31),(32) and the values (25) yield formula (14).

□

3. Improved Kato inequality

We begin with some remarks on (9).

Remark 3.1. *a) The singular weight* $\frac{1}{|x|}$ *in the right-hand side of* (9) *is optimal, in the sense that it may not be replaced by* $\frac{1}{|x|^\alpha}$ *with* $\alpha > 1$. *This can be easily seen by choosing* $\varphi \in H^1(\mathbb{R}^N_+)$ *such that* $\varphi(x) = |x|^{-\frac{N-2}{2}+\frac{\alpha-1}{2}}$ *in a neighborhood of the origin.*

Moreover, the infimum in (10) *is not achieved.*

b) In dimension $N = 2$ *the infimum* (10) *is zero, see* [15]. *Nonetheless, if the test-functions* φ *are required to vanish on the half line* $x_1 > 0$ *then the infimum has been computed in* [15] *:*

$$\inf\left\{\frac{\int_{\mathbb{R}^2_+}|\nabla\varphi|^2}{\int_{\partial\mathbb{R}^2_+}\frac{\varphi^2}{|x|}} : \varphi \in H^1(\mathbb{R}^2_+), \varphi(x_1,0) = 0 \quad \text{if} \quad x_1 > 0, \varphi\Big|_{\partial\mathbb{R}^2_+} \not\equiv 0\right\} = \frac{1}{\pi} \tag{33}$$

c) Using Stirling's formula we see that

$$H_N = \frac{N-3}{2} + O(\frac{1}{N}) \qquad \text{as } N \to \infty. \tag{34}$$

Indeed, since $\Gamma(z) = \sqrt{2\pi/z}\left(\frac{z}{e}\right)^z\left(1+\frac{1}{12z}+O\left(\frac{1}{z^2}\right)\right)$ *for* $z > 0$,

$$\begin{aligned}
H_N &= 2\frac{\Gamma(\frac{N}{4})^2}{\Gamma(\frac{N-2}{4})^2} = 2\frac{N-2}{N}\frac{\left(\frac{N}{4e}\right)^{N/2}}{\left(\frac{N-2}{4e}\right)^{(N-2)/2}}\frac{\left(1+\frac{1}{3N}+O(1/N^2)\right)^2}{\left(1+\frac{1}{3(N-2)}+O(1/N^2)\right)^2} \\
&= 2\left(1-\frac{2}{N}\right)(4e)^{\frac{N-2}{2}-\frac{N}{2}}\left(\frac{N}{N-2}\right)^{N/2}(N-2)\left(1+O(1/N^2)\right) \\
&= \frac{N}{2e}\left(1-\frac{2}{N}\right)^{2-N/2}\left(1+O(1/N^2)\right) \\
&= \frac{N}{2}\left(1-\frac{3}{N}+O(1/N^2)\right)\left(1+O(1/N^2)\right) = \frac{N-3}{2}+O(1/N).
\end{aligned}$$

d) The estimates

$$\frac{N-3}{2} \leq H_N \leq \frac{\sqrt{(N-3)^2+1}}{2} \tag{35}$$

can be obtained in a more straightforward way using particular test functions. We give a proof of this at the end of Section 3. Also observe that (34) *could be deduced from* (35).

Let us explain first informally the idea behind the proof of Theorem 1.4, assuming for a moment that a minimizer $\overline{w} \in H^1(\mathbb{R}^N_+)$ of (10) exists. $\overline{w}$ then satisfies the associated Euler-Lagrange equation:

$$\begin{cases} \Delta \overline{w} = 0 & \text{in } \mathbb{R}^N_+, \\ \dfrac{\partial \overline{w}}{\partial \nu} = H_N \dfrac{\overline{w}}{|x|} & \text{on } \partial\mathbb{R}^N_+. \end{cases} \tag{36}$$

Elementary changes of variables show that given $R > 0$ and a rotation $e \in \mathcal{O}(N-1)$, $\overline{w_R} := R^{\frac{2-N}{2}}\,\overline{w}(Rx)$ and $\overline{w_e} := \overline{w}(e(x'), x_N)$ are also minimizers of (10). Thus it is natural to assume $\overline{w} = \overline{w_R} = \overline{w_e}$ for all $R > 0$ and $e \in \mathcal{O}(N-1)$. In particular a constant multiple of $\overline{w}$ solves

$$\begin{cases} \Delta w = 0 & \text{in } \mathbb{R}^N_+, \\ w = |x|^{-\frac{N-2}{2}} & \text{on } \partial\mathbb{R}^N_+. \end{cases}$$

Unfortunately, such a function w does not belong to $H^1(\mathbb{R}^N_+)$. Let $w = w_\alpha$ with $\alpha = \frac{N-2}{2}$ as defined in (12). Observe that $C(N, \frac{N-2}{2}) = H_N$ by (16) and hence w is indeed a solution of(36).

Following an idea of Brezis and Vázquez (equation (4.6) on page 453 of [4]), we restate (9) in terms of the new variable $v = \varphi/w$.

Proof of Theorem 1.4. When $N \geq 3$, $C_0^\infty(\mathbb{R}^N_+ \setminus \{0\})$ is dense in $H^1(\mathbb{R}^N_+)$. So it suffices to prove (9) for $\varphi \in C_0^\infty(\mathbb{R}^N_+ \setminus \{0\})$. Fix such a $\varphi \not\equiv 0$ and let w be the function defined by (12). Notice that, on supp φ, w is smooth and bounded from above and from below by some positive constants. Hence $v := \frac{\varphi}{w} \in C_0^\infty(\overline{\mathbb{R}^N_+})$ is well defined. Now, $\varphi = vw$, $\nabla\varphi = v\nabla w + w\nabla v$ and

$$|\nabla\varphi|^2 = v^2|\nabla w|^2 + w^2|\nabla v|^2 + 2vw\nabla v\nabla w.$$

Integrating

$$\int_{\mathbb{R}^N_+} |\nabla\varphi|^2 = \int_{\mathbb{R}^N_+} v^2|\nabla w|^2 + \int_{\mathbb{R}^N_+} w^2|\nabla v|^2 + 2\int_{\mathbb{R}^N_+} vw\nabla v\nabla w$$

and by Green's formula

$$\begin{aligned} \int_{\mathbb{R}^N_+} v^2|\nabla w|^2 &= \int_{\partial\mathbb{R}^N_+} v^2 w\frac{\partial w}{\partial\nu} - \int_{\mathbb{R}^N_+} w\nabla(v^2\nabla w) \\ &= \int_{\partial\mathbb{R}^N_+} v^2 w\frac{\partial w}{\partial\nu} - 2\int_{\mathbb{R}^N_+} wv\nabla w\nabla v, \end{aligned}$$

since w is harmonic in $\mathbb{R}^N_+$. Thus,

$$\int_{\mathbb{R}^N_+} |\nabla\varphi|^2 = \int_{\mathbb{R}^N_+} w^2|\nabla v|^2 + \int_{\partial\mathbb{R}^N_+} v^2 w\frac{\partial w}{\partial\nu} = \int_{\mathbb{R}^N_+} w^2|\nabla v|^2 + \int_{\partial\mathbb{R}^N_+} \frac{\varphi^2}{w}\frac{\partial w}{\partial\nu}. \tag{37}$$

But by (16) $\frac{\frac{\partial w}{\partial\nu}(x)}{w(x)} = \frac{H_N}{|x|}$ for $x \in \partial\mathbb{R}^N_+$ and hence,

$$\int_{\mathbb{R}^N_+} |\nabla\varphi|^2 \geq H_N \int_{\partial\mathbb{R}^N_+} \frac{\varphi^2}{|x|} + \int_{\mathbb{R}^N_+} w^2|\nabla v|^2 \quad \forall\varphi \in H^1(\mathbb{R}^N_+). \tag{38}$$

The second term in the right hand side of the above inequality yields the improvement of Kato's inequality when φ has support in the unit ball.

