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Locally inaccessible information as a fundamental ingredient to quantum information

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New Journal of Physics **14** (2012) 013027 (13pp)

Received 4 August 2011

Published 16 January 2012

Online at <http://www.njp.org/>

doi:10.1088/1367-2630/14/1/013027

Abstract. Quantum discord (QD) measures the fraction of the pairwise mutual information that is locally inaccessible in a multipartite system. Fundamental aspects related to two important measures in quantum information theory, namely the entanglement of formation (EOF) and the conditional entropy, can be understood in terms of the distribution of this form of local inaccessible information (LII). As such, the EOF for an arbitrarily mixed bipartite system AB can be related to the gain or loss of LII due to the extra knowledge that a purifying ancillary system E has on the pair AB . Similarly, a clear meaning of the negativity of the conditional entropy for AB is given. We employ these relations to elucidate important and yet not well-understood quantum features, such as the bipartite entanglement sudden death and the distinction between EOF and QD for quantifying quantum correlation. For that we introduce the concept of LII flow that quantifies the LII shared in a multipartite system when sequential local measurements are carried out.

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Contents

1. Introduction	2
2. Locally inaccessible information (LII)	2
3. Entanglement of formation and the flow of LII	5
4. Difference between entanglement of formation and quantum discord	7
5. Example: LII and entanglement sudden death	9
6. Additional fundamental relations	10
7. Summary	12
Acknowledgments	12
References	12

1. Introduction

Different ways of measuring quantum correlations have been widely studied over the last few years [1–3]. Among these quantum correlations, quantum discord (QD) [1] has played an important role. Based on the difference between two distinct definitions of mutual information (MI), Ollivier and Zurek [1] developed a new measure of quantum correlations. This new feature of correlations was explored in its various aspects [4, 5], intriguing the community with its peculiar properties—for instance, asymmetry and sudden changes [6]. It was recently shown that the entanglement of formation (EOF) and QD obey a very special monogamic relation [7]. This important result gives rise to new operational aspects for QD, such as the net amount of entanglement processed in a quantum computer [7], the difference between the entanglement cost and entanglement distillation [8], and the amount of entanglement consumed in the state merging protocol [9].

In contrast to classical systems, a fraction of the quantum MI cannot be accessed locally. Based on this idea, other interesting operational interpretations of QD emerge—as a measure of the MI fraction that is not accessible locally or, shortly, the locally inaccessible information (LII) [10]. In this paper, we explore the properties of the LII to derive fundamental relations—we show that EOF between any two subsystems A and B can always be written exclusively as a function of the LII. Moreover, it is possible to write the EOF between two subsystems A and B as average LII of the pair minus the balance of LII of the pair with a purifying environment E , giving to EOF a new operational meaning. We derive several relations between EOF and symmetrized and antisymmetrized versions of the LII that essentially quantify the average of the LII and the directional balance of LII, when measurements are made at A and B , respectively. This allows, for example, to understand the difference between EOF and QD for a bipartite system, and elucidates important aspects of the entanglement sudden death. Furthermore, we relate the QD to the conditional entropy in a simple manner for an arbitrary bipartite system. Such a relation gives a new way of understanding the negative signal of the conditional entropy.

2. Locally inaccessible information (LII)

In classical information theory, MI measures the amount of correlation between two stochastic variables, as measured by the Shannon entropy. The same concept when extended to quantum

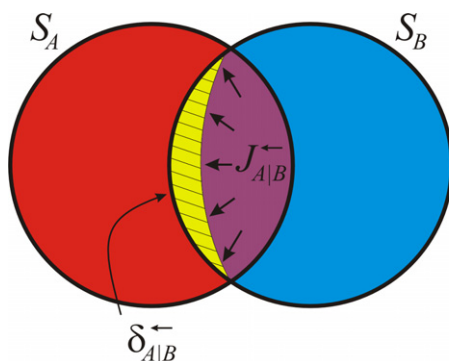


Figure 1. An extended Venn diagram where the quantum entropies are exposed. Here a part of the MI is not locally accessible and it is divided into two parts: the classical correlation and the QD.

