

**Structure of potentials with  $N$  Higgs doublets**

C. C. Nishi\*

*Instituto de Física Teórica, UNESP, São Paulo State University, Rua Pamplona, 145, 01405-900, São Paulo, Brasil  
and Instituto de Física Gleb Wataghin, UNICAMP, PO Box 6165, 13083-970, Campinas, SP, Brasil*

(Received 19 June 2007; published 28 September 2007)

Extensions of the standard model with  $N$  Higgs doublets are simple extensions presenting a rich mathematical structure. An underlying Minkowski structure emerges from the study of both variable space and parameter space. The former can be completely parametrized in terms of two future lightlike Minkowski vectors with spatial parts forming an angle whose cosine is  $-(N-1)^{-1}$ . For the parameter space, the Minkowski parametrization enables one to impose sufficient conditions for bounded below potentials, characterize certain classes of local minima, and distinguish charge breaking vacua from neutral vacua. A particular class of neutral minima presents a degenerate mass spectrum for the physical charged Higgs bosons.

DOI: [10.1103/PhysRevD.76.055013](https://doi.org/10.1103/PhysRevD.76.055013)

PACS numbers: 12.60.Fr, 11.30.Qc, 14.80.Cp

**I. INTRODUCTION**

The scalar sector of the standard model (SM) is the only directly untested part of this successful model which accounts for all the variety of phenomena involving subnuclear particles [1]. The proper knowledge of the only elementary scalar in the SM, the Higgs, is critically important to test one of the major features of the SM, the Higgs mechanism, responsible to give masses to all massive gauge bosons and fermions and to hide the  $SU(2)_L \otimes U(1)_Y$  symmetry [2]. The discovery of the Higgs is eagerly awaited to happen in the LHC experiment [2].

Several theoretical reasons, however, force us to consider the possibility of more than one elementary scalar [3–5]. One of the reasons is the increasingly accepted notion that the SM is possibly a low energy manifestation of a more fundamental, yet unknown, theory such as grand unified theories, with or without supersymmetry, or extra-dimensional theories, which contain more scalars in general [6]. The search for physics beyond the SM is well motivated by several theoretical incompleteness features or problems the SM faces [6]. For example, the minimal supersymmetric SM (MSSM) requires two Higgs doublets from supersymmetry [7]. Another particular mechanism, the spontaneous  $CP$  breaking [8], generally needs more scalars to be implemented. Historically, the quest for alternative or additional  $CP$  violating sources was the reason to consider simple extensions of the SM containing more than one Higgs doublets, in particular, two and three Higgs doublets [8–10].

This work aims the study of the scalar potential of extensions of the SM with  $N$  Higgs doublets (NHDMs) [11–13]. Such models contain a reparametrization freedom [14] induced by  $SU(N)_H$  transformations on the  $N$  Higgs doublets which is physically irrelevant because they are in the same representation of the gauge group, i.e., they possess the same quantum numbers. Such reparametriza-

tion transformations are called horizontal transformations, acting on the horizontal space formed by the  $N$ -Higgs doublets [13]. Hence, two different potentials defined by two different sets of parameters but connected by some reparametrization transformation are physically equivalent. Properties such as  $CP$  symmetry or asymmetry is also independent of reparametrization which means any  $CP$  invariant potential, even with complex parameters, can be connected to a potential where all coefficients are real, i.e., manifestly  $CP$  symmetric [15]. Thus, reparametrization invariant quantities, such as the Jarlskog invariant [16] in the SM, can be constructed to quantify  $CP$  violation [13,17,18]. In Ref. [13], we tried to solve the question: what are the necessary and sufficient conditions for explicit and spontaneous  $CP$  violation for a given NHDM potential? We could solve partially the explicit  $CP$  violation conditions but the study of the different minima of the potential were not considered.

Concerning general NHDM potentials we can pose two questions: (1) how to find all the minima for a given potential specified by given parameters and (2) how to parametrize all physically permissible or interesting NHDM potentials and sweep all their parameter space. This work solves neither question (1) nor question (2) completely, but some sufficient physical conditions can be implemented and several consistency criteria can be formulated concerning question (2) while question (1) can be solved in some classified cases. Following the formalism adopted in Ref. [13] to study  $CP$  violation, and the extension for 2HDMs studied in Ref. [19], we will study the structure of NHDMs and the properties of the different nontrivial minima. These different minima can be first classified into two types: the usual neutral (N) minimum and the charge breaking (CB) minimum. The former can be further classified into neutral normal (NN) and  $CP$  breaking (CPB) minimum. With only one Higgs doublet, only the neutral normal minimum is possible. With more than one doublet, emerges the possibility of breaking also the electromagnetic symmetry (CB)

\*ccnishi@ift.unesp.br

[20,21] or the  $CP$  symmetry (CPB) [8] spontaneously. Since the  $CP$  properties were already considered in Ref. [13], we will concentrate on the differences between charge breaking and neutral vacua.

A rich mathematical structure also emerges from the study of the NHDM potential. We will see an underlying Minkowski structure will emerge, analogously to the 2HDM potential [19]. For example, it will be shown that the variable space lies inside and on the future light cone for an appropriately chosen set of  $N^2$  real variables. The Minkowski structure, however, will not be sufficient to characterize all the vacuum properties for  $N > 2$ . Nevertheless, it is possible to consider the Lorentz group  $SO(1, N^2 - 1)$ , containing the group  $SU(N)_H$ , as a powerful parametrization tool. For example, a sufficient condition for bounded below potentials can be formulated within this context. Various properties of the vacuum, such as the distinction between charge breaking and neutral vacuum, can be also formulated in a Minkowskian language. Using a certain gauge choice, we will also see that the variable space can be parametrized by two future lightlike vectors for which the cosine of the angle between their spatial parts is the rational number  $-(N - 1)^{-1}$ .

From the physical point of view, interesting predictive information can be extracted for certain limits. For example, there are models preserving EM symmetry which exhibits a degenerate mass spectrum for physical charged Higgs bosons.

The outline is as follows: In Sec. II we analyze the Minkowski structure of the NHDM potentials and introduce some useful mathematical definitions. The section is divided in the analysis of the variable space (Sec. II A) and the parameter space (Sec. II B). In Sec. III we analyze the stationary points of the potential, introduce the physical charged Higgs basis (Sec. III A), and analyze the properties of charge breaking (Sec. III B) and neutral (Sec. III C) vacua. The conclusions are discussed in Sec. IV.

## II. THE STRUCTURE OF THE NHDM POTENTIAL

In a previous work [13], it was shown that a general gauge invariant potential with  $N \geq 2$  SM Higgs doublets  $\Phi_a = (\phi_{a1}, \phi_{a2})^T$ ,  $a = 1, \dots, N$ , can be written solely in terms of the real variables

$$\mathbb{A}^\mu \equiv \frac{1}{2} \Phi_a^\dagger (\lambda^\mu)_{ab} \Phi_b, \quad \mu = 0, 1, \dots, d, \quad (1)$$

where  $\lambda^0 = \sqrt{\frac{2}{N}} \mathbb{1}$  and  $\{\lambda^i\}$  are the  $d = N^2 - 1$  Hermitian generators of  $SU(N)_H$  in the fundamental representation, obeying the normalization  $\text{Tr}[\lambda^\mu \lambda^\nu] = 2\delta^{\mu\nu}$ . There is, nevertheless, a more appropriate normalization of the variable  $\mathbb{A}^0$  in Eq. (1), when  $N > 2$ , which allows us to uncover a Minkowski structure in the variable space of the NHDM potential, extending then the 2HDM case [19].

Defining

$$r^\mu(\Phi) \equiv \Phi_a^\dagger (T^\mu)_{ab} \Phi_b, \quad \mu = 0, 1, \dots, d, \quad (2)$$

where

$$T^\mu \equiv \left( \sqrt{\frac{N-1}{2N}} \mathbb{1}_N, \frac{1}{2} \lambda^i \right), \quad (3)$$

it is proved in Appendix A that

$$r_\mu r^\mu = \phi_{a1}^* \phi_{a1} \phi_{b2}^* \phi_{b2} - |\phi_{a1}^* \phi_{a2}|^2 \geq 0, \quad (4)$$

assuming the usual Minkowski metric  $g_{\mu\nu} = \text{diag}(1, -\mathbb{1}_d)$ , the definition of the covariant vector  $r_\mu \equiv g_{\mu\nu} r^\nu$ , and the conventional sum over repeated indices. Equation (4) then restricts the space of the variables  $r^\mu$  to be inside and on the future light cone

$$LC^\dagger \equiv \{x^\mu \in \mathbb{R}^{1,d} | x^\mu x_\mu \geq 0, x_0 > 0\}, \quad (5)$$

in a Minkowski spacetime  $\mathbb{R}^{1,d}$ . We will see in Sec. II A that the variables  $r^\mu = r^\mu(\Phi)$  in Eq. (2) do not cover the whole  $LC^\dagger$  neither do they form a vector subspace. It is important to stress that the quantity in Eq. (4) calculated for the vacuum expectation value signals a charge breaking vacuum for nonzero values [19].