Now we assume $\varphi \in C_0^\infty(\overline{\mathbb{R}^N_+} \setminus \{0\} \cap B)$ and, as before, set $v = \frac{\varphi}{w}$. Our aim is to prove that given $1 \le q < 2$ there exists $C > 0$ such that

$$I := \int_{\mathbb{R}^N_+} w^2|\nabla v|^2 \ge \frac{1}{C}\|\varphi\|_{W^{1,q}}. \tag{39}$$

In spherical coordinates

$$I = \int_0^1 r^{N-1}\int_{S_1^+} w^2(r\theta)|\nabla v(r\theta)|^2 d\theta\, dr$$

where $S_1^+ = S_1 \cap \mathbb{R}^N_+$ and $S_1 = \{x \in \mathbb{R}^N \,/\, |x| = 1\}$ is the sphere of radius 1. From (15) we have $w(x) \ge \frac{1}{C}|x|^{-\frac{N-2}{2}}$ for some $C > 0$ and all $x \in B \cap \mathbb{R}^N_+$. Hence

$$I \ge \frac{1}{C}\int_0^1 r \int_{S_1^+} |\nabla v(r\theta)|^2 d\theta\, dr.$$

Let us compute the Sobolev norm of φ :

$$\begin{aligned}
\|\varphi\|^q_{W^{1,q}} &= \int_{\mathbb{R}^N_+\cap B} |\nabla\varphi|^q dx = \int_0^1 r^{N-1}\int_{S_1^+}|\nabla\varphi(r\theta)|^q d\theta\, dr\\
&= \int_0^1 r^{N-1}\int_{S_1^+}|\nabla v(r\theta)\; w(r\theta) + \nabla w(r\theta)\; v(r\theta)|^q d\theta\, dr\\
&\le C_q\int_0^1 r^{N-1}\int_{S_1^+}|\nabla v(r\theta)|^q|w(r\theta)|^q + |\nabla w(r\theta)|^q|v(r\theta)|^q d\theta\, dr.
\end{aligned}$$

Define

$$I_1 := \int_0^1 r^{N-1}\int_{S_1^+}|\nabla v(r\theta)|^q|w(r\theta)|^q d\theta\, dr$$

$$I_2 := \int_0^1 r^{N-1}\int_{S_1^+}|\nabla w(r\theta)|^q|v(r\theta)|^q d\theta\, dr.$$

Since $w(x) \le C|x|^{-\frac{N-2}{2}}$ we have by Hölder's inequality

$$\begin{aligned}
I_1 &\le C\int_0^1 r^{N-1-\frac{(N-2)q}{2}}\int_{S_1^+}|\nabla v(r\theta)|^q d\theta\, dr\\
&\le C\Big[\int_0^1 r\int_{S_1^+}|\nabla v(r\theta)|^2 d\theta dr\Big]^{\frac{q}{2}}\Big[\int_0^1 r^{(N-1-\frac{Nq}{2}+\frac{q}{2})\frac{2}{2-q}}dr\Big]^{\frac{2-q}{2}} = CI^{\frac{q}{2}},
\end{aligned} \tag{40}$$

since $q < 2$.

Using $|\nabla w(x)| \le C|x|^{-\frac{N}{2}}$ we estimate I_2 :

$$I_2 \le C\int_{S_1^+}\int_0^1 r^{N-1-\frac{Nq}{2}}|v(r\theta)|^q\, dr\, d\theta.$$

From the classical Hardy inequality

$$\int_0^1 r^\gamma |f(r)|^p\, dr \le \Big(\frac{p}{\gamma+1}\Big)^p\int_0^1 r^{\gamma+p}|f'(r)|^p\, dr$$

$(p \geq 1, \gamma > -1, f \in C_0^\infty(0,1))$ we deduce

$$\int_0^1 r^{N-1-\frac{Nq}{2}} |v(r\theta)|^q \, dr \leq C \int_0^1 r^{N-1-\frac{Nq}{2}+q} |\nabla v(r\theta)|^q \, dr$$

and therefore

$$I_2 \leq C \int_{S_1^+} \int_0^1 r^{N-1-\frac{Nq}{2}+q} |\nabla v(r\theta)|^q \, dr \, d\theta.$$

Hölder's inequality yields

$$I_2 \leq C \Big[\int_{S_1^+} \int_0^1 r |\nabla v(r\theta)|^2 \, dr \, d\theta \Big]^{\frac{q}{2}} \Big[\int_{S_1^+} \int_0^1 r^{(N-1-\frac{Nq}{2}+\frac{q}{2})\frac{2}{2-q}} \, dr \, d\theta \Big]^{1-\frac{q}{2}} = C I^{\frac{q}{2}}, \tag{41}$$

where we have used $q < 2$. Gathering (40) and (41) we conclude that (39) holds. □

Now we pass to the proof of item (d) of Remark 3.1.

Proof of (35)**.** We shall first show the inequality

$$\frac{N-3}{2} \leq H_N, \quad \forall N \geq 4.$$

One may assume that $u = u(r,t)$ where $r = |(x_1, \ldots, x_{N-1})|$ and $t = x_N$. Then

$$\int_{\partial \mathbb{R}^N_+} \frac{u^2}{|x|} = (N-1)\omega_{N-1} \int_0^\infty u(r,0)^2 r^{N-3} \, dr,$$

But

$$u(r,0) = -2 \int_0^\infty u(r,t) \frac{\partial u}{\partial t}(r,t) \, dt.$$

So,

$$\begin{aligned}
\int_{\partial \mathbb{R}^n_+} \frac{u^2}{|x|} &= -2(N-1)\omega_{N-1} \int_0^\infty \int_0^\infty u(r,t) \frac{\partial u}{\partial t}(r,t) r^{N-3} \, dr \, dt \\
&\leq 2(N-1)\omega_{N-1} \int_0^\infty \Big(\int_0^\infty u(r,t)^2 r^{N-4} \, dr \Big)^{1/2} \Big(\int_0^\infty \big(\frac{\partial u}{\partial t}(r,t) \big)^2 r^{N-2} \, dr \Big)^{1/2} \, dt.
\end{aligned}$$

We use now the inequality

$$\int_0^\infty u(r,t)^2 r^{N-4} \, dr \leq \frac{4}{(N-3)^2} \int_0^\infty \big(\frac{\partial u}{\partial r}(r,t) \big)^2 r^{N-2} \, dr,$$

which is one of the classical version of Hardy's inequality (in dimension $N-1$). We obtain

$$\begin{aligned}
&\int_{\partial \mathbb{R}^N_+} \frac{u^2}{|x|} \\
&\leq \frac{4}{N-3}(N-1)\omega_{N-1} \int_0^\infty \Big[\int_0^\infty \big(\frac{\partial u}{\partial r}(r,t) \big)^2 r^{N-2} \, dr \Big]^{\frac{1}{2}} \Big[\int_0^\infty \big(\frac{\partial u}{\partial t}(r,t) \big)^2 r^{N-2} \, dr \Big]^{\frac{1}{2}} \, dt \\
&\leq \frac{2}{N-3}(N-1)\omega_{N-1} \int_0^\infty \int_0^\infty \Big[\big(\frac{\partial u}{\partial t} \big)^2 + \big(\frac{\partial u}{\partial r} \big)^2 \Big] r^{N-2} \, dr \, dt \\
&= \frac{2}{N-3} \int_{\mathbb{R}^N_+} |\nabla u|^2.
\end{aligned}$$