systems, in terms of von Neumann entropy, allows the interpretation of MI as the quantity of information shared by two quantum systems. It is generally accepted as the measure of the total amount of correlation (quantum and classical) of a quantum state. For a bipartite state ρ_{AB} , the quantum MI I_{AB} accepts the extension of the standard form of the classical MI as

$$I_{AB} = S_A + S_B - S_{AB}, \quad (1)$$

where $S_{AB} \equiv S(\rho_{AB})$, $S_A \equiv S(\text{Tr}_B\{\rho_{AB}\})$ and $S_B \equiv S(\text{Tr}_A\{\rho_{AB}\})$, where $S(\cdot)$ denotes the von Neumann entropy. However, the very definition of the MI, $S(A : B) = S_A - S_{A|B}$, in terms of the conditional entropy $S_{A|B} = S_{AB} - S_B$ shows that there may be a problem with this simple extension above. In fact, this second definition of the quantum MI is measurement dependent, so depends on which system the measurement is made on. Thus, from the start, it seems that $S(A : B)$ is not necessarily symmetric, i.e. generally $S(A : B) \neq S(B : A)$. Moreover, local measurement over a subsystem depends on the basis of the meter, and even with a good basis choice, generally the total MI cannot be accessed. Therefore a fraction of this MI is non-local, the so-called local inaccessible information.

Given this peculiarity of a quantum system, Henderson and Vedral [2] and, independently, Ollivier and Zurek [1] defined a quantity that measures the maximum amount of locally accessible information [2],

$$J_{AB}^{\leftarrow} = \max_{\{\Pi_k\}} \left[S_A - \sum_k p_k S_{A|k} \right], \quad (2)$$

where $S_{A|k}$ is the conditional entropy after a measurement in B . Explicitly, $S_{A|k} \equiv S(\rho_{A|k})$ where $\rho_{A|k} = \text{Tr}_B(\Pi_k \rho_{AB} \Pi_k) / \text{Tr}_{AB}(\Pi_k \rho_{AB} \Pi_k)$ is the reduced state of A after obtaining the outcome k in B and $\{\Pi_k\}$ is a complete set of positive operator valued measurements that results in the outcome k with probability $p_k = \text{Tr}_{AB}(\Pi_k \rho_{AB} \Pi_k)$. In this case, since a measurement might give different results depending on the basis choice, a maximization is required. Thus J_{AB}^{\leftarrow} is the locally accessible MI and gives the maximum amount of AB MI that one can extract by measuring at B only [10]. An illustration of that is shown in figure 1, where the arrows represent the maximization involved in the calculation of the locally accessible MI. Note that a fraction of the MI is not locally accessible because it can be divided into two terms: one given by the

J_{AB}^{\leftarrow} and another given by the LII. The LII is then given by the MI minus J_{AB}^{\leftarrow} , which is exactly the definition of the QD,

$$\delta_{AB}^{\leftarrow} = I_{AB} - J_{AB}^{\leftarrow}. \quad (3)$$

In other words, the QD above gives the amount of information that is not accessible locally by measurements on B . It is easy to see that δ_{AB}^{\leftarrow} , in fact, measures the difference between the conditional entropy given by the second term of equation (2), $S_q(A|B) \equiv \min_{\{\Pi_k\}} \sum_k p_k S_{A|k}$, under optimal measurements⁷ on B , and the conditional entropy $S_{A|B} = S(A, B) - S(B)$ prior to measurement,

$$\delta_{AB}^{\leftarrow} = S_q(A|B) - S(A|B). \quad (4)$$

If $S_q(A|B) = S(A|B)$ all the available information about ρ_{AB} was acquired locally. So the QD has a strikingly simple meaning as a measure of how much a bipartite system state is affected by local measurements. In fact, the QD, δ_{AB}^{\leftarrow} , vanishes if and only if the density matrix of the composed system ρ_{AB} remains unaffected by a measurement in B . In this case, all the MI between the pair is locally accessible. Based on this fact, we can rephrase the definition of δ_{AB}^{\leftarrow} as *the fraction of the AB mutual information locally inaccessible by B*.

While in δ_{AB}^{\leftarrow} the measurements over the basis that minimizes the inaccessible information are made over B (meaning that it is the MI of AB that is inaccessible by B , which is being minimized), in δ_{BA}^{\leftarrow} those measurements are made over A (meaning that the MI of AB is inaccessible by A). Indeed, there are states such that $\delta_{BA}^{\leftarrow} \neq 0$ although $\delta_{AB}^{\leftarrow} = 0$ and vice versa.