Using the Minkowski variables of Eq. (2) we can write the most general gauge invariant potential in the form

$$V(r) = M_\mu r^\mu + \frac{1}{2} \Lambda_{\mu\nu} r^\mu r^\nu, \quad (6)$$

where  $M^\mu$  is a general vector and  $\Lambda^{\mu\nu}$  is a general symmetric rank-2 tensor in Minkowski space. The relation between the parameters  $M$  and  $\Lambda$  and the more usual parameters  $Y$  and  $Z$ , used to write the potential in the form [15,22]

$$V(\Phi) = Y_{ab} \Phi_a^\dagger \Phi_b + \frac{1}{2} Z_{(ab)(cd)} (\Phi_a^\dagger \Phi_b)^* (\Phi_c^\dagger \Phi_d), \quad (7)$$

can be found in Appendix B. The explicit parametrization for the 2HDM can be found in Ref. [13].

### A. Variable space

The vector  $r^\mu$  in Eq. (2) defines a particular mapping of  $\{\Phi_a\}$  in  $\mathbb{C}^N \otimes \mathbb{C}^2$  into  $\mathbb{R}^{1,d}$ . The former space can be parametrized by  $4(N-1)$  real parameters, with the  $SU(2)_L \otimes U(1)_Y$  gauge freedom already taken into account, while the latter space requires  $N^2 = d+1$  parameters. Since  $N^2 \geq 4(N-1)$  for  $N \geq 2$ , the mapping is obviously not surjective. The image of such mapping defines therefore a space

$$\mathcal{V}_\Phi \equiv \{x^\mu \in LC^\dagger | x^\mu = r^\mu(\Phi)\}, \quad (8)$$

contained in  $LC^\dagger$ . We will then analyze the properties of  $\mathcal{V}_\Phi$  and seek a criterion to identify if a vector  $x^\mu$  in  $LC^\dagger$  is also in  $\mathcal{V}_\Phi$ .

First, define the bijective mapping  $f^\mu$  from the set of Hermitian complex  $N \times N$  matrices, denoted by

$\mathcal{M}_h(N, c)$ , into  $\mathbb{R}^{1,d}$ :

$$f^\mu(h) \equiv \text{Tr}[T^\mu h]. \quad (9)$$

This mapping is invertible and therefore bijective, since, defining

$$\tilde{x} \equiv 2x^\mu \tilde{T}_\mu, \quad (10)$$

where

$$\tilde{T}^\mu \equiv \left( \frac{T^0}{N-1}, -T^i \right), \quad (11)$$

we identify

$$h = \tilde{x}, \quad (12)$$

once the equality  $f^\mu(h) = x^\mu$  holds. Such identity can be easily verified by using the relation

$$2 \text{Tr}[T^\mu \tilde{T}^\nu] = g^{\mu\nu}. \quad (13)$$

We can express the Minkowski inner product in  $\mathcal{M}_h(N, c)$  by defining a function  $\Delta$  of a Hermitian matrix  $h$  as

$$\Delta(h) \equiv \frac{1}{2}[(\text{Tr}h)^2 - \text{Tr}(h^2)]. \quad (14)$$

It is easy to verify using the trace properties of  $\tilde{T}^\mu$  that

$$x^\mu x_\mu = \Delta(\tilde{x}). \quad (15)$$

It is only for the particular case of  $N = 2$  that we have  $\Delta(\tilde{x}) = \det \tilde{x}$ , allowing the extension from  $SU(2)$  to  $SL(2, c)$  that preserves the Minkowski metric and therefore can represent the group of proper Lorentz transformations.

Now we can realize the definition in Eq. (2) corresponds to the  $f^\mu$  mapping of a particular class of Hermitian matrices. Defining vectors  $u$  and  $w$  in  $\mathbb{C}^N$  such that

$$u_a \equiv \phi_{a1}, \quad (16)$$

$$w_a \equiv \phi_{a2}, \quad (17)$$

we can see that

$$r^\mu(\Phi) = f^\mu(uu^\dagger + ww^\dagger) = f^\mu(uu^\dagger) + f^\mu(ww^\dagger). \quad (18)$$

From Eq. (15) and the property  $h^2 = \text{Tr}[h]h$  for  $h = uu^\dagger$ , we see  $f^\mu(uu^\dagger)$  and  $f^\mu(ww^\dagger)$  lie on the future light cone. Thus  $r^\mu(\Phi)$  is a sum of two future lightlike vectors:

$$r^\mu(\Phi) = x^\mu + y^\mu, \quad (19)$$

where  $x^\mu x_\mu = 0$ ,  $y^\mu y_\mu = 0$ ,  $x^0, y^0 > 0$ . Note that the splitting of Eq. (19) into the sum of  $x^\mu = f^\mu(uu^\dagger)$  and  $y^\mu = f^\mu(ww^\dagger)$  is not gauge invariant since  $SU(2)_L$  gauge transformations can mix  $u$  with  $w$ .

Now we can state the *criterion*:

a vector  $x^\mu$  in  $LC^\dagger$  is also in  $\mathcal{V}_\Phi$  if, and only if, the corresponding matrix  $\tilde{x}$  has rank two or less and its nonzero eigenvalues are positive. A vector  $x^\mu$  in  $\mathcal{V}_\Phi$  is future lightlike if, and only if,  $\tilde{x}$  has rank one.

The proof for necessity is trivial, since any matrix of the form  $h = uu^\dagger + ww^\dagger$  has rank two or less and its non-null eigenvalues are positive. The converse can be proved by diagonalizing  $\tilde{x}$ . If  $\tilde{x}$  has rank two or less and its non-null eigenvalues are positive, it can be written in the form

$$\tilde{x} = \lambda_1^2 v_1 v_1^\dagger + \lambda_2^2 v_2 v_2^\dagger, \quad (20)$$

where  $\lambda_i^2$  are the positive eigenvalues and  $v_i$  their respective normalized eigenvectors. With the identification  $u = \lambda_1 v_1$  and  $w = \lambda_2 v_2$  we see  $x^\mu = f^\mu(uu^\dagger + ww^\dagger)$  is in  $\mathcal{V}_\Phi$  and we complete our proof. Setting  $\lambda_2$  to zero and using Eq. (15), we obtain the rank one subcase. One last remark concerns the ambiguity in associating  $\tilde{x}$  with  $h = uu^\dagger + ww^\dagger$ , since  $u$  and  $w$  need not be orthogonal. However, the gauge freedom allows us to choose a particular representative of  $\Phi$  for which  $u, w$  are orthogonal, i.e.,

$$u^\dagger w = 0. \quad (21)$$

The proof is shown in Appendix C. With the choice of Eq. (21), the mapping between  $x^\mu$  in  $\mathcal{V}_\Phi$  and  $h = uu^\dagger + ww^\dagger$  in  $\mathcal{M}_h(N, c)$  is unambiguous, once an ordering for the eigenvalues of  $\tilde{x}$  is defined, hence  $\Phi$  may also be determined uniquely, except for rephasing transformations on  $u, w$  which does not alter the condition (21). Thus  $4(N-1)$  real parameters are necessary to parametrize  $u, w$  faithfully considering condition (21) and the rephasing freedom for  $u$  and  $w$ . Therefore, the same number of parameters are necessary to parametrize  $\mathcal{V}_\Phi$ . We will adopt the choice of Eq. (21) from this point on. Since the sum of two rank two Hermitian matrices can be equal or greater than two, we also see  $\mathcal{V}_\Phi$  does not form a vector subspace of  $\mathbb{R}^{1,d}$ . The exception happens for  $N = 2$  when they form a subspace and  $\mathcal{V}_\Phi = LC^\dagger$ .