To prove

$$H_N \le \frac{\sqrt{(N-3)^2+1}}{2}, \tag{42}$$

we consider, for fixed $a > 0$ and $\varepsilon \downarrow 0$, the function

$$\tilde{\varphi}(r, x_N) = \begin{cases} r^{\frac{2-N}{2}} e^{-a x_N / r} & \text{if } r > \varepsilon \\ \varepsilon^{\frac{2-N}{2}} e^{-a x_N/\varepsilon} & \text{if } r \le \varepsilon, \end{cases}$$

where $x = (x', x_N) \in \mathbb{R}_+^{N-1} \times \mathbb{R}_+$, $r = |x'|$. Let $\eta \in C_0^\infty(\mathbb{R}^N)$, $0 \le \eta \le 1$, $\eta \equiv 1$ in $B_1(0)$ and $\eta \equiv 0$ outside of $B_2(0)$ and set

$$\varphi = \eta \tilde{\varphi}.$$

Then

$$\int_{\partial \mathbb{R}_+^N} \frac{\varphi^2}{|x|} = (N-1)\omega_{N-1} \log(\frac{1}{\varepsilon}) + O(1) \tag{43}$$

and

$$\int_{\mathbb{R}_+^N} \eta^2 |\nabla \tilde{\varphi}|^2 = (N-1)\omega_{N-1} \left(\frac{1}{8a}((N-3)^2+1) + \frac{a}{2} \right) \log \frac{1}{\varepsilon} + O(1),$$

where $O(1)$ is bounded as $\varepsilon \to 0$. The value of a that minimizes the expression above is $a = \frac{1}{2}\sqrt{(N-3)^2+1}$, and this yields

$$\int_{\mathbb{R}_+^N} |\nabla \varphi|^2 = (N-1)\omega_{N-1} \frac{1}{2}\sqrt{(N-3)^2+1} \log \frac{1}{\varepsilon} + O(1)$$

which combined with (43) proves (42).

4. The exponential case

We need the following result that characterizes extremal singular solutions belonging to $H^1(\Omega)$.

Lemma 4.1. *Assume that $v \in H^1(\Omega)$ is an unbounded solution of* (1) *for some $\lambda > 0$. Assume furthermore the stability condition*

$$\lambda \int_{\Gamma_1} f'(v)\varphi^2 \le \int_\Omega |\nabla \varphi|^2 \quad \forall \varphi \in C^1(\overline{\Omega}), \varphi = 0 \text{ on } \Gamma_2. \tag{44}$$

Then $\lambda = \lambda^*$ and $v = u^*$.

Remark 4.2. *The above lemma is an adaptation of* [4, Theorem 3.1]. *The proof of* [4, Theorem 3.1] *relies on a result of Martel (see* [22]*), stating that the extremal solution of* (5) *is unique in the class of weak solutions. This uniqueness result is not known in our context, unless we require in addition that Ω is smooth, see* [12, Theorem 3.11]. *Below we bypass this difficulty and prove Lemma 4.1, even if Ω has a corner at the interface $\Gamma_1 \cap \Gamma_2$ (as is the case in Theorem 1.3).*

Proof. The fact that $\lambda = \lambda^*$ can be proved exactly in the same way as in [4, Theorem 3.1]. Hence we have to show that $v = u^*$. Note that v is a supersolution of (1), and therefore $u_\lambda \le v$ for all $0 < \lambda < \lambda^*$. Thus $u^* \le v$.

By density, (44) holds for $\varphi \in H^1(\Omega)$ such that $\varphi = 0$ on Γ_2. By assumption, $v \in H^1(\Omega)$ and since f satisfies (3), we also have $u^* \in H^1(\Omega)$. Thus we may choose $\varphi = v - u^*$ in (44) and obtain

$$\int_{\Gamma_1} [f(u^*) - f(v) - f'(v)(u^* - v)]\,(v - u^*) \le 0.$$

But the integrand is nonnegative since $v \ge u^*$ a.e. and f is convex. Therefore

$$f(u^*) = f(v) + f'(v)(u^* - v) \quad \text{a.e. on } \Gamma_1.$$

It follows that f is linear in intervals of the form $[u^*(x), v(x)]$ for a.e. $x \in \Gamma_1$. If $u^* \not\equiv v$ then the union of such intervals is an interval of the form $[a, \infty)$ for some $a \ge 0$, which can be proved as in [17] or [12]. This is a contradiction with (3) and we conclude that $u^* \equiv v$. □

To prove Theorems 1.3 and 1.5 it will be convenient to study the function u_0 defined by

$$u_0(x) = \int_{\partial\mathbb{R}^N_+} K(x,y) \log \frac{1}{|y|}\, dy \qquad \text{for } x \in \mathbb{R}^N_+, \tag{45}$$

where as before $K(x,y) = \frac{2x_N}{N\omega_N}|x-y|^{-N}$. Then u_0 is harmonic in $\mathbb{R}^N_+$ and

$$u_0(x) = \log \frac{1}{|x|} \quad \text{for } x \in \partial\mathbb{R}^N_+, x \neq 0.$$

Note that

$$u_0(Rx) = u_0(x) + \log \frac{1}{R}.$$

Let $r = |x'|$. Then

$$u_0(x', x_N) = v\left(\frac{x_N}{r}\right) + \log \frac{1}{r}, \tag{46}$$

for some $v : [0,\infty) \to \mathbb{R}$ such that $v(0) = 0$. We see that

$$\frac{\partial u_0}{\partial \nu} = -\frac{\partial u_0}{\partial x_N}\Big|_{x_N=0} = -\frac{1}{r} v'(0)$$

so

$$\frac{\partial u_0}{\partial \nu} = \lambda_{0,N} e^{u_0} \qquad \text{on } \partial\mathbb{R}^N_+,$$

where we let

$$\lambda_{0,N} = -v'(0).$$

Let

$$\Omega_0 = \{x \in \mathbb{R}^N_+ : u_0(x) > 0\}$$
$$\Gamma_1 = \partial\Omega \cap \partial\mathbb{R}^N_+ \qquad \Gamma_2 = \partial\Omega \setminus \partial\mathbb{R}^N_+.$$

The boundary $\partial\Omega_0$ is not smooth itself but Γ_1, Γ_2 are, and it can be checked that Proposition 1.1 still holds in this case.

It can be verified that Ω_0 can be written as $\Omega_0 = \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}_+ : |x'| < e^{v(x_N/|x'|)}\}$.