By using the asymmetry of δ_{AB}^{\leftarrow} and δ_{BA}^{\leftarrow} , we can define two important quantities: the first one is the *average* of the LII when measurements are made on A and B ,

$$\varpi_{A|B}^+ = \frac{1}{2} (\delta_{AB}^{\leftarrow} + \delta_{BA}^{\leftarrow}), \quad (5)$$

and the second one is the *balance* of LII when measurements are made on A and B ,

$$\varpi_{A|B}^- = \frac{1}{2} (\delta_{AB}^{\leftarrow} - \delta_{BA}^{\leftarrow}). \quad (6)$$

The *average* LII (equation (5)) is a symmetric function since $\varpi_{A|B}^+ = \varpi_{B|A}^+$ and quantifies how much a system state is disturbed by *any* local measurement. On the other hand, the LII *balance* [10] is asymmetric and gives the difference in the efficiency with which each subsystem has to determine the MI by local measurements, which in a sense quantifies the asymmetry of a given bipartite state under local measurements. Suppose, for example, that $\varpi_{A|B}^- > 0$. In this case, a well-chosen measurement in A is more efficient for inferring MI of AB than a well-chosen measurement in B . Thus, A has less LII than B and this imbalance increases as $\varpi_{A|B}^-$ increases. On the other hand, if $\varpi_{A|B}^- < 0$, then measurements in A are less efficient at inferring the state of B than vice versa. As seen below, these quantities are very useful in uniquely relating EOF to LII. To present the relation between EOF and the LII, we begin by considering a pure joint state $|\psi_{AB}\rangle$. In this case, QD is symmetric ($\delta_{AB}^{\leftarrow} = \delta_{BA}^{\leftarrow}$) and is equal to EOF. Thus we can write

$$E_{AB} = \varpi_{A|B}^+, \quad (7)$$

where $\varpi_{A|B}^+$ is given by equation (5), and so for an arbitrary pure bipartite state the EOF is simply the average LII. Now we extend our consideration to an arbitrary mixed state ρ_{AB} shared by A and B . In such a case, a new subsystem E that purifies the pair A and B must be considered. In

⁷ Optimal measurements are those that account for base state projections that minimize $S(\rho_A|k)$.

this new situation, an informational cost must be paid to include an additional subsystem—the exceeded knowledge that the environment E has over the pair needs to be considered. As seen below, the EOF for the resulting mixed state ρ_{AB} cannot be simply written as in equations (7). Instead, it is given by the average LII of the pair (A, B) minus the LII balance of each of the subsystems A and B with E . To prove this relationship, let us suppose a pure state described by $\rho_{ABE} = |\phi_{ABE}\rangle\langle\phi_{ABE}|$ where $\rho_{AB} = \text{Tr}_E\{\rho_{ABE}\}$. We begin with a conservation relation for the distributed EOF and QD derived in a previous work [7],

$$E_{AB} + E_{AE} = \delta_{AB}^{\leftarrow} + \delta_{AE}^{\leftarrow}, \quad (8)$$

$$E_{AB} + E_{BE} = \delta_{BA}^{\leftarrow} + \delta_{BE}^{\leftarrow}, \quad (9)$$

$$E_{AE} + E_{BE} = \delta_{EA}^{\leftarrow} + \delta_{EB}^{\leftarrow}. \quad (10)$$

Rearranging equations (8)–(10) and writing them out as a function of the average LII, given by equation (5), and the LII balance (equation (6)), we can rewrite E_{AB} as

$$E_{AB} = \varpi_{A|B}^+ - \varpi_{E|A}^- - \varpi_{E|B}^-. \quad (11)$$

We can see that when compared to the pure state version from equation (7) the EOF in equation (11) decreases if the local measurements at the ancilla E have less access to the MI with A and B than the subsystems A and B together. So the EOF is not only given by the shared non-local information as in equation (7), but also by the balance of the bipartite system AB LII with the ancilla E . This relation allows an alternative interpretation of the EOF, which is independent of the number of system copies [12]—*The EOF E_{AB} , for an arbitrarily mixed quantum state ρ_{AB} , is the average LII of A, B minus the LII balance between each subsystem A and B with a purifying ancilla E .* In simple words, the EOF of the pair A, B is their average LII minus the loss (or gain) of LII due to correlation with E .

We shall return to discuss the implications of equation (11) soon, but first we must define another way of interpreting how the LII is distributed in the system.