An interesting feature arises with the adoption of Eq. (21): the cosine of the angle between the spatial parts of  $x^\mu = f^\mu(uu^\dagger)$  and  $y^\mu = f^\mu(ww^\dagger)$  is a rational number. Such property can be seen by

$$r^\mu(\Phi)r_\mu(\Phi) = 2x_\mu y^\mu = 2x^0 y^0 (1 - \cos\theta), \quad (22)$$

where  $\cos\theta = \frac{x \cdot y}{|x||y|}$ . Equations (4) and (15) imply

$$\Delta(uu^\dagger + ww^\dagger) = |u|^2 |w|^2 = \frac{2N}{N-1} x^0 y^0, \quad (23)$$

which yields the relation

$$\cos\theta = \frac{-1}{N-1}. \quad (24)$$

The specific angles vary from  $\theta = \pi$  ( $N = 2$ ) to  $\theta \rightarrow \pi/2^+$  ( $N \rightarrow \infty$ ). In particular, for  $N = 2$ , the vectors  $x^\mu$  and  $y^\mu$  lie in opposite directions on the future light cone.

## B. Parameter space

There are two advantages of parametrizing the potential in the form of Eq. (6) compared with the parametrization of Eq. (7). First, we can consider any vector  $M$  with  $N^2$  components and any symmetric tensor  $\Lambda$  with  $N^2 \times N^2$  entries as parameters, restricted only by physical requirements which will be further discussed, while the tensor  $Z_{(ab)(cd)}$  in Eq. (7) contains redundancies by index exchange [13]. Therefore, we can adopt the parametrization of Eq. (6) as the starting point to analyze physical features such as the requirement of bounded below potential or the possibility of having CB or CPB vacua.

We have at our disposal  $N^2(N^2 + 1)/2$  real parameters in  $\Lambda$  and  $N^2$  real parameters in  $M$ . The number of physically significant parameters, however, is fewer due to the reparametrization freedom which identifies all potentials connected by horizontal transformations as physically equivalent. In this context, the relevant horizontal group is  $SU(N)_H$  [13], acting on the horizontal space spanned by the Higgs doublets. The action of a horizontal transformation  $U$  in the fundamental representation  $\mathbf{N}$  of  $SU(N)_H$  can be written as

$$\Phi_a \rightarrow U_{ab} \Phi_b. \quad (25)$$

While the quadratic variables  $r^\mu$ , transform leaving  $r^0$  invariant and  $r^i$  transforming according to the adjoint representation  $\mathbf{d}$  of  $SU(N)_H$ , in accordance to the branching  $\bar{\mathbf{N}} \otimes \mathbf{N} = \mathbf{d} \oplus \mathbf{1}$ . Since  $\text{adj}SU(N)_H$  can be obtained by exponentiation of the algebra spanned by  $i(T_j)_{kl} = f_{jkl}$  which is real and antisymmetric,  $\text{adj}SU(N)_H$  forms a subgroup of  $SO(d)$ .

Because of the  $SU(N)_H$  reparametrization freedom, since the action of  $\text{adj}SU(N)_H$  is effective on  $LC^\dagger$ , i.e., some orbits in  $LC^\dagger$  are not trivial, the physically distinct potentials can be parametrized by only  $N^2 + \frac{1}{2}N^2(N^2 + 1) - (N^2 - 1) = \frac{1}{2}N^2(N^2 + 1) + 1$  real parameters.<sup>1</sup> For  $N = 2$ , such minimal number of parameters can be easily achieved by diagonalizing the  $3 \times 3$  matrix  $\Lambda_{ij}$ , which gives 11 parameters needed to define  $M$  (4),  $\Lambda_{00}$  (1),  $\Lambda_{0i}$  (3), and  $\Lambda_{ij}$  (3). When the potential exhibits  $CP$  invariance, such a basis, called canonical  $CP$  basis in Ref. [13], coincides with the real basis [15] for which all coefficients in the potential are real. The minimal parametrization for  $N > 2$  is not explicitly known [13].

The second advantage of Eq. (6) concerns the possibility of extending  $\text{adj}SU(N)_H$  to  $SO(d)$  and then to  $SO(1, d)$  which is the group of homogeneous proper Lorentz transformations in  $\mathbb{R}^{1,d}$ . The importance of such extension relies on the fact that  $SO(1, d)$  leaves  $LC^\dagger$  invariant and acts

transitively on it, i.e., any two vectors  $x^\mu, y^\mu$  in  $LC^\dagger$  can be connected by  $SO(1, d)$ . If the parameter space generated by  $r^\mu(\Phi)$  covered the whole  $LC^\dagger$ , we could parametrize all physically inequivalent NHDM potentials by parametrizing the cosets  $SO(1, d)/\text{adj}SU(N)_H$  acting on some fixed representative classes of  $\{M, \Lambda\}$ . For example, for  $N = 2$ , all  $LC^\dagger$  can be covered by  $r^\mu(\Phi)$  and all physically bounded below potentials can be parametrized by parameters  $M$  (4 parameters),  $\Lambda = \text{diag}(\Lambda_0, \Lambda_i)$  (4 parameters), with  $\Lambda_i > -\Lambda_0$ , and a boost parameter  $\xi$  (3 parameters), needed to generate the  $\Lambda_{0i}$  components [19]. Boosts belong to  $SO(1, 3)/\text{adj}SU(2)_H$  and, furthermore, specially for  $N = 2$ , they can be implemented over  $\Phi$  with the extension of  $SU(2)_H$  to  $SL(2, c)$ .

Nevertheless, although the permissible variable space only covers  $\mathcal{V}_\Phi$ , which is smaller than  $LC^\dagger$  when  $N > 2$ , we can cover a large class of physically acceptable potentials by considering all  $r^\mu$  in  $LC^\dagger$  and imposing the physical restrictions on the set  $\{M, \Lambda\}$ . The physical restrictions to consider are (i) bounded below potential and (ii) the existence of nontrivial extrema,  $\langle \Phi \rangle \neq 0$ .

We can impose the restriction (i) by requiring [19]

P1:  $\Lambda$  is diagonalizable by  $SO(1, d)$ , i.e., there is a basis where

$$\Lambda_{\mu\nu} = \text{diag}(\Lambda_0, \Lambda_i), \quad (26)$$

P2:  $\Lambda_0 > 0$  and  $\Lambda_i > -\Lambda_0$ .

The conditions P1 and P2 are necessary and sufficient to guarantee the quartic part of the potential in Eq. (6) to be positive definite for all  $r^\mu$  in  $LC^\dagger$ . Since the variable space does not cover the whole  $LC^\dagger$  but only  $\mathcal{V}_\Phi$ , for  $N > 2$ , the above conditions are only *sufficient* to guarantee the positivity of the quartic part of the potential. Obviously, the class of potentials with the quartic part positive definite for all  $r^\mu$  in  $\mathcal{V}_\Phi$  is larger. The proof of P1 and P2 follows analogously to the 2HDM case where the group is  $SO(1, 3)$  [19]. The treatment of general diagonalizable tensors in  $SO(1, n - 1)$  can be found in Ref. [23].

The restriction (ii) of nontrivial extrema will be considered in the next section where the properties of stationary points will be analyzed. One can say, however, that to ensure the existence of nontrivial stationary points ( $\langle \Phi \rangle \neq 0$ ), it is necessary to have the quadratic part of the potential acquiring negative values for some  $\Phi$ . The latter is only possible when  $Y$  in Eq. (7) has at least one negative eigenvalue.

## III. STATIONARY POINTS

To find the stationary points we differentiate  $V$  in Eq. (6):

$$\frac{\partial}{\partial \phi_{ai}^*} V(\Phi) = \frac{\partial}{\partial r^\mu} V(r) \frac{\partial r^\mu}{\partial \phi_{ai}^*} = \mathbb{M}_{ab} \phi_{bi}, \quad (27)$$

<sup>1</sup>For a  $CP$  invariant potential, in the real basis, we can count  $N(N + 1)/2$  (quadratic) and  $N^2(N^2 + 3)/4$  (quartic) parameters. If we take the  $SO(N)_H$  reparametrization freedom into account, the potential can be parametrized by a total of  $N + N^2(N^2 + 3)/4$  real parameters.

where

$$\mathbb{M} \equiv X_\mu T^\mu, \quad (28)$$

$$X_\mu(r^\mu) \equiv M_\mu + \Lambda_{\mu\nu} r^\nu. \quad (29)$$

The stationary points  $\langle \Phi \rangle$  correspond to the roots of Eq. (27), i.e., solutions of

$$\langle (\mathbb{M} \otimes 1_2) \Phi \rangle = 0, \quad (30)$$

which requires

$$\det \langle \mathbb{M} \rangle = 0 \quad (31)$$

for nontrivial solutions  $\langle \Phi \rangle \neq 0$ . The brackets  $\langle \rangle$  mean to take expectation values on all fields  $\Phi$ , including on  $\mathbb{M}$ .