Lemma 4.3. *We have*

$$\lambda_{0,N} = \begin{cases} (N-3)\frac{\sqrt{\pi}\,\Gamma(\frac{N}{2}-\frac{3}{2})}{2\Gamma(\frac{N}{2}-1)} & \textit{if } N \ge 4, \\ 1 & \textit{if } N = 3. \end{cases}$$

Proof. We give details for $N \geq 4$, the case $N = 3$ being similar. We need to compute $v'(0)$. Calculating Δu_0 in terms of v (see (46)) we obtain that v satisfies

$$(1 + t^2)v''(t) + (4 - N)tv'(t) + 3 - N = 0$$

and thus v' is given by

$$v'(t) = (N - 3)(1 + t^2)^{\frac{N-4}{2}} \int_0^t (1 + s^2)^{\frac{2-N}{2}} \, ds + (1 + t^2)^{\frac{N-4}{2}} v'(0).$$

Integrating and using $v(0) = 0$ yields

$$v(t) = (N - 3) \int_0^t (1 + \tau^2)^{\frac{N-4}{2}} \int_0^\tau (1 + s^2)^{\frac{2-N}{2}} \, ds \, d\tau + v'(0) \int_0^t (1 + \tau^2)^{\frac{N-4}{2}} \, d\tau. \tag{47}$$

We look at the asymptotics of the two integrals above, as $t \to \infty$. For the second integral, we have

$$\lim_{t\to\infty} \frac{\int_0^t (1 + \tau^2)^{\frac{N-4}{2}} \, d\tau}{t^{N-3}} = \frac{(1 + t^2)^{\frac{N-4}{2}}}{(N - 3)t^{N-4}} = \frac{1}{N - 3}.$$

And for the first integral,

$$\begin{aligned}
\lim_{t\to\infty} \frac{\int_0^t (1 + \tau^2)^{\frac{N-4}{2}} \int_0^\tau (1 + s^2)^{\frac{2-N}{2}} \, ds \, d\tau}{t^{N-3}} &= \lim_{t\to\infty} \frac{(1 + t^2)^{\frac{N-4}{2}} \int_0^t (1 + s^2)^{\frac{2-N}{2}} \, ds}{(N - 3)t^{N-4}} \\
&= \frac{1}{N - 3} \int_0^\infty (1 + s^2)^{\frac{2-N}{2}} \, ds \\
&= \frac{1}{N - 3} \frac{\sqrt{\pi}\Gamma(\frac{N}{2} - \frac{3}{2})}{2\Gamma(\frac{N}{2} - 1)}.
\end{aligned}$$

Going back to (47), we obtain that

$$v(t) = \left(\frac{\sqrt{\pi}\Gamma(\frac{N}{2} - \frac{3}{2})}{2\Gamma(\frac{N}{2} - 1)} + \frac{v'(0)}{N - 3} \right) t^{N-3} + o(t^{N-3}).$$

Now, recall that for $x_N > 0$, $\lim_{r\to0} v(x_N/r) + \log \frac{1}{r} = u_0(0, x_N) \in \mathbb{R}$ exists and is finite. Hence, we must have

$$v'(0) = -(N - 3)\frac{\sqrt{\pi}\Gamma(\frac{N}{2} - \frac{3}{2})}{2\Gamma(\frac{N}{2} - 1)}.$$

□

Proof of Theorem 1.3. We have shown that u_0 defined in (45) is a solution to (1) with $\Omega = \Omega_0$ and $\lambda = \lambda_{0,N}$. This solution satisfies the stability condition (44) if and only if (by scaling)

$$\lambda_{0,N} \int_{\partial\mathbb{R}^N_+} \frac{\varphi^2}{|x|} \leq \int_{\mathbb{R}^N_+} |\nabla\varphi|^2, \qquad \forall\varphi \in C^1_0(\overline{\mathbb{R}^N_+} \setminus \{0\}).$$

In the Appendix we prove that

$$H_N \geq \lambda_{0,N} \text{ if and only if } N \geq 10 \tag{48}$$

and this completes the proof of the theorem. □

Proof of Theorem 1.5.

We prove the theorem by contradiction, assuming that u^* is unbounded. We use an idea of Crandall and Rabinowitz [11], but with different test functions.

Let $\phi(x) = \int_{\partial\mathbb{R}^N_+} K(x,y)|y|^{2-N+\varepsilon}dy$ and $\psi(x) = \int_{\partial\mathbb{R}^N_+} K(x,y)|y|^{\frac{2-N+\varepsilon}{2}}dy$. Then,

$$\frac{\partial\phi}{\partial\nu} = K_\phi|x|^{1-N+\varepsilon} \qquad \frac{\partial\psi}{\partial\nu} = K_\psi|x|^{\frac{-N+\varepsilon}{2}}, \tag{49}$$

where the constants K_ϕ, K_ψ are given by

$$K_\phi = \lambda_{0,N}\varepsilon + O(\varepsilon^2) \quad \text{and} \quad K_\psi = H_N + O(\varepsilon).$$

Indeed, since u_0 and ϕ are harmonic in Ω,

$$\int_{\partial\Omega} u_0\frac{\partial\phi}{\partial\nu} = \int_{\partial\Omega}\phi\frac{\partial u_0}{\partial\nu}.$$

Clearly, $\int_{\Gamma_2}\left|\phi\frac{\partial u_0}{\partial\nu}\right| \le C$, for some constant C independent of ε. So

$$K_\phi\int_0^1 \ln\left(\frac{1}{r}\right)\frac{1}{r}r^{2-N+\varepsilon}r^{N-2}dr = \lambda_{0,N}\int_0^1\frac{1}{r}r^{2-N+\varepsilon}r^{N-2}dr + O(1) = \frac{\lambda_{0,N}}{\varepsilon} + O(1).$$

Now, $\int_0^1 \ln\frac{1}{r}r^{-1+\varepsilon}dr = \frac{1}{\varepsilon^2}$ so we end up with

$$K_\phi = \lambda_{0,N}\varepsilon + O(\varepsilon^2).$$

Similarly, since ψ and w (defined in (12)) are harmonic in Ω, we have

$$\int_{\partial\Omega} w\frac{\partial\psi}{\partial\nu} = \int_{\partial\Omega}\psi\frac{\partial w}{\partial\nu}.$$

As before the boundary terms on Γ_2 are bounded independently of ε so

$$K_\psi\int_0^1 r^{-1+\varepsilon}dr = H_N\int_0^1 r^{-1+\varepsilon}dr + O(1).$$

Hence,

$$K_\psi = H_N + O(\varepsilon).$$

For $0 < \lambda < \lambda^*$, let u_λ denote the minimal solution of (1). Integrating by parts twice against ϕ yields :

$$\int_{\partial\Omega} u_\lambda\frac{\partial\phi}{\partial\nu} = \int_{\partial\Omega}\phi\frac{\partial u_\lambda}{\partial\nu} = \lambda\int_{\Gamma_1}\phi e^{u_\lambda} + \int_{\Gamma_2}\phi\frac{\partial u_\lambda}{\partial\nu} \le \lambda\int_{\Gamma_1}\phi e^{u_\lambda}.$$

Recall that $u_\lambda \nearrow u^*$ and $e^{u^*} \in L^1(\Gamma_1)$. Furthermore, ϕ is bounded away from the origin : given $R > 0$, $\phi \le R^{2-N}\mathrm{diam}(\Gamma_1)^\varepsilon \le R^{2-N}(\mathrm{diam}(\Gamma_1)+1)$ in $\Gamma_1 \setminus B_R(0)$. So,

$$\int_{\partial\Omega} u^*\frac{\partial\phi}{\partial\nu} \le \lambda^*\int_{\Gamma_1}\phi e^{u^*} \le \int_{\Gamma_1\cap B_R(0)}\phi e^{u^*} + C. \tag{50}$$