3. Entanglement of formation and the flow of LII

Since the QD and consequently the LII functions essentially quantify the difference between the conditional entropy after and before optimized measurements, it is useful to quantify the LII amount involved when measurements are made in a sequential closed form, e.g. $E \rightarrow B \rightarrow A$. In that sequence the LII in the pure tripartite system ABE is computed by adding the pairwise QD contributions when measurements are carried out on E, B and A to infer the MI of the pairs EB, BA and AE , respectively,

$$\mathcal{L}_{\odot} \equiv \delta_{BE}^{\leftarrow} + \delta_{AB}^{\leftarrow} + \delta_{EA}^{\leftarrow}. \quad (12)$$

The resulting amount represents, as shown in figure 2, a clockwise, \mathcal{L}_{\odot} , flow of pairwise LII [13], and it represents how much the joint ABE system state is affected by the sequential optimized measurements on E, B and A . Conversely, the computation of the pairwise QDs for the sequence of measurements $A \rightarrow B \rightarrow E$ represents a counterclockwise (see figure 2), \mathcal{L}_{\ominus} , pairwise flow of LII,

$$\mathcal{L}_{\ominus} \equiv \delta_{BA}^{\leftarrow} + \delta_{EB}^{\leftarrow} + \delta_{AE}^{\leftarrow}. \quad (13)$$

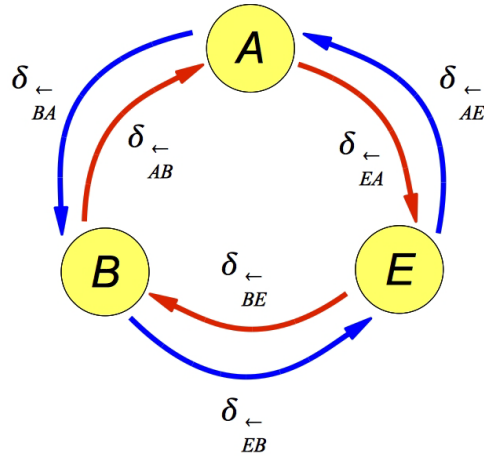


Figure 2. Depiction of clockwise (red arrows) and counterclockwise (blue arrows) flow of LII. The sum of the two possible directions of LII flow results in the sum of all possible EOF between pairs A , B and E .

Through equation (11), we can extend equations (8)–(10) to see that, for an arbitrary pure tripartite quantum system, the sum of all bipartite EOF is equal to the sum of all average LII,

$$E_{AB} + E_{AE} + E_{BE} = \varpi_{A|B}^+ + \varpi_{A|E}^+ + \varpi_{B|E}^+ \quad (14)$$

or

$$E_{AB} + E_{AE} + E_{BE} = \frac{1}{2} (\mathcal{L}_{\circlearrowleft} + \mathcal{L}_{\circlearrowright}). \quad (15)$$

So the sum of all possible EOF between pairs A , B and E is the sum of the clockwise and counterclockwise flows of LII. But the difference between (12) and (13) LII flows gives

$$\frac{\mathcal{L}_{\circlearrowleft} - \mathcal{L}_{\circlearrowright}}{2} = (E_{AB} - \delta_{AE}^{\leftarrow}) + (E_{AE} - \delta_{EB}^{\leftarrow}) + (E_{BE} - \delta_{BA}^{\leftarrow}). \quad (16)$$

Interestingly, the right-hand side of equation (16) is equal to the sum $S_{A|E} + S_{E|B} + S_{B|A}$, which vanishes for all pure ABE joint states [7, 14]. So, for pure states, $\mathcal{L}_{\circlearrowleft} = \mathcal{L}_{\circlearrowright}$ and equation (15) results in

$$E_{AB} + E_{AE} + E_{BE} = \mathcal{L}_{\circlearrowleft}. \quad (17)$$

Therefore, for a given tripartite pure state ρ_{ABE} the sum of the pairwise EOF between A , B and E is simply given by the LII flow in a closed cycle. The implication of $\mathcal{L}_{\circlearrowleft} = \mathcal{L}_{\circlearrowright}$ in terms of the LII balance is that

$$\omega_{A|B}^- + \omega_{B|E}^- + \omega_{E|A}^- = 0, \quad (18)$$

i.e. all the cyclic sums of the LII balance ($E \rightarrow B \rightarrow A \rightarrow E$ or $E \rightarrow A \rightarrow B \rightarrow E$) vanish. This is simply a feature of the purity of the system—since the system is closed, there is no LII missing and the balance is null. In other words, the amount of information contained in the cyclic sum of the conditional entropies balance is not disturbed by local measurements. As discussed below, this result is the basis to obtain the most fundamental expressions relating the EOF and discord. Furthermore, it gives a very simple relation between the conditional entropy and QD.