Rewriting Eq. (30) in terms of  $u$ ,  $w$  in Eqs. (16) and (17), we have

$$\langle \mathbb{M} u \rangle = 0, \quad \langle \mathbb{M} w \rangle = 0. \quad (32)$$

If  $\langle u \rangle$  and  $\langle w \rangle$  are non-null and noncollinear, Eq. (32) means that  $\langle \mathbb{M} \rangle$  has two zero eigenvalues and  $\langle u \rangle$ ,  $\langle w \rangle$  are the respective eigenvectors. From

$$\langle \Phi^\dagger \Phi \rangle = \langle u^\dagger u \rangle + \langle w^\dagger w \rangle, \quad (33)$$

it is necessary that at least one of  $\langle u \rangle$  or  $\langle w \rangle$  be non-null to have a nontrivial vacuum expectation value (VEV). We can then classify CB and N stationary points depending on

- (i) cond. CB:  $\langle r^\mu r_\mu \rangle > 0$ . Equivalently, both  $\langle u \rangle$  and  $\langle w \rangle$  are non-null and noncollinear.
- (ii) cond. N:  $\langle r^\mu r_\mu \rangle = 0$ . Equivalently, either  $\langle u \rangle$  or  $\langle w \rangle$  is null or they are collinear.

On the other hand, multiplying  $\langle \Phi \rangle^\dagger$  on the left of Eq. (30) yields

$$\langle X_\mu r^\mu \rangle = 0. \quad (34)$$

For  $\langle r^\mu \rangle$  timelike, any vector orthogonal, with respect to the Minkowski metric, have to be spacelike [24]. For  $\langle r^\mu \rangle$  lightlike only (a) lightlike collinear vectors and (b) spacelike vectors can be orthogonal [24]. Then we can classify the solutions of Eq. (34) into three types, when  $\langle r^\mu \rangle \neq 0$  and in  $LC^\dagger$ :

- (I) Trivial solution with  $\langle X_\mu \rangle = 0$  and  $\langle \mathbb{M} \rangle = 0$ ; EM symmetry can be broken or not.<sup>2</sup>
- (II) Solution with  $\langle X_\mu X^\mu \rangle = 0$ ,  $\langle X_\mu \rangle \neq 0$ ; EM symmetry is always preserved and  $\langle X \rangle^\mu = \alpha \langle r^\mu \rangle$  corresponding to case (a).
- (III) Solution with  $\langle X_\mu X^\mu \rangle < 0$ ,  $\langle X_\mu \rangle \neq 0$ ; EM symmetry can be broken ( $\langle r^\mu r_\mu \rangle > 0$ ) or not ( $\langle r^\mu r_\mu \rangle = 0$ ).

Note that type (I) solutions also correspond to the stationary points of  $V(r^\mu)$  with respect to  $r^\mu$ .

<sup>2</sup>Note that, for neutral solutions,  $\langle \mathbb{M} \rangle = 0$  means that all charged scalars, i.e., one charged Goldstone and  $N - 1$  physical charged Higgs bosons, are massless.

Let us consider some special cases: For  $N = 2$ , for which the identity  $\det \tilde{x} = \Delta(\tilde{x})$  is valid, there are only solutions of type (I) and (II) since Eq. (31) implies  $\langle X_\mu X^\mu \rangle = 0$ . Furthermore, any charge breaking solution is of type (I). For  $N = 3$ , the type (III) solution is present and because we need two null eigenvalues for  $\langle \mathbb{M} \rangle$ ,  $\langle X^\mu \rangle$  must be in the cone defined by  $(N - 1)^2 X_0^2 - \mathbf{X}^2 = 0$ , i.e.,  $\langle X^\mu \rangle$  is spacelike. The proof is shown in Appendix D.

Now we can seek the explicit solutions. For type (I) solutions, an explicit expression can be given,

$$\langle r^\mu \rangle = -(\Lambda^{-1})^\mu{}_\nu M^\nu. \quad (35)$$

Of course,  $\langle r^\mu \rangle$  should be restricted to  $\mathcal{V}_\Phi$  which only happens when  $-M^\mu$  is in the image of  $\mathcal{V}_\Phi$  by  $\Lambda^\mu{}_\nu$  [19]. If  $\Lambda$  is not invertible, it is necessary to take the inverse only over the non-null space.

For type (II) solutions,  $\langle r^\mu \rangle$  should satisfy

$$\langle \Lambda_{\mu\nu} r^\nu - \alpha r_\mu \rangle = -M_\mu, \quad (36)$$

where  $\alpha$  is an unknown parameter which has to be determined from Eq. (36) and the constraint that  $r^\mu$  should be in  $\mathcal{V}_\Phi$ . Obviously, there may be more than one of such solutions with different  $\alpha$ , as it is for the  $N = 2$  case [19].

The type (III) solutions are not explicitly expressible and involve nonlinear equations in Eq. (32).

Let us analyze the general properties of the potential expanded around any stationary point. The expansion is induced by the replacements

$$\Phi \rightarrow \Phi + \langle \Phi \rangle, \quad (37)$$

$$r^\mu \rightarrow r^\mu + \langle r^\mu \rangle + s^\mu, \quad (38)$$

where

$$s^\mu \equiv \langle \Phi \rangle^\dagger T^\mu \Phi + \Phi^\dagger T^\mu \langle \Phi \rangle, \quad (39)$$

$$= f^\mu (u \langle u \rangle^\dagger) + f^\mu (w \langle w \rangle^\dagger) + \text{H.c.} \quad (40)$$

Thus,

$$V(\Phi + \langle \Phi \rangle) = V_0 + V_2 + V_3 + V_4, \quad (41)$$

where

$$V_0 = V(\langle r^\mu \rangle), \quad (42)$$

$$V_2 = \Phi^\dagger \langle \mathbb{M} \rangle \Phi + \frac{1}{2} \Lambda_{\mu\nu} s^\mu s^\nu, \quad (43)$$

$$V_3 = \Lambda_{\mu\nu} s^\mu r^\nu, \quad (44)$$

$$V_4 = \frac{1}{2} \Lambda_{\mu\nu} r^\mu r^\nu. \quad (45)$$

To guarantee the stationary point is a local minimum, it is necessary and sufficient to have the mass matrix after spontaneous symmetry breaking (SSB), extractable from Eq. (43), to be positive semidefinite. On the other hand, due to Eq. (34) and the positivity of  $V_4$ , we have

$$V_0 = \frac{1}{2} M_\mu \langle r^\mu \rangle = -\frac{1}{2} \Lambda_{\mu\nu} \langle r^\mu \rangle \langle r^\nu \rangle < 0. \quad (46)$$

The last inequality means any nontrivial stationary point lies deeper than the trivial extremum  $\langle \Phi \rangle = 0$ .

### A. Physical charged Higgs basis

We can write the potential (41) in an explicit basis where the physical degrees of freedom can be more easily extracted. For such a purpose we choose the physical charged Higgs (PCH) basis [25] where

$$\begin{aligned} \langle w \rangle &= \begin{pmatrix} 0 \\ \vdots \\ 0 \\ |\langle w \rangle| \end{pmatrix} = |\langle w \rangle| e_N, \\ \langle u \rangle &= \begin{pmatrix} 0 \\ \vdots \\ |\langle u \rangle| \\ 0 \end{pmatrix} = |\langle u \rangle| e_{N-1}, \end{aligned} \quad (47)$$

where  $e_i$ ,  $i = 1, \dots, N$  defined by  $(e_i)_j = \delta_{ij}$  are the canonical basis vectors. The module  $|\langle w \rangle|$  denotes the square root of  $\langle w^\dagger w \rangle = \langle w \rangle^\dagger \langle w \rangle$ . Such a choice is always allowed by the  $SU(N)_H$  reparametrization freedom, once the condition (21) is met. Although there is an additional  $SU(N-1)$  or  $SU(N-2)$  reparametrization freedom in the subspace orthogonal to  $\langle w \rangle \neq 0$  or  $\langle w \rangle, \langle u \rangle \neq 0$ , which need to be fixed to specify the PCH basis. Conventionally, we choose  $\langle w \rangle$  to be always non-null from the requirement of non-trivial vacuum. Therefore,  $\langle u \rangle \neq 0$  or  $\langle u \rangle = 0$  correspond, respectively, to the charge breaking vacuum (CBV) and the neutral vacuum (NV) solutions.