Let $\eta \in C^\infty(\mathbb{R}^N)$ be such that $\eta \equiv 1$ in $B_R(0)$ where $R > 0$ is small and fixed, and $\eta = 0$ on Γ_2. Using the stability condition (7) with $\eta\psi$ yields

$$\begin{aligned}\lambda^*\int_{\Gamma_1\cap B_R(0)} e^{u^*}\psi^2 &\le \int_\Omega|\nabla(\eta\psi)|^2 = \int_{\partial\Omega}\frac{\partial}{\partial\nu}(\eta\psi)(\eta\psi) - \int_\Omega(\eta\psi)\Delta(\eta\psi)\\ &\le \int_{\Gamma_1\cap B_R(0)}\frac{\partial\psi}{\partial\nu}\psi + C\end{aligned} \tag{51}$$

where the constant C does not depend on ε. Since $\psi^2 = \phi$ on $\partial\mathbb{R}^N_+$ combining (50) and (51) we obtain

$$\int_{\partial\Omega} u^*\frac{\partial\phi}{\partial\nu} \le \int_{\Gamma_1\cap B_R(0)}\frac{\partial\psi}{\partial\nu}\psi + C.$$

Using (49) we arrive at

$$K_\phi \int_{\Gamma_1\cap B_R(0)} u^*|x|^{1-N+\varepsilon} \le K_\psi \int_{\Gamma_1\cap B_R(0)} |x|^{1-N+\varepsilon} + C$$

and thus

$$\int_{\Gamma_1\cap B_R(0)} u^*|x|^{1-N+\varepsilon} \le (N-1)\omega_{N-1}\frac{H_N}{\lambda_{0,N}}\frac{1}{\varepsilon^2} + O(\frac{1}{\varepsilon}).$$

This last equation can be rewritten as

$$\int_0^R r^{-1+\varepsilon} \fint_{S^{N-2}} u^* d\omega\, dr \le \frac{H_N}{\lambda_{0,N}}\frac{1}{\varepsilon^2} + O(\frac{1}{\varepsilon}), \tag{52}$$

Next we claim that for any given $0 < \sigma < 1$ there exists $r(\sigma) > 0$ such that

$$u^*(x) \ge (1-\sigma)\log\frac{1}{|x|} \qquad \forall x \in \Gamma_1,\ |x| \le r(\sigma). \tag{53}$$

Observe first that for all $0 < \lambda < \lambda^*$ the minimal solution u_λ is symmetric in the variables $x_1, \ldots, x_{N-1}$ by uniqueness of the minimal solution and symmetry of Ω. Using the symmetry and convexity assumptions on Ω combined with the moving plane method (see Proposition 5.2 in [7]), we also have that u_λ achieves its maximum at the origin.

Assume by contradiction that (53) is false. Then there exists $\sigma > 0$ and a sequence $x_k \in \Gamma_1$ with $x_k \to 0$ such that

$$u^*(x_k) < (1-\sigma)\log\frac{1}{|x_k|}. \tag{54}$$

Let $s_k = |x_k|$ and choose $0 < \lambda_k < \lambda^*$ such that

$$\max_{\overline{\Omega}} u_{\lambda_k} = u_{\lambda_k}(0) = \log\frac{1}{s_k}. \tag{55}$$

Note that $\lambda_k \to \lambda^*$, otherwise u_{λ_k} would remain bounded. Let

$$v_k(x) = \frac{u_{\lambda_k}(s_k x)}{\log\frac{1}{s_k}} \qquad x \in \Omega_k \equiv \frac{1}{s_k}\Omega.$$

Then $0 \le v_k \le 1$, $v_k(0) = 1$, $\Delta v_k = 0$ in Ω_k and

$$\begin{aligned}\frac{\partial v_k}{\partial \nu}(x) &= \frac{1}{\log\frac{1}{s_k}} s_k\lambda_k \exp(u_{\lambda_k}(s_k x))\\ &\le \frac{\lambda_k}{\log\frac{1}{s_k}} \to 0.\end{aligned}$$

by (55). By elliptic regularity $v_k \to v$ uniformly on compact sets of $\overline{\mathbb{R}^N_+}$ to a function v satisfying $0 \le v \le 1$, $v(0) = 1$, $\Delta v = 0$ in $\mathbb{R}^N_+$, $\frac{\partial v}{\partial \nu} = 0$ on $\partial\mathbb{R}^N_+$. Extending v evenly to $\mathbb{R}^N$ we deduce that $v \equiv 1$. Since $|x_k| = s_k$ we deduce that

$$\frac{u_{\lambda_k}(x_k)}{\log\frac{1}{s_k}} \to 1,$$

which contradicts (54).

Going back to (52) and using (53) we find

$$(1-\sigma)\int_0^{r(\sigma)} \log\frac{1}{r}\, r^{\varepsilon-1}\, dr \le \frac{K_\psi}{K_\phi}\frac{1}{\varepsilon} + C = \frac{H_N}{\lambda_{0,N}}\frac{1}{\varepsilon^2} + O(\frac{1}{\varepsilon}). \tag{56}$$

Integrating

$$(1-\sigma)\left(\frac{1}{\varepsilon^2}r(\sigma)^\varepsilon + \frac{1}{\varepsilon}r(\sigma)^\varepsilon \log\frac{1}{r(\sigma)}\right) \le \frac{H_N}{\lambda_{0,N}}\frac{1}{\varepsilon^2} + O(\frac{1}{\varepsilon}).$$

Letting $\varepsilon \to 0$ yields

$$(1-\sigma) \le \frac{H_N}{\lambda_{0,N}}.$$

As σ is arbitrarily small we deduce $\frac{H_N}{\lambda_{0,N}} \ge 1$ which by (48) forces $N \ge 10$, a contradiction. □

Proof of Proposition 1.7. Let indeed $u = u_\lambda$ be the minimal solution of (1). Working as in [11] we take $\varphi = e^{ju}-1$, $j > 0$ in (6) and multiply (1) by $\psi = e^{2ju}-1$. We obtain

$$\frac{\lambda}{j^2}\int_{\Gamma_1} e^u\left(e^{ju}-1\right)^2 ds \le \frac{\lambda}{2j}\int_{\Gamma_1} e^u\left(e^{2ju}-1\right) ds.$$

It follows that

$$\begin{aligned}\left(\frac{1}{j}-\frac{1}{2}\right)\int_{\Gamma_1} e^{(2j+1)u}\, ds &\le \frac{2}{j}\int_{\Gamma_1} e^{(j+1)u}\, ds\\ &\le \frac{2}{j}\int_{\Gamma_1\cap A} e^{(j+1)u}\, ds + \frac{2}{j}\int_{\Gamma_1\cap B} e^{(j+1)u}\, ds,\end{aligned}$$

where $A = [(1/j-1/2)e^{(2j+1)u} < \frac{4}{j}e^{(j+1)u}]$ and $B = [(1/j-1/2)e^{(2j+1)u} \ge \frac{4}{j}e^{(j+1)u}]$. Given $j \in (0,2)$, we see that u remains uniformly bounded on A, while

$$\frac{2}{j}\int_{\Gamma_1\cap B} e^{(j+1)u}\, ds \le \frac{1}{2}\left(\frac{1}{j}-\frac{1}{2}\right)\int_{\Gamma_1} e^{(j+1)u}\, ds.$$

We conclude that e^u is bounded in $L^{2j+1}(\partial\Omega)$ independently of λ. If $2j+1 > N-1$ we obtain by elliptic estimates a bound for u in $C^\alpha(\overline{\Omega})$, for some $\alpha \in (0,1)$. Thus if $N < 6$ we can choose $j \in (0,2)$ such that $N-1 < 2j+1 < 5$ and obtain a bound for u in $C^\alpha(\overline{\Omega})$ independent of λ. However, for $N = 6,7,8$ or 9, this argument does not prove that $u^* \in L^\infty(\Omega)$. □

5. The power case

Proof of Theorem 1.10. We shall give here the proof of the case $pC(N,\frac{1}{p-1}) > H_N$. If $p < \frac{N}{N-2}$, the boundedness of u^* follows from standard techniques, using the Sobolev trace embedding theorem $H^1(\Omega) \to L^{\frac{2(N-1)}{N-2}}(\partial\Omega)$.