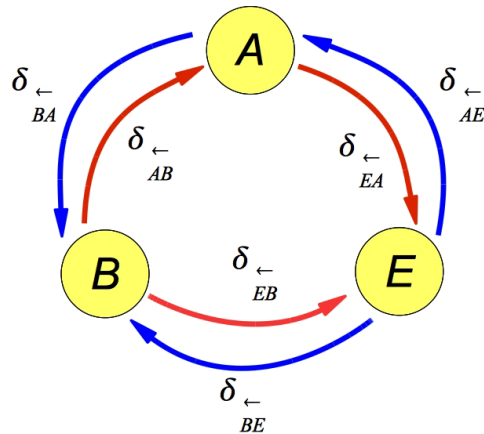


Figure 3. Depiction of flow of the locally inaccessible information departing from measurements in E (blue arrows) and concentrating in E (red arrows). The net result of these two flows is the difference between the EOF and the QD for AB when measurements are made on B .

4. Difference between entanglement of formation and quantum discord

An intriguing aspect is the difference between entanglement and quantum correlation. Once QD can be different from zero for separable states, it is usually assumed that it could include extra quantum correlations when compared to entanglement. For example, for a typically separable state of the form

$$\rho_{AB} = \sum_i p_i \rho_A^i \rho_B^i, \quad (19)$$

while $E_{AB} = 0$, the QD vanishes if, and only if, the set of states $\{\rho_B^i\}$ is a set of orthogonal projectors (with the measurements in B). On the other hand, for mixed entangled states, there are some situations where the QD is smaller than the EOF. Thus, a fundamental question emerges: what in fact measures the difference between them? By using equation (11), we can write it as the difference between the EOF and the QD:

$$E_{AB} - \delta_{AB}^{\leftarrow} = \varpi_{B|A}^- + \varpi_{A|E}^- + \varpi_{B|E}^-, \quad (20)$$

and so exclusively in terms of the LII balance. Note that $E_{AB} - \delta_{AB}^{\leftarrow}$ can be either larger or smaller than zero, once it depends on the efficiency of determining the locally MI by carrying out measurements on each subsystem. It is natural that depending on the quantum state ρ_{ABE} , the efficiency with which measurements were carried out in E to determine the MI of the pairs AE and BE is different from the efficiency of measurements carried out in B to determine the MI of BE and AB (as well as from that of measurements made in A to determine the MI of AB and AE). Thus, the difference between the EOF and the QD gives the balance of such an efficiency. Furthermore, it is interesting to note that equation (8) implies that if $E_{AB} - \delta_{AB}^{\leftarrow}$ is positive, then certainly $E_{AE} - \delta_{AE}^{\leftarrow}$ is negative and vice versa.

Equation (20) tells that the EOF and QD differ by the amount of the LII balance in the system, but yet does not offer a clear meaning to it. This issue can be clarified by the flow of LII as follows. As depicted in figure 3, all pairwise LII contributions in equation (20) can be split into two forms. The first one is the sum of the QD with an initial measurement on E and,

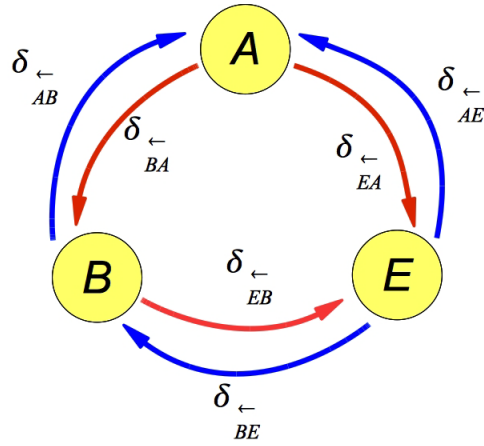


Figure 4. Depiction of flow of locally inaccessible information departing from measurements in E (blue arrows) and concentrating in E (red arrows). The net result of these two flows is the difference between the EOF and the QD for AB when measurements are made on A .

subsequently, on A concentrating in B , $\delta_{AE}^{\leftarrow} + \delta_{BA}^{\leftarrow}$, and with a measurements on E concentrating in B directly, δ_{BE}^{\leftarrow} . Similarly to what we have developed previously, we can define an LII flow from E to B as

$$\mathcal{L}_{E \rightarrow A \rightarrow B} \equiv \delta_{BE}^{\leftarrow} + \delta_{AE}^{\leftarrow} + \delta_{BA}^{\leftarrow}. \quad (21)$$

The second one accounts for the inverse flow of LII, i.e. the sum of the QDs with an initial measurements on B and, subsequently, on A concentrating in E , $\delta_{AB}^{\leftarrow} + \delta_{EA}^{\leftarrow}$, and with a measurement on B concentrating in E directly, δ_{EB}^{\leftarrow} . Similarly to equation (21), we define the flow from B to E as

$$\mathcal{L}_{B \rightarrow A \rightarrow E} \equiv \delta_{EB}^{\leftarrow} + \delta_{AB}^{\leftarrow} + \delta_{EA}^{\leftarrow}. \quad (22)$$