In the PCH basis

$$f^\mu(\langle w w^\dagger \rangle) = \langle w^\dagger w \rangle \sqrt{\frac{N-1}{2N}} n^\mu, \quad (48)$$

$$f^\mu(\langle u u^\dagger \rangle) = \langle u^\dagger u \rangle \sqrt{\frac{N-1}{2N}} n'^\mu, \quad (49)$$

where  $n^\mu$  and  $n'^\mu$  have non-null components

$$(n^0, n^{N-2}, n^{N-1}) = (1, 0, -1), \quad (50)$$

$$(n'^0, n'^{N-2}, n'^{N-1}) = \left(1, -\frac{\sqrt{N(N-2)}}{N-1}, \frac{1}{N-1}\right), \quad (51)$$

given the ordering of  $\mu$  following

$$\{T^0, h_a, \mathcal{S}_{ab}, \mathcal{A}_{ab}\}, \quad (52)$$

with  $a = 1, \dots, N-1$ ,  $b = 1, \dots, N$ , and  $a < b$ , denoting the non-null entries of  $2(\mathcal{S}_{ab})_{ab} = 2(\mathcal{S}_{ab})_{ba} = 1$  and  $2(\mathcal{A}_{ab})_{ab} = -2(\mathcal{A}_{ab})_{ba} = -i$  [13], which are the combination of ladder operators analogous to  $\sigma^1$  and  $\sigma^2$  for  $SU(2)$ . The matrices  $h_a$  form the Cartan subalgebra which

can be chosen diagonal. Notice that Eqs. (50) and (51) satisfy Eq. (24).

From Eq. (32), we have for  $\langle w \rangle \neq 0$

$$\langle \mathbb{M}_{aN} \rangle = \langle \mathbb{M}_{Na} \rangle = 0, \quad (53)$$

for all  $a = 1, \dots, N$ . In addition, if  $\langle u \rangle \neq 0$  (CBV), we have

$$\langle \mathbb{M}_{a,N-1} \rangle = \langle \mathbb{M}_{N-1,a} \rangle = 0, \quad (54)$$

reducing the non-null matrix to its upper-left  $(N-1) \times (N-1)$  ( $\langle u \rangle = 0$ ) or  $(N-2) \times (N-2)$  ( $\langle u \rangle \neq 0$ ) submatrix. In both cases we can use the remaining reparametrization freedom to choose  $\langle \mathbb{M} \rangle$  diagonal

$$\langle \mathbb{M} \rangle = \begin{cases} \text{diag}(m_a^2, 0, 0), & a = 1, \dots, N-2 \text{ for } \langle u \rangle \neq 0, \\ \text{diag}(m_a^2, 0), & a = 1, \dots, N-1 \text{ for } \langle u \rangle = 0. \end{cases} \quad (55)$$

This form can be always achieved because the remaining  $SU(N-1)$  or  $SU(N-2)$  reparametrization freedom leaves  $\langle r^\mu \rangle$  invariant. Equation (55) defines the PCH basis uniquely if the eigenvalues  $m_a^2$  are ordered, assuming they are not degenerate.

The null eigenvalues of Eq. (55) correspond to the Goldstone modes for the combination of fields not present in  $s^\mu(\Phi)$  in Eq. (43). The four massless Goldstone modes are

$$\sqrt{2} \text{Im}(w_N), \quad \sqrt{2} \text{Im}(u_{N-1}), \quad (56)$$

and the fields  $R, I$  proportional, by real normalization constants, to

$$R \propto |\langle u \rangle| \text{Re}(w_{N-1}) - |\langle w \rangle| \text{Re}(u_N), \quad (57)$$

$$I \propto |\langle u \rangle| \text{Im}(w_{N-1}) + |\langle w \rangle| \text{Im}(u_N). \quad (58)$$

To find the Goldstone modes for the neutral vacuum solution it is sufficient to set  $\langle u \rangle = 0$  in the equations above and disconsider Eq. (54) which makes  $\sqrt{2} \text{Im}(u_{N-1})$  also massive. The explicit form of  $s^\mu(\Phi)$  in this basis is shown in Appendix E.

### B. Charge breaking vacuum

A vacuum expectation value  $\langle \Phi \rangle$  breaking EM symmetry (CBV) [20], is characterized by cond. CB stated in Sec. III. They can be of type (I) or (III). To assure two zero eigenvalues we must have

$$\det \langle \mathbb{M} \rangle = (-1)^{N-1} \gamma_N(\langle \mathbb{M} \rangle) = 0, \quad \gamma_{N-1}(\langle \mathbb{M} \rangle) = 0. \quad (59)$$

The explicit forms of the matricial functions  $\gamma_k$  are unimportant here, except that knowing the traces  $\text{Tr}[\langle \mathbb{M} \rangle^j]$  from  $j = 1, \dots, k$  determines  $\gamma_k$  uniquely. The explicit form can be found in Eq. (D5). Equation (59) defines two equations for  $\langle r^\mu \rangle$  in addition to the restriction that  $\langle r^\mu \rangle$  belongs to  $\mathcal{V}_\Phi$ . Then, the possible vectors  $\langle u \rangle$  and  $\langle w \rangle$  extracted from

the possible  $\langle r^\mu \rangle$ , through the procedure in Eq. (20), should be the eigenvectors of  $\langle \mathbb{M} \rangle$  with eigenvalue zero.

Some conditions, however, can be extracted in the PCH basis. From  $\langle w^\dagger \mathbb{M} w \rangle = 0$  and  $\langle u^\dagger \mathbb{M} u \rangle = 0$ , we have, respectively,

$$\langle X_0 \rangle = \langle X_{N-1} \rangle, \quad (60)$$

$$\sqrt{N(N-2)} \langle X_{N-2} \rangle = (N-1) \langle X_0 \rangle + \langle X_{N-1} \rangle = N \langle X_0 \rangle. \quad (61)$$

Then,

$$-\langle X_\mu X^\mu \rangle \geq \langle X_{N-1}^2 + X_{N-2}^2 - X_0^2 \rangle = \frac{N}{N-2} \langle X_0^2 \rangle, \quad (62)$$

confirming, for  $N > 2$ , that all charge breaking solutions are of type (III) unless  $\langle X_i \rangle = 0$ , which implies a type (I) solution.

For type (I) solutions, one can see from Eq. (43) that the masses of all scalars will depend only on  $\Lambda$  which has to be positive definite in the basis defined by the non-Goldstone fields; such a condition assures the stationary point is a local minimum. In the PCH basis we can extract the mass matrix from the field combinations  $s^\mu(\Phi)$  in Appendix E. The only non-null combinations are  $s^\mu(\Phi)$  with

$$T^\mu = T^0, h_{N-2}, h_{N-1}, \mathcal{S}_{aN}, \mathcal{S}_{bN-1}, \mathcal{A}_{aN}, \mathcal{A}_{bN-1}, \quad (63)$$

for  $a = 1, \dots, N-1$  and  $b = 1, \dots, N-2$ . These field combinations can be considered as independent except for

$$s^{N-2} = -\sqrt{\frac{N-2}{N}}(s^0 + s^{N-1}). \quad (64)$$

The mass matrix  $(M_{CB}^2)_{ab}$  can then be extracted from  $\Lambda_{\mu\nu}$  eliminating all components  $\mu, \nu$  not contained in Eq. (63) and eliminating the component  $\mu = N-2$  or  $\nu = N-2$  using Eq. (64). The resulting matrix, which is  $4(N-1)$  dimensional  $[1+1+2(N-1)+2(N-2)]$ , should be positive definite. For  $N=2$ ,  $(M_{CB}^2)_{ab}$  is four dimensional and is  $\Lambda_{\mu\nu}$  itself, identifying  $a, b = \mu+1, \nu+1 = 1, 2, 3, 4$ , except for normalization factors for  $s^\mu(\Phi)$  [19].

For type (III) solutions, in addition to the second term of Eq. (43), which is the same as for type (I) solutions, we have to add the first term given by

$$\sum_{a=1}^{N-2} m_a^2 (|u_a|^2 + |w_a|^2), \quad (65)$$

using Eq. (55). Notice that the coefficients of  $\Lambda_{\mu\nu}$ , not present in Eq. (43), do not contribute to the masses but only to the trilinear and quartic interactions in Eqs. (44) and (45).