Let $v = C(N,\frac{1}{p-1})^{\frac{1}{p-1}} w_{\frac{1}{p-1}}$. Then v satisfies

$$\begin{cases}\Delta v = 0 & \text{in } \mathbb{R}^N_+\\ \dfrac{\partial v}{\partial \nu} = v^p & \text{on } \partial\mathbb{R}^N_+.\end{cases}$$

Observe that $pv^{p-1} = \frac{pC(N,\frac{1}{p-1})}{|x|} > \frac{H_N}{|x|}$ on $\partial\mathbb{R}^N_+ \setminus \{0\}$ and hence

$$\inf \frac{\int_{\partial\mathbb{R}^N_+} |\nabla\varphi|^2 - p\int_{\partial\mathbb{R}^N_+} v^{p-1}\varphi^2}{\int_{\partial\mathbb{R}^N_+}\varphi^2} = -\infty \tag{57}$$

where the infimum is taken over the functions $\varphi \in C_0^\infty(\overline{\mathbb{R}^N_+})$ that do not vanish identically on $\partial\mathbb{R}^N_+$.

Assume that u^* is singular. For $R > 0$ and $0 < \lambda < \lambda^*$ let

$$u_R(x) = \lambda^{\frac{1}{p-1}} R^{\frac{1}{p-1}} u_\lambda(Rx + x_\lambda),$$

where x_λ denotes a point of maximum of u_λ. Observe that since u_λ is positive and harmonic in Ω, $x_\lambda \in \Gamma_1$.

For $0 < \lambda < \lambda^*$, we choose R such that $u_R(0) = 1$ i.e. such that $\lambda^{\frac{1}{1-p}} R^{\frac{1}{p-1}} u_\lambda(x_\lambda) = 1$. Since $u_\lambda(x_\lambda) \to \infty$ as $\lambda \uparrow \lambda^*$ we have $R \to 0$ as $\lambda \uparrow \lambda^*$.

Then u_R verifies

$$\begin{cases} \Delta u_R = 0 & \text{in } \Omega_R \\ \dfrac{\partial u_R}{\partial \nu} = (\lambda^{\frac{1}{p-1}} R^{\frac{1}{p-1}} + u_R)^p & \text{on } \Gamma_1^R \\ u_R = 0 & \text{on } \Gamma_2^R, \end{cases}$$

where

$$\Omega_R = (\Omega - x_\lambda)/R, \quad \Gamma_1^R = (\Gamma_1 - x_\lambda)/R, \quad \Gamma_2^R = (\Gamma_2 - x_\lambda)/R.$$

Furthermore u_R satisfies the stability condition

$$\int_{\Omega_R} |\nabla\varphi|^2 \ge p \int_{\Gamma_1^R} (\lambda^{\frac{1}{p-1}} R^{\frac{1}{p-1}} + u_R)^{p-1}\varphi^2 \qquad \forall\varphi \in C_0^\infty(\Omega_R \cup \Gamma_1^R).$$

Let

$$\phi(R) = \sup\{r > 0 \,/\, B_r \cap \partial\mathbb{R}^N_+ \subset \Gamma_1^R\}.$$

Using the convexity assumptions on Ω, the moving plane method implies that the distance of the point $x_\lambda \in \Gamma_1$ to $\Gamma_1 \cap \Gamma_2$ stays bounded away from zero, see [7] for this method in the context of non-linear Neumann condition. This implies that

$$\phi(R) \to +\infty \quad \text{as } R \to 0. \tag{58}$$

Step 1. We have

$$u_R \le v \quad \text{in } \Gamma_1^R \cap B_{\phi(R)}. \tag{59}$$

Proof of Step 1. Suppose not. Define

$$r_0 = \sup\{\, r > 0 \,|\, r < \phi(R),\ u_R \le v \text{ in } B_r \cap \Gamma_1^R \,\}.$$

Since v is singular at 0, $r_0 > 0$ and we have $u_R \le v$ in $B_{r_0} \cap \Gamma_1^R$. Furthermore, there exists $x_0 \in \partial\mathbb{R}^N_+$ such that $|x_0| = r_0$ and $u_R(x_0) = v(x_0)$.

Let $x = (x', 0) \in \Gamma_1^R$ be such that $|x| = r_0$. If $u_R(x', 0) = v(x', 0)$ then $\frac{\partial v}{\partial \nu}(x', 0) = v(x', 0)^p < (\lambda^{\frac{1}{1-p}} R^{\frac{1}{p-1}} + u_R(x', 0))^p = \frac{\partial u_R}{\partial \nu}(x', 0)$ and hence for some $\delta_x > 0$

$$\frac{\partial v}{\partial x_N}(y, t) > \frac{\partial u_R}{\partial x_N}(y, t) \qquad |y - x'|^2 + t^2 < \delta_x^2. \tag{60}$$

It follows that for some $m_x > 0$ (and decreasing if necessary δ_x)

$$u_R(y, t) < v(y, t), \qquad \text{if } |y - x'|^2 + t^2 < \delta_x^2,\ t > m(\,|y| - r_0),\ t > 0. \tag{61}$$

Indeed, because of (60) and $u_R(y, 0) \le v(y, 0)$ for $|y| \le r_0$ we immediately obtain

$$u_R(y, t) < v(y, t) \quad \text{ for } |y - x'|^2 + t^2 < \delta_x^2, |y| \le r_0,\ t > 0.$$

If (61) is false, then there are sequences $y_k \to x'$, $t_k \to 0$ with $|y_k| > r_0$ and $\frac{t_k}{|y_k - x'|} \to \infty$ such that $v(y_k, t_k) \le u_R(y_k, t_k)$. Then by the mean value theorem there exists a point ξ_k in the segment from (y_k, t_k) to $(x', 0)$ such that $\nabla(v(\xi_k) - u_R(\xi_k)) \cdot w_k \le 0$ where w_k is the unit vector parallel to $(y_k - x', t_k)$. Taking the limit we obtain $\frac{\partial}{\partial x_N}(v(x',0) - u_R(x',0)) \le 0$ which contradicts (60) (recall that $\frac{\partial}{\partial \nu} = -\frac{\partial}{\partial x_N}$). This proves (61).

Suppose $x \in \Gamma_1^R$ is such that $|x| = r_0$ and $u_R(x',0) < v(x',0)$. The by continuity there is $\delta_x > 0$ such that (61) still holds.

Then by compactness for some $\delta > 0$ and $m > 0$ we have

$$u_R(x',t) < v(x',t), \quad \text{if } \left(|x'| - r_0\right)^2 + t^2 < \delta^2,\ t > m(|x'| - r_0),\ t > 0. \tag{62}$$

Now consider

$$z(x',t) = \rho^\alpha(\sin(\alpha(\theta - \theta_0)) + b(\theta - \theta_0)^2),$$

where (ρ, θ) are polar coordinates around $(r_0, 0)$ i.e.

$$|x'| = r_0 + \rho\cos(\theta), \quad t = \rho\sin(\theta).$$

We choose $\theta_0 \in (0, \frac{\pi}{2})$ is close enough to $\frac{\pi}{2}$ so that $\tan\theta_0 > m$. The parameters α, b are chosen later on.