Note that the definitions in equations (21) and (22) are asymmetric and so quite distinct from the cyclic LII flux given in equations (12) and (13). With that it is possible to write equation (20) as

$$E_{AB} - \delta_{AB}^{\leftarrow} = \frac{1}{2} (\mathcal{L}_{E \rightarrow A \rightarrow B} - \mathcal{L}_{B \rightarrow A \rightarrow E}), \quad (23)$$

i.e. the difference of the EOF and the QD for the pair A and B when measurements are made in B is the difference between the flow of LII from and to the purifying ancilla E . This difference is the net, or residual, LII shared with E . Similarly, we can write

$$E_{AB} - \delta_{BA}^{\leftarrow} = \frac{1}{2} (\mathcal{L}_{E \rightarrow B \rightarrow A} - \mathcal{L}_{A \rightarrow B \rightarrow E}), \quad (24)$$

where the order of A and B has been changed to explicitly differentiate it from equation (23) due to the distinct sequence of measurements, as depicted in figure 4. Combining the last two equations, it is easy to rewrite a symmetrized form for them, which turns out to be an equivalent version of equation (11) as

$$E_{AB} - \varpi_{A|B}^+ = \frac{1}{2} (\mathcal{L}_{E \rightarrow \binom{A}{B}} - \mathcal{L}_{\binom{A}{B} \rightarrow E}), \quad (25)$$

where

$$\mathcal{L}_{E \rightarrow \binom{A}{B}} \equiv \delta_{AE}^{\leftarrow} + \delta_{BE}^{\leftarrow} \quad (26)$$

and

$$\mathcal{L}_{(B) \rightarrow E}^{(A)} \equiv \delta_{EA}^{\leftarrow} + \delta_{EB}^{\leftarrow}. \quad (27)$$

The form of equation (25) is appropriate since it is symmetric accounting for the difference between the EOF and the average LII for the pair AB . Entanglement, as measured by the EOF, is a typical correlation of quantum nature as well as the average LII, i.e. the average amount of MI locally inaccessible by measurements on A and B . Now the difference between these quantities for the pair AB is equal to the net flow of LII in and out of the ancilla E . Since $\mathcal{L}_{E \rightarrow (B)}^{(A)}$ accounts for how much the state ρ_{AB} is disturbed by measurements on the purifying ancilla E and, similarly, $\mathcal{L}_{(B) \rightarrow E}^{(A)}$ accounts for how much the state ρ_E is disturbed by local measurements on A and B , the net LII flux $\mathcal{L}_{E \rightarrow (B)}^{(A)} - \mathcal{L}_{(B) \rightarrow E}^{(A)}$ computes the asymmetry in this process. In fact, the asymmetry captures the notion that some extra local inaccessible information of the pair AB is being shared with E , being the reason for the difference between E_{AB} and $\varpi_{A|B}^+$. Indeed $E_{AB} = \omega_{A|B}^+$ when the system is symmetric so that $\delta_{AE}^{\leftarrow} = \delta_{EA}^{\leftarrow}$ and $\delta_{BE}^{\leftarrow} = \delta_{EB}^{\leftarrow}$. But the net flow of LII in and out of the ancilla E can vanish as well when $\delta_{AE}^{\leftarrow} = \delta_{EB}^{\leftarrow}$, and $\delta_{BE}^{\leftarrow} = \delta_{EA}^{\leftarrow}$. In such a case, even though $E_{AB} \neq \delta_{BA}^{\leftarrow} \neq \delta_{AB}^{\leftarrow}$, the EOF E_{AB} is equal to the average LII of the pair AB . So, whenever the net flow of LII in and out of the ancilla E is null, even though there might be some LII for the subsystem AB missing for being shared with E , it is compensated for and so $\varpi_{A|B}^+$ computes all the LII which is useful for nonlocal tasks as entanglement of A and B can be.