### C. Neutral vacuum

A neutral vacuum is characterized by cond. N stated in Sec. III. These solutions have  $\langle r^\mu(\Phi) \rangle$  lightlike,  $\langle w \rangle \neq 0$

but  $\langle u \rangle = 0$  and they can be of types (I), (II), or (III). We can set  $\langle u \rangle = 0$  in all previous calculations where charge breaking was assumed. We can promptly see that  $s^\mu$  in Eq. (40) does not depend on  $u_a$ . Hence, from Eq. (43) we conclude that  $\langle \mathbb{M} \rangle$  is the mass matrix for the charged Higgs bosons, i.e., the matrix whose eigenvalues are the squared masses of the charged Higgs bosons, combinations of  $u_a$ . The single null eigenvalue corresponds to the charged Goldstone. This conclusion can be also reached by taking the matrix of second derivatives of  $V$  with respect to  $\phi_{ai}^*$  and  $\phi_{bj}$ , and take the VEV for  $i = j = 1$ . On the other hand, the mass matrix for neutral Higgs bosons, combinations of  $w_a$ , depends explicitly on  $\Lambda$  in addition to the contribution of  $\langle \mathbb{M} \rangle$ . In the PCH basis, the three Goldstone modes are the neutral  $\sqrt{2} \text{Im} w_N$  and charged  $u_N$ . The SM Higgs is  $\sqrt{2} \text{Re} w_N$ .

Let us analyze type (III) solutions for which the following proposition can be proved.

*Proposition 1:* For all  $N \geq 3$ , any type (III) solution which preserves EM symmetry must have  $\langle X^\mu \rangle$  in the region defined by

$$LC_N = \{x^\mu \in \mathbb{R}^{1,d} | (N-1)^2 x_0^2 - \mathbf{x}^2 \geq 0 \text{ and } x_\mu x^\mu < 0\}. \quad (66)$$

This condition is not Lorentz invariant but  $SU(N)_H$  invariant. Such a proposition means neutral type (III) solutions cannot have arbitrarily spacelike  $\langle X^\mu \rangle$ . The proof is shown in Appendix D.

The type (II) solutions are the most predictive ones for we have  $\langle X^\mu \rangle = \alpha \langle r^\mu \rangle$ ,  $\alpha > 0$ . From

$$2x_\mu T^\mu = \frac{N}{N-1} T^0 x_0 - \tilde{x}, \quad (67)$$

for any  $x^\mu$  in  $\mathbb{R}^{1,d}$ , we can conclude that

$$\langle \mathbb{M} \rangle = \alpha \langle r_\mu \rangle T^\mu = \frac{\alpha \langle \Phi^\dagger \Phi \rangle}{2} \left[ \mathbb{1} - \frac{\langle w w^\dagger \rangle}{\langle w^\dagger w \rangle} \right], \quad (68)$$

where  $\langle w^\dagger w \rangle = \langle \Phi^\dagger \Phi \rangle = v^2/2$  and  $v = 246$  GeV is the electroweak symmetry breaking scale, considering the basis where  $\langle u \rangle = 0$ . Obviously,  $\langle w \rangle$  is an eigenvector of  $\langle \mathbb{M} \rangle$  with eigenvalue zero. Notice that Eq. (68) implies  $\langle \mathbb{M} \rangle$  satisfies the matricial equation

$$\langle \mathbb{M} \rangle^2 = \frac{\alpha}{4} v^2 \langle \mathbb{M} \rangle. \quad (69)$$

With the simple structure of Eq. (68), a remarkable result can be proved: *all charged physical Higgs bosons have the same mass*. Such a result can be more easily seen in the PCH basis where Eq. (47) is valid. Then, from Eq. (68), the physical charged Higgs bosons are the fields  $u_i$ , with  $i = 1, \dots, N-1$ , and they all have mass squared

$$m_{H^+}^2 = \frac{\alpha}{4} v^2. \quad (70)$$

Although the exact value of  $\alpha$  should be a complicated

function of the parameters  $M$ ,  $\Lambda$  derived from Eq. (36), the degenerate mass spectrum is a testable prediction.

The mass matrix for the neutral fields can be also straightforwardly constructed from  $\langle \mathbb{M} \rangle$  and  $\Lambda$  using Eq. (43) but usually nondegenerate because of the contribution of  $\Lambda$ . The procedure of construction, in the PCH basis, is analogous to the one in Sec. III B but the non-null components of  $s^\mu$ , instead of Eq. (63), correspond to

$$T^\mu = T^0, h_{N-1}, \mathcal{S}_{aN}, \mathcal{A}_{aN}, \quad (71)$$

$a = 1, \dots, N-1$ , with the non-null  $s^\mu$  all functionally independent and depending solely on  $w_a$ . The procedure is the same for type (III) solutions.

Comparing neutral type (II) solutions with neutral type (III) solutions, we see  $-\langle X^\mu X_\mu \rangle \geq 0$  is a measure of how degenerate the masses are of the physical charged bosons. Knowing the mass matrix  $\langle \mathbb{M} \rangle$ , we can recover  $\langle X_\mu \rangle$  from

$$\langle X_\mu \rangle = 2 \text{Tr}[\tilde{T}_\mu \langle \mathbb{M} \rangle]. \quad (72)$$

The properties of neutral type (I) solutions can be analyzed setting  $\alpha \rightarrow 0$  in the type (II) solutions. We can conclude that all charged Higgs bosons are massless. Therefore, there are  $N-1$  charged pseudo Goldstone bosons and one genuine charged Goldstone contributing to the Higgs mechanism.

#### IV. CONCLUSIONS

The study of the NHDM potentials performed here reveals a very rich underlying structure. In terms of the set of variables defined in Eq. (2), the variable space is limited to a subregion contained inside and on the future light cone  $LC^\dagger$  of a  $1+d=N^2$  dimensional Minkowski space. Furthermore, imposing the gauge condition (21), the variable space can be parametrized by two lightlike vectors whose spatial parts form an angle for which the cosine is  $-(N-1)^{-1}$ . The Minkowski structure also enabled us to find a sufficient, yet very general, criterion to require a bounded below potential. The Lorentz group can be also used as a powerful parametrization tool using the cosets  $SO(1, d)/\text{adj}SU(N)_H$  to avoid reparametrization redundancies. Charge breaking vacuum and neutral vacuum can be distinguished by calculating the Minkowski length of  $r^\mu(\Phi)$  for VEVs. The stationary points can be classified according to the Minkowski length of  $\langle X_\mu \rangle$ , in Eq. (29), into types (I), (II), and (III).

The Minkowski structure would also help to seek the type (II) minima. The method of caustics presented in Ref. [19] may be generalized to count the number of type (II) solutions for  $r^\mu$  restricted to  $LC^\dagger$ . The restriction to  $\mathcal{V}_\Phi$ , however, would need more mathematical tools. For example, the proper parametrization of  $SO(1, d)/\text{adj}SU(N)_H$  would be very important to the complete study of the NHDM potential minima.

The knowledge of the matrix  $\langle \mathbb{M} \rangle$  (or  $\langle X_\mu \rangle$ ) and  $\Lambda$  is sufficient to construct the mass matrix for all the scalars. In particular, when EM symmetry is not broken,  $\langle \mathbb{M} \rangle$  is itself the mass matrix of the charged Higgs bosons while the mass matrix of neutral bosons also requires the information of  $\Lambda$ . In view of the privileged information contained in  $\langle \mathbb{M} \rangle$ , one can try to parametrize any physical NHDM potential by attributing to  $\langle \mathbb{M} \rangle$  a general  $N \times N$  Hermitian matrix (positive semidefinite if NV) with one (NV) or two (CBV) null eigenvalues and attributing to  $\Lambda$  a general  $N^2 \times N^2$  real symmetric matrix which keeps  $V_4$  of Eq. (45) positive definite. The quadratic coefficient before SSB,  $Y = M_\mu T^\mu$ , can be obtained from

$$Y = \langle \mathbb{M} \rangle - \Lambda_{\mu\nu} \langle r^\mu \rangle T^\nu, \quad (73)$$

where

$$\langle r^\mu \rangle = \alpha_1 f^\mu(\langle v_1 v_1^\dagger \rangle) + \alpha_2 f^\mu(\langle v_2 v_2^\dagger \rangle), \quad (74)$$

with  $\alpha_1, \alpha_2$  nonnegative and  $v_1, v_2$  orthonormal eigenvectors of  $\langle \mathbb{M} \rangle$  with eigenvalue zero. The parameters  $\alpha_1, \alpha_2$  should be constrained by  $\alpha_1 + \alpha_2 = \langle \Phi^\dagger \Phi \rangle = v^2/2$ . This parametrization is not minimal but it assures that the stationary point (74) is a local minimum and has the advantage that some physical parameters, such as the masses of the charged Higgs bosons, can be chosen as parameters. On the other hand, nothing prevents the potential, defined with general  $\Lambda$  and  $Y$ , as in Eq. (73), to have a minimum  $\langle r'^\mu \rangle$  that lies deeper than the original  $\langle r^\mu \rangle$ , in Eq. (74), used for parametrization. Such possibility limits the potentialities of this parametrization fixed by  $\{\langle \mathbb{M} \rangle, \Lambda\}$  since the original minimum must be checked if it is the absolute minimum. In the 2HDM, for example, potentials with two neutral vacua lying in different depths can be constructed [26].