We shall use the maximum principle to prove that for sufficiently small $\delta > 0$, $\varepsilon > 0$ we have

$$v(x',0) - u_R(x',0) \ge \varepsilon z(x',0), \quad r_0 - \delta < |x'| < r_0.$$

We have

$$\begin{aligned}
\Delta z = \frac{\rho^{\alpha-2}}{r_0 + \rho\cos(\theta)}\Big[&2br_0 + \alpha^2 br_0(\theta - \theta_0)^2 + \rho\Big(- (N-2)\alpha\sin(\theta)\cos(\alpha(\theta - \theta_0)) \\
&- 2b(N-2)\sin(\theta)(\theta - \theta_0) + 2b\cos(\theta) + (N-2)\alpha\cos(\theta)\sin(\alpha(\theta - \theta_0)) \\
&+ b\alpha(\alpha + N - 2)\cos(\theta)(\theta - \theta_0)^2\Big)\Big] \\
= \rho^{\alpha-2}&\Big(2b + \alpha^2 b(\theta - \theta_0)^2 + O(\rho)\Big), \quad \text{as } \rho \to 0
\end{aligned} \tag{63}$$

and, observing that for $\theta = \pi$ we are on $\partial\mathbb{R}^N_+$

$$\frac{\partial z}{\partial \nu} = \frac{1}{\rho}\frac{\partial z}{\partial \theta}\Big|_{\theta=\pi} = \rho^{\alpha-1}(\alpha\cos(\alpha(\pi - \theta_0)) + 2b(\pi - \theta_0)).$$

But $\pi - \theta_0 > \frac{\pi}{2}$. We fix $0 < \alpha < 1$ close enough to 1 such that $\cos(\alpha(\pi - \theta_0)) < 0$, and then take $b > 0$ small enough so that $\alpha\cos(\alpha(\pi - \theta_0)) + 2b(\pi - \theta_0) < 0$. Thus, setting

$$a \equiv -(\alpha\cos(\alpha(\pi - \theta_0)) + 2b(\pi - \theta_0)) > 0,$$

we have

$$\frac{\partial z}{\partial \nu} = -a\rho^{\alpha-1} \quad \text{for } \rho > 0 \text{ small.}$$

On the other hand, by (63)

$$\Delta z \ge \rho^{\alpha-2}\,(2b + O(\rho)) \quad \text{as } \rho \to 0 \tag{64}$$

uniformly for $\theta_0 < \theta < \pi$. Now consider the region

$$D = \{\,(x',t) \mid (|x'| - r_0)^2 + t^2 < \delta^2,\ t > m(|x'| - r_0),\ t > 0\,\} = \{\,(\rho,\theta) \mid 0 < \rho < \delta,\ \theta_0 < \theta < \pi\,\}$$

and write $\partial D = S_1 \cup S_2 \cup A$ where

$$S_1 = \{\,(\rho,\theta) \mid 0 < \rho < \delta,\ \theta = \theta_0\,\}, \quad S_2 = \{\,(\rho,\theta) \mid 0 < \rho < \delta,\ \theta = \pi\,\}$$

and

$$A = \{\,(\rho,\theta) \mid \rho = \delta,\ \theta_0 < \theta < \pi\,\}.$$

By (64) and choosing $\delta > 0$ smaller if necessary we achieve $\Delta z > 0$ in D. On S_1 we have $z = 0$ and $v - u_R > 0$. Now we seek $\varepsilon > 0$, $\delta > 0$ smaller than before such that

$$\inf_{S_2}\Big(\frac{\partial v}{\partial \nu} - \frac{\partial u_R}{\partial \nu}\Big) \geq \varepsilon \sup_{S_2} \frac{\partial z}{\partial \nu} = -a\varepsilon\delta^{\alpha-1}$$

and

$$\inf_A (v - u_R) > \varepsilon \sup_A z = \varepsilon \delta^{\alpha} C_1$$

where $C_1 = \sin(\alpha(\pi - \theta_0)) + b(\pi - \theta_0)^2$. Writing $K = -\inf_{S_2}(\frac{\partial v}{\partial \nu} - \frac{\partial u_R}{\partial \nu}) < \infty$ and $c_0 = \inf_A (v - u_R) > 0$ we first choose $\delta > 0$ small such that

$$\delta \frac{K}{a} < \frac{c_0}{C_1}$$

and then ε such that $\delta^{1-\alpha}\frac{K}{a} \leq \varepsilon < \frac{c_0}{C_1}\delta^{-\alpha}$.

The calculations above and the maximum principle then yield $v - u_R \geq \varepsilon z$ in D, which was the desired conclusion. Now this implies that $v - u_R$ is not differentiable at $(x_0', 0)$, a contradiction.

Step 2. We let $\lambda \uparrow \lambda^*$ and hence $R \to 0$. Since $0 \leq u_R \leq u_R(0) = 1$, $u_R \to u$ uniformly on compact sets of $\overline{\mathbb{R}^N_+}$ and u satisfies

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^N_+ \\ \dfrac{\partial u}{\partial \nu} = u^p & \text{on } \partial\mathbb{R}^N_+ \end{cases} \tag{65}$$

and

$$u(0) = 1.$$

Also, u satisfies

$$\int_{\mathbb{R}^N_+} |\nabla \varphi|^2 \geq p \int_{\partial\mathbb{R}^N_+} u^{p-1}\varphi^2 \qquad \forall \varphi \in C_0^\infty(\overline{\mathbb{R}^N_+}). \tag{66}$$

By (58) and the previous step we deduce

$$u \leq v \quad \text{in } \partial\mathbb{R}^N_+.$$

Let

$$\mu = \sup_{\partial\mathbb{R}^N_+} \frac{u}{v} \leq 1. \tag{67}$$

We claim that $\mu = 1$. Indeed, let

$$\tilde{u}(x) = c_1 \int_{\partial\mathbb{R}^N_+} \frac{u(y)^p}{|x-y|^{N-2}}\, dy \qquad x \in \mathbb{R}^N_+.$$

Then $\tilde{u}$ is harmonic in $\mathbb{R}^N_+$ and agrees with u on $\partial\mathbb{R}^N_+$. Since u is bounded by 1 and $\tilde{u}$ is bounded, we see that $\tilde{u} - u$ must be a constant. But then, since $\tilde{u}(x', 0) \to 0$ and $u(x', 0) \to 0$ as $|x'| \to \infty$ we see that $u \equiv \tilde{u}$.

Thus

$$u(x) = \int_{\partial\mathbb{R}^N_+} \frac{u(y)^p}{|x-y|^{N-2}}\, dy \leq c_1\mu^p \int_{\partial\mathbb{R}^N_+} \frac{v(y)^p}{|x-y|^{N-2}}\, dy = \mu^p v(x) \qquad x \in \partial\mathbb{R}^N_+.$$

This implies $\mu \leq \mu^p$ and since $\mu \neq 0$, we conclude $\mu = 1$.