5. Example: LII and entanglement sudden death

By using the relations presented here, we can investigate another important aspect of the distribution of the entanglement and the QD in a multipartite system. We consider a four-qubit system where two initially pure entangled qubits A and B interact individually with their own reservoir R_A and R_B , respectively (for details see [15]). We suppose an amplitude damping channel at temperature $T = 0$ K and we write a map to each qubit as

$$\begin{aligned} \Sigma(|0\rangle_A|0\rangle_{R_A}) &\rightarrow |0\rangle_A|0\rangle_{R_A}, \\ \Sigma(|1\rangle_A|0\rangle_{R_A}) &\rightarrow \sqrt{1-p}|1\rangle_A|0\rangle_{R_A} + \sqrt{p}|0\rangle_A|1\rangle_{R_A}, \end{aligned} \quad (28)$$

where $p = 1 - e^{-\Gamma t}$ and identically for B interacting with R_B . We choose as the initial condition $|\Psi(0)\rangle = \frac{2}{\sqrt{3}}|0\rangle_A|0\rangle_B + \frac{1}{\sqrt{3}}|1\rangle_A|1\rangle_B$, which is an example where the phenomenon known as entanglement sudden death [16] occurs. As one can observe by equation (11), the entanglement between AB suddenly vanishes when the average LII between AB is equal to the balance of the LII between the environment and each subsystem (A and B). Actually, as soon as measurements over the environment allow more inference about the MI with the pair A, B , their entanglement decreases. As illustrated in figure 5, when the entanglement between A and B vanishes (entanglement sudden death), the excess of knowledge that the environment E has about the subsystems A and B , as measured by

$$\varpi_{R_A R_B|A}^- + \varpi_{R_A R_B|B}^- = \mathcal{L}_{R_A R_B \rightarrow (B)}^{(A)} - \mathcal{L}_{(B) \rightarrow R_A R_B}^{(A)},$$

becomes equal to the average LII in a finite time.

To obtain the results plotted in figure 5, we analytically solve the dynamics of E_{AB} and the QD between each subsystem A and B with the whole environment $E \equiv R_A \otimes R_B$. In this

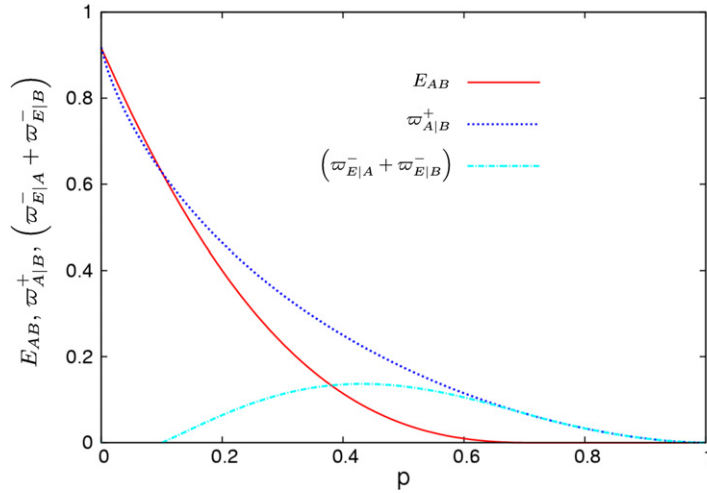


Figure 5. The red curve (solid) shows the entanglement between the pair AB , whereas the blue curve (dotted) shows the average LII $\varpi_{A|B}^+$. The cyan curve (traced) represents the sum of the balance LII between the environment and the pair AB . When $p \approx 0.65$, the average becomes equal to the sum of the balance and the entanglement sudden death occurs.

case, we use our relations to analytically calculate the QD for a system of dimension 2×4 . For example, to calculate the QD between A and the whole environment $R_A \otimes R_B$, we have that

$$\delta_{A(R_A R_B)}^{\leftarrow} = E_{AB} + S_{A|B}, \quad (29)$$

where $S_{A|B}$ is the conditional entropy and both E_{AB} and $S_{A|B}$ can be calculated analytically by means of the density matrix ρ_{AB} . These results extend further the investigation in [5] as it provides a way of calculating the QD and the EOF for different partitions (e.g. $A(R_A R_B)$) and for higher dimensional systems. Indeed, the monogamic relation can be used to calculate the QD and the EOF between two subsystems with dimension $2 \times N$ and rank 2 (see also [17]). It is true because the extra system that purifies a rank 2 density matrix is always a qubit. For example, given a qubit A and an environment E with dimension N , a rank 2 density matrix ρ_{AE} can be purified in a density matrix ρ_{ABE} where the dimension of the subsystem B is always two. Noting that AB is thus a system composed of two qubits, we have [7]

$$E_{AE} = \delta_{AB}^{\leftarrow} + S_{A|B} \quad (30)$$

and

$$\delta_{AE}^{\leftarrow} = E_{AB} + S_{A|B}. \quad (31)$$

Equation (30) shows that the EOF between a qubit and a qudit for any rank 2 density matrix can be calculated numerically by means of the QD of the two qubits AB . More importantly, equation (31) shows that the QD between a qubit and a qudit for any rank 2 density matrix can be calculated analytically by means of the EOF of the two-qubit system AB .