For parametrization purposes, the form of Eq. (6) is also very advantageous since it avoids the redundancies contained in  $Z_{(ab)(cd)}$  when written in the form of Eq. (7). Other several quantities can guide, for instance, numerical studies to distinguish charge breaking vacua from neutral vacua or local minima from saddle points. To identify the absolute minimum, however, is still a difficult question.

The interesting case of mass degenerate charged Higgs bosons, the type (II) vacuum, may have testable phenomenological implications. Because of the same mass we could have an enhancement of production of physical charged Higgs bosons for large  $N$ . However, even in this case, because some parameters in  $\Lambda$  can be functionally free in the trilinear and quartic interactions, the predictions for its width can be very difficult and variable. Usually, as  $N$  grows, we rapidly lose predictability unless we impose some symmetries or approximations. The mass degeneracy is then a very predictive result for certain NHDMs.

Even without the knowledge of an explicit minimum of the potential, writing the theory in the PCH basis presents various advantages. The two main advantages are the



easily extractable physical information and the minimality of parametrization. For example, the VEVs in the PCH basis depend only on two real nonnegative parameters,  $|\langle u \rangle|$  and  $|\langle w \rangle|$ , a smaller number than the four real parameters needed in the basis shown in Ref. [11]. Obviously, since the VEVs are real, the  $CP$  properties of the vacuum should be encoded in the parameters  $M'$  and  $\Lambda'$  transformed by  $SU(N)_H$  in the PCH basis. Thus, if the original parameters  $M$  and  $\Lambda$  are invariant by the canonical  $CP$  reflections [13], only the real subgroup  $SO(N)_H$  should connect the original basis to the PCH basis, besides rephasing transformations.

In conclusion, the results presented here uncover a rich structure contained in the NHDM potential and illuminates the properties of the possible vacua. A complete study of certain specific models should be guided by more restrictive ingredients and interesting phenomenology. The study performed here, however, is sufficiently general to cover a large class of physically possible NHDMs.

### ACKNOWLEDGMENTS

This work was supported by *Fundação de Amparo à Pesquisa do Estado de São Paulo* (Fapesp). The author would like to thank Professor Juan Carlos Montero and Professor Vicente Pleitez for critical discussions. The author would also like to thank I.P. Ivanov for useful discussions.

### APPENDIX A: PROOF OF EQ. (4)

First, we recall the completeness formulas for  $SU(N)_H$  and  $SU(2)_L$ , respectively [27]

$$\frac{1}{2}(\lambda^\mu)_{ab}(\lambda^\mu)_{cd} = \delta_{ad}\delta_{cb}, \quad (\text{A1})$$

$$\frac{1}{2}(\sigma^\mu)_{ij}(\sigma^\mu)_{kl} = \delta_{il}\delta_{kj}. \quad (\text{A2})$$

Then, the combination of both relations yields

$$\begin{aligned} & \Phi^\dagger(\lambda^\mu \otimes \mathbb{1}_2)\Phi\Phi^\dagger(\lambda^\mu \otimes \mathbb{1}_2)\Phi \\ &= \Phi^\dagger(\mathbb{1}_N \otimes \sigma^\mu)\Phi\Phi^\dagger(\mathbb{1}_N \otimes \sigma^\mu)\Phi, \end{aligned} \quad (\text{A3})$$

where the left-hand side of the equation is a shorthand for  $\phi_{ai}^*(\lambda^\mu)_{ab}\phi_{bi}\phi_{ck}^*(\lambda^\mu)_{ab}\phi_{dk}$ , the indices  $a, b, c, d = 1, \dots, N$  label the doublets (horizontal space), and  $i, k = 1, 2$  label each field in the doublet [representation space for  $SU(2)_L$ ]. We can explicitly verify the relation

$$\begin{aligned} \sigma &\equiv \phi_{a1}^*\phi_{a1}\phi_{b2}^*\phi_{b2} - |\phi_{a1}^*\phi_{a2}|^2 \\ &= \frac{1}{4}[(\Phi^\dagger\Phi)^2 - (\Phi^\dagger\mathbb{1}_N \otimes \vec{\sigma}\Phi)^2], \end{aligned} \quad (\text{A4})$$

which is always nonnegative due to Schwartz inequality. Finally, substituting  $(\Phi^\dagger\mathbb{1}_N \otimes \vec{\sigma}\Phi)^2$  of Eq. (A4) into Eq. (A3) yields

$$\frac{N-1}{2N}(\Phi^\dagger\Phi)^2 - \left(\frac{1}{2}\Phi^\dagger\vec{\lambda} \otimes \mathbb{1}_2\Phi\right)^2 = \sigma \geq 0, \quad (\text{A5})$$

which is the desired relation.

### APPENDIX B: TRANSLATION RULES

The relation between the parameters  $Y, Z$  of Eq. (7) and the parameters  $M, \Lambda$  of Eq. (6) is [13]

$$M_\mu = 2\text{Tr}[Y\tilde{T}_\mu], \quad (\text{B1})$$

$$\Lambda_{\mu\nu} = 4(\tilde{T}_\mu)_{ab}Z_{(ab)(cd)}(\tilde{T}_\nu)_{cd}^*, \quad (\text{B2})$$

where  $\tilde{T}^\mu$  is defined in Eq. (11). The relations above can be obtained from the inverse of Eq. (2),

$$\Phi_a^\dagger\Phi_b = 2(\tilde{T}_\mu)_{ba}r^\mu(\Phi) = 2(\tilde{T}_\mu r^\mu)_{ab}^*, \quad (\text{B3})$$

derived from the completeness relation (A1) written in terms of  $T^\mu$  and  $\tilde{T}^\mu$ , i.e.,

$$2(T^\mu)_{ab}(\tilde{T}_\mu)_{cd} = \delta_{ad}\delta_{cb}. \quad (\text{B4})$$

### APPENDIX C: GAUGE CHOICE FOR $\Phi$

We will prove here we can always perform a gauge transformation  $SU(2)_L \otimes U(1)_Y$  on  $\Phi = u \otimes e_1 + w \otimes e_2$  that renders  $u$  and  $w$  orthogonal, i.e., Eq. (21) is satisfied.

First, we recall a gauge transformation  $U$  acts equally on all the doublets as

$$\Phi_a \rightarrow U\Phi_a. \quad (\text{C1})$$

We know some gauge transformations in  $SU(2)_L$  mix the vectors  $u_a = \phi_{a1}$  and  $w_a = \phi_{a2}$ , which induces complicated transformations on  $f^\mu(uu^\dagger)$  and  $f^\mu(ww^\dagger)$ . We know, however, the combinations

$$z_A = \begin{pmatrix} u^\dagger & w^\dagger \end{pmatrix} \tau_A \begin{pmatrix} u \\ w \end{pmatrix}, \quad A = 1, 2, 3, \quad (\text{C2})$$

transform as vectors in  $\mathbb{R}^3$  by ordinary rotations, where  $\tau_A$  are Pauli matrices, generators of  $SU(2)_L$ . Therefore, we can always rotate the vector in Eq. (C2) to its third component. Requiring  $z_1 = 2\text{Re}(u^\dagger w) = 0$  and  $z_2 = 2\text{Im}(u^\dagger w) = 0$  implies  $u^\dagger w = 0$ .

### APPENDIX D: CHARACTERISTIC EQUATION FOR MATRICES

First, comparing

$$\Delta(2x_\mu T^\mu) = (N-1)^2 x_0^2 - \mathbf{x}^2 \quad (\text{D1})$$

with  $\Delta(2x_\mu \tilde{T}^\mu) = x_\mu x^\mu$ , we obtain

$$x_\mu x^\mu = \Delta(2x_\mu T^\mu) - N(N-2)x_0^2. \quad (\text{D2})$$

Now we can equate  $x_\mu = \langle X_\mu \rangle$  of Eq. (29) and require semidefinite positiveness for  $\langle \mathbb{M} \rangle$ , Eq. (28), i.e., all eigenvalues are nonnegative, since  $\langle \mathbb{M} \rangle$  corresponds to the mass

matrix of the charged Higgs bosons when EM symmetry is preserved. Then, the semidefinite positivity of  $\langle \mathbb{M} \rangle$  implies  $\Delta(\langle \mathbb{M} \rangle) \geq 0$ , hence

$$\langle X_\mu X^\mu \rangle \geq -N(N-2)\langle X_0 \rangle^2, \quad (\text{D3})$$

which is equivalent to state that  $\langle X^\mu \rangle$  is in  $LC_N$ , when  $\langle X \rangle^\mu$  is spacelike. The equality holds for  $N=3$ , for charge breaking solutions, as will be proved in the following.