Step 3. Observe that the supremum in (67) is not attained. Otherwise $v - u$ would achieve a minimum at a point $x \in \partial\mathbb{R}^N_+$, where the normal derivative would be zero. By Hopf's lemma, we would have $u \equiv v$, which is impossible since u is bounded and v is not. Let $x_k \in \partial\mathbb{R}^N_+$ be such that $|x_k| \to \infty$ and $\frac{u(x_k)}{v(x_k)} \to 1$. Let

$$u_k(x) = |x_k|^{\frac{1}{p-1}} u(|x_k|x).$$

Since v is invariant under the above transformation we have $u_k \le v$ in $\partial\mathbb{R}^N_+$. Thus, for a subsequence we have $u_k \to u_0$ and u_0 solves (65). Since $u_k(\frac{x_k}{|x_k|}) \to v(y)$ where $y = \lim \frac{x_k}{|x_k|}$, again using Hopf's lemma we see that $u_0 \equiv v$. But u_0 satisfies the condition (66), contradicting (57).

Proof of Theorem 1.8. Set $u = w_{\frac{1}{p-1}} - 1$ so that

$$\Delta u = 0 \qquad \text{in } \mathbb{R}^N_+ \tag{68}$$

$$\frac{\partial u}{\partial \nu} = C(N, \tfrac{1}{p-1})(1+u)^p \qquad \text{on } \partial\mathbb{R}^N_+ \tag{69}$$

Let $\Omega = \{x \in \mathbb{R}^N_+ \,|\, u(x) > 0\}$, $\Gamma_1 = \partial\Omega \cap \partial\mathbb{R}^N_+$, $\Gamma_2 = \partial\Omega \setminus \partial\mathbb{R}^N_+$. Then u is a singular solution to (68), (69) with

$$u = 0 \quad \text{on } \Gamma_2.$$

To apply Lemma 4.1 we need to verify that $u \in H^1(\Omega)$. We are assuming that $pC(N, \frac{1}{p-1}) \le H_N$ and $p \ge \frac{N}{N-2}$. Actually we must have $p > \frac{N}{N-2}$. For this it is convenient to observe that :

$$C(N,\alpha) = C(N, N-2-\alpha) \quad \forall 0 < \alpha < N-2, \quad \text{and} \tag{70}$$

$$\alpha \mapsto C(N,\alpha) \quad \text{is increasing for } 0 < \alpha < \tfrac{N-2}{2}. \tag{71}$$

Property (70) is direct from (16) and we leave (71) to the appendix. From these properties we deduce that $p > \frac{N}{N-2}$ and therefore $u \in H^1(\Omega)$. This solution satisfies the stability condition (44) if and only if (by scaling)

$$pC(N, \tfrac{1}{p-1}) \int_{\partial\mathbb{R}^N_+} \frac{\varphi^2}{|x|} \le \int_{\mathbb{R}^N_+} |\nabla\varphi|^2, \qquad \forall \varphi \in C^1_0(\overline{\mathbb{R}^N_+} \setminus \{0\})$$

which is guaranteed by Kato's inequality (9). Thus we may apply Lemma 4.1 and conclude that u is the extremal solution.

□

Appendix

Proof of (48)**.** We have $H_N = 2\left[\frac{\Gamma(\frac{N}{4})}{\Gamma(\frac{N}{4}-\frac{1}{2})}\right]^2 = 2f(N/4)^2$ where

$$f(z) = \frac{\Gamma(z)}{\Gamma(z-\frac{1}{2})} \tag{72}$$

and similarly we have $\lambda_{0,N} = \sqrt{\pi}\frac{N-3}{2}\frac{\Gamma\left(\frac{N}{2}-\frac{3}{2}\right)}{\Gamma\left(\frac{N}{2}-1\right)} = \sqrt{\pi}\frac{\Gamma\left(\frac{N}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{N}{2}-1\right)} = \sqrt{\pi} f\left(\frac{N}{2}-\frac{1}{2}\right)$. Then

$$\frac{H_N}{\lambda_{0,N}} = \frac{2f(N/4)^2}{\sqrt{\pi} f\left(\frac{N}{2}-\frac{1}{2}\right)}.$$

Since $H_{10} = \frac{9\pi}{8} > \lambda_{0,10} = \frac{35\pi}{32}$ it follows that

$$\frac{H_{10}}{\lambda_{0,10}} > 1.$$

On the other hand

$$\frac{H_9}{\lambda_{0,9}} = \frac{\left(\frac{5\Gamma(\frac{1}{4})^2}{12\pi}\right)^2}{\frac{16}{5}} \approx \frac{3.039}{3.333} < 1.$$

Let us compute

$$\begin{aligned}\frac{d}{dx}\frac{f(\frac{x}{4})^2}{f\left(\frac{x}{2}-\frac{1}{2}\right)} &= \frac{\frac{1}{2}f(\frac{x}{4})f'(\frac{x}{4})f\left(\frac{x}{2}-\frac{1}{2}\right) - \frac{1}{2}f(\frac{x}{4})^2 f'\left(\frac{x}{2}-\frac{1}{2}\right)}{f\left(\frac{x}{2}-\frac{1}{2}\right)^2} \\ &= \frac{f(\frac{x}{4})^2}{2f\left(\frac{x}{2}-\frac{1}{2}\right)}\left[\frac{f'(\frac{x}{4})}{f(\frac{x}{4})} - \frac{f'\left(\frac{x}{2}-\frac{1}{2}\right)}{f\left(\frac{x}{2}-\frac{1}{2}\right)}\right]\end{aligned}$$

Recall that

$$\Gamma'(z) = \psi_0(z)\Gamma(z), \qquad \text{where } \psi_0(z) = -\left(\frac{1}{z} + \gamma + \sum_{n=1}^{\infty}\left(\frac{1}{n+z} - \frac{1}{n}\right)\right).$$

In particular, ψ_0 is a positive increasing function on $]\Xi, \infty[$ where $\Xi \approx 1.4$ is the unique zero of ψ_0 in $\mathbb{R}_+$, and

$$f'(z) = \left[\psi_0(z) - \psi_0\left(z - \frac{1}{2}\right)\right]\frac{\Gamma(z)}{\Gamma(z-\frac{1}{2})} > 0 \qquad \text{for } z \in]1/2, \infty[.$$

Calculating

$$\frac{d}{dx}\frac{f'(x)}{f(x)} = \psi_0'(x) - \psi_0'(x - \frac{1}{2}) = \sum_{n=0}^{\infty}\frac{1}{(n+x)^2} - \frac{1}{(n+x-\frac{1}{2})^2} < 0 \quad \text{for } x > \frac{1}{2}. \tag{73}$$

So $\frac{f'(x)}{f(x)}$ is decreasing for $x > \frac{1}{2}$ and it follows that $\frac{f'(\frac{x}{4})}{f(\frac{x}{4})} - \frac{f'\left(\frac{x}{2}-\frac{1}{2}\right)}{f\left(\frac{x}{2}-\frac{1}{2}\right)} > 0$ for $x > 2$. Hence

$$H_N > \lambda_{0,N} \quad \text{only for } N \geq 10.$$

□

Proof of (71)**.** Using the notation (72) we see that

$$C(N,\alpha) = f\left(\frac{\alpha}{2} + \frac{1}{2}\right) f\left(\frac{N-1}{2} - \frac{\alpha}{2}\right).$$

Hence, for $0 < \alpha < \frac{N-2}{2}$ we have

$$\frac{d}{d\alpha}C(N,\alpha) = \frac{1}{2}C(N,\alpha)\left[\frac{f'\left(\frac{\alpha}{2}+\frac{1}{2}\right)}{f\left(\frac{\alpha}{2}+\frac{1}{2}\right)} - \frac{f'\left(\frac{N-1}{2}-\frac{\alpha}{2}\right)}{f\left(\frac{N-1}{2}-\frac{\alpha}{2}\right)}\right] \geq 0$$

by (73). □

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