6. Additional fundamental relations

Based on the previous results, we are able to obtain additional fundamental expressions relating the EOF and the QD as well as the conditional entropy. First of all, we combine equations (11)

and (18) to show that

$$E_{AB} = \delta_{AB}^{\leftarrow} + \delta_{BE}^{\leftarrow} - \delta_{EB}^{\leftarrow}, \quad (32)$$

$$E_{AB} = \delta_{BA}^{\leftarrow} + \delta_{AE}^{\leftarrow} - \delta_{EA}^{\leftarrow}. \quad (33)$$

These equations are the simplest expressions relating EOF exclusively to QD. They show that the difference between the entanglement and the QD is proportional (twice) to the LII balance of one of the subsystems with the environment. In equation (32), we see that for an arbitrarily mixed system AB , when the subsystem B is measured in order to know about the MI of the pair AB , some additional information is acquired about the pair BE , and so it needs to be taken into account ($2\omega_{B|E}^-$ needs to be summed). We can also derive explicitly an important result discussed in the previous section but in a simpler form: *if the systems B and E are symmetric, we have that $E_{AB} = \delta_{AB}^{\leftarrow}$* , which is a direct consequence of the fact that for this case $\delta_{BE}^{\leftarrow} = \delta_{EB}^{\leftarrow}$. Then, the QD δ_{AB}^{\leftarrow} is equal to the EOF E_{AB} not only when the system AB is pure but also when the systems E and B are symmetric. The same is valid for equation (33). The δ_{AB}^{\leftarrow} is equal to E_{AB} not only when the system AB is pure but also when the systems E and A are symmetric. Of course, if we have full symmetry between A , B and E then $E_{AB} = \omega_{A|B}^+$.

By using the results above, we are able to find a very useful relation between the QD and the conditional entropy. The conditional entropy is an important quantity in information theory that is intimately related to the entanglement distillation and irreversibility. In addition, it is fundamental in the state merging protocol [18]. In this protocol, given two parties A and B with a shared state ρ_{AB} , the conditional entropy measures the amount of quantum communication that is needed to transfer the part A to the part B such that part B ends with the state ρ_{AB} , keeping possible correlations of ρ_{AB} with any external system. Interestingly, the conditional entropy can be negative and this means that B can obtain the full state AB using only classical communication. Additionally, A and B will be able to transfer quantum information in the future at no further cost [18]. First of all, let us pay attention to the case of pure states. For a pure state, we can write the conditional entropy as

$$-S_{A|B} = \delta_{BA}^{\leftarrow}. \quad (34)$$

As exposed above, δ_{BA}^{\leftarrow} measures the amount of MI of AB inaccessible by measurements on A . Clearly, A cannot transfer this information to the subsystem B and consequently it is preserved for a future communication. In this sense, what could we say about mixed states? To calculate the conditional entropy for mixed states in terms of the LII, we use the relation [7, 14] $E_{AB} = \delta_{BE}^{\leftarrow} + S_{B|E}$ and equation (32). Based on this equation, it is simple to show that

$$-S_{A|B} = \delta_{BA}^{\leftarrow} - \delta_{EA}^{\leftarrow}. \quad (35)$$

By observing equation (35), we can clearly understand what in fact happens for mixed states. As one can see, there is an amount of LII that A shares with E once it cannot be sent to B . Furthermore, since this information mutually belongs to A and E , it cannot be used jointly with B for further tasks. In fact, it has to be subtracted from δ_{BA}^{\leftarrow} . Moreover, by using equation (35), it is easy to analyze the negativity of the conditional entropy, which depends on the balance of LII. The sign of the conditional entropy has an important meaning in important tasks such as quantum state merging and entanglement distillation. Again, as for the EOF relations, more important than the amount of LII is the balance of LII between the subsystems A and B and the purifying ancilla (environment). By using the balance of LII, we are able to identify the signal of the conditional entropy. For instance, if the subsystem A shares the same amount of

LII (independently of the amount) with B and E , the conditional entropy is null, $S_{A|B} = 0$. If the A LII with B is larger than the LII with E , the conditional entropy is negative. Obviously, $S_{A|B}$ is positive when the A LII with E is larger than the LII with B .

7. Summary

To conclude, we have presented alternative forms to interpret the EOF in terms of the locally inaccessible information functions. Our relations based on average LII and LII balance demonstrate that the EOF can be understood for a general quantum system exclusively as a function of the LII being