For any square matrix  $A$ , the characteristic equation can be written as [13]

$$\det(A - \lambda \mathbb{1}) = (-1)^n \left[ \lambda^n - \sum_{k=1}^n \gamma_k(A) \lambda^{n-k} \right], \quad (\text{D4})$$

where

$$\gamma_k(A) = \frac{1}{k} \text{Tr} \left[ A^k - \sum_{j=1}^{k-1} \gamma_j(A) A^{k-j} \right]. \quad (\text{D5})$$

In particular, the function  $\Delta$  defined in Eq. (14) is related to  $\gamma_2$  by

$$\Delta(A) = -\gamma_2(A). \quad (\text{D6})$$

For  $3 \times 3$  matrices we have then

$$\det(A - \lambda \mathbb{1}) = (-1)[\lambda^3 - \gamma_1(A)\lambda^2 - \gamma_2(A)\lambda - \gamma_3(A)]. \quad (\text{D7})$$

To have two null eigenvalues we must have  $\gamma_2(A) = \gamma_3(A) = 0$ .

### APPENDIX E: $s^\mu$ IN THE PHYSICAL CHARGED HIGGS BASIS

The combination of fields in  $s^\mu(\Phi)$ , Eq. (39), determines the non-Goldstone modes. In the physical charged Higgs basis, they can be written explicitly. The list of all  $T^\mu$  was shown in Eq. (52), as well as the explicit representation for  $\mathcal{S}_{ab}$  and  $\mathcal{A}_{ab}$ . For the Cartan subalgebra formed by  $h_a$  we can adopt [13]

$$h_a = \frac{1}{\sqrt{2a(a+1)}} \text{diag}(\mathbb{1}_a, -a, 0, \dots, 0), \quad (\text{E1})$$

$$a = 1, \dots, N-1.$$

A more detailed description of the parametrization of the  $SU(N)$  algebra in the fundamental representation can be found in Ref. [13]. In the following, we list only the non-null components of  $s^\mu$ , according to  $T^\mu$ .

(i)  $T^0$ :

$$s^0 = \sqrt{\frac{2(N-1)}{N}} [|\langle w \rangle| \text{Re}(w_N) + |\langle u \rangle| \text{Re}(u_{N-1})]. \quad (\text{E2})$$

(ii)  $T^i = h_a$ ,  $a = 1, \dots, N-1$ :

$$s^{N-1} = \sqrt{\frac{2(N-1)}{N}} [(N-1)|\langle u \rangle| \text{Re}(u_{N-1}) - |\langle w \rangle| \text{Re}(w_N)], \quad (\text{E3})$$

$$s^{N-2} = -\sqrt{\frac{2(N-2)}{N-1}} |\langle u \rangle| \text{Re}(u_{N-1}). \quad (\text{E4})$$

(iii)  $T^i = \mathcal{S}_{ab}$ ,  $a = 1, \dots, N-1$ ,  $b = N$ ,  $s^\mu(\Phi) \rightarrow s_+^{ab}(\Phi)$ :

$$s_+^{a,N} = |\langle w \rangle| \text{Re}(w_a) + \delta_{a,N-1} |\langle u \rangle| \text{Re}(u_N), \quad (\text{E5})$$

$$s_+^{a,N-1} = |\langle u \rangle| \text{Re}(u_a), \quad a < N-1. \quad (\text{E6})$$

(iv)  $T^i = \mathcal{A}_{ab}$ ,  $a = 1, \dots, N-1$ ,  $b = N$ ,  $s^\mu(\Phi) \rightarrow s_-^{ab}(\Phi)$ :

$$s_-^{a,N} = -|\langle w \rangle| \text{Im}(w_a) + \delta_{a,N-1} |\langle u \rangle| \text{Im}(u_N), \quad (\text{E7})$$

$$s_-^{a,N-1} = -|\langle u \rangle| \text{Im}(u_a), \quad a < N-1. \quad (\text{E8})$$

From Eqs. (53) and (54), the fields  $u_N, u_{N-1}, w_N, w_{N-1}$  are absent in the first term of the quadratic part of the potential in Eq. (43). The fields  $\text{Im}w_N$  and  $\text{Im}u_{N-1}$  are also absent in  $s^\mu(\Phi)$  which make them massless. The real components  $\text{Re}w_N$  and  $\text{Re}u_{N-1}$  are present in Eqs. (E2)–(E4). The reminder of the fields involving  $w_{N-1}$  and  $u_N$  are only present in the combinations of Eqs. (E5) and (E7) for  $a = N-1$ . The orthogonal combinations shown in Eqs. (57) and (58) are then massless.

- 
- [1] S. Eidelman *et al.* (Particle Data Group), Phys. Lett. B **592**, 1 (2004).  
[2] C. Quigg, arXiv:0704.2045.  
[3] J.F. Gunion, H.E. Haber, G. Kane, and S. Dawson, *The Higgs Hunter's Guide* (Perseus, Cambridge, MA, 1990); arXiv:hep-ph/9302272.  
[4] G.L. Landsberg (CDF and D0 Collaborations), *42nd Rencontres de Moriond on QCD and High-Energy*

*Hadronic Interactions, La Thuile, Italy, 2007*, arXiv:0705.2855 (unpublished).

- [5] T. Binoth, S. Karg, N. Kauer, and R. Ruckl, Phys. Rev. D **74**, 113008 (2006).  
[6] H. Murayama, Int. J. Mod. Phys. A **19**, 1265 (2004).  
[7] M. Carena and H.E. Haber, Prog. Part. Nucl. Phys. **50**, 63 (2003).  
[8] T.D. Lee, Phys. Rev. D **8**, 1226 (1973); Phys. Rep. **9**, 143

- (1974).
- [9] S. Weinberg, Phys. Rev. Lett. **37**, 657 (1976).
- [10] Y.-L. Wu and L. Wolfenstein, Phys. Rev. Lett. **73**, 1762 (1994); J. Liu and L. Wolfenstein, Nucl. Phys. **B289**, 1 (1987).
- [11] A. Barroso, P. M. Ferreira, R. Santos, and J. P. Silva, Phys. Rev. D **74**, 085016 (2006).
- [12] L. Lavoura and J. P. Silva, Phys. Rev. D **50**, 4619 (1994).
- [13] C. C. Nishi, Phys. Rev. D **74**, 036003 (2006).
- [14] I. F. Ginzburg and M. Krawczyk, Phys. Rev. D **72**, 115013 (2005).
- [15] J. F. Gunion and H. E. Haber, Phys. Rev. D **72**, 095002 (2005).
- [16] C. Jarlskog, Phys. Rev. Lett. **55**, 1039 (1985); Also, Phys. Rev. D **35**, 1685 (1987).
- [17] F. J. Botella and J. P. Silva, Phys. Rev. D **51**, 3870 (1995).
- [18] G. C. Branco, M. N. Rebelo, and J. I. Silva-Marcos, Phys. Lett. B **614**, 187 (2005).
- [19] I. P. Ivanov, Phys. Rev. D **75**, 035001 (2007).
- [20] P. M. Ferreira, R. Santos, and A. Barroso, Phys. Lett. B **603**, 219 (2004); **629**, 114(E) (2005).
- [21] A. Barroso, P. M. Ferreira, and R. Santos, Phys. Lett. B **632**, 684 (2006).
- [22] The possibility of additional quartic terms like  $|\Phi_2^\dagger i\sigma_2 \Phi_1|^2$  was raised by Professor A. G. Dias, but the relation  $|\Phi_2^\dagger i\sigma_2 \Phi_1|^2 = \Phi_2^\dagger \Phi_2 \Phi_1^\dagger \Phi_1 - \Phi_2^\dagger \Phi_1 \Phi_1^\dagger \Phi_2$  holds.
- [23] M. Renardy, Linear Algebra Appl. **236**, 53 (1996).
- [24] A. Das, *The Special Theory of Relativity—A Mathematical Exposition* (Springer-Verlag, Berlin, 1993).
- [25] This name is justified because in this basis, when EM symmetry is preserved, the components  $u_a$ ,  $a = 1, \dots, N - 1$  are already the physical charged Higgs bosons.
- [26] A. Barroso, P. M. Ferreira, and R. Santos, Phys. Lett. B **652**, 181 (2007).
- [27] C. C. Nishi, Am. J. Phys. **73**, 1160 (2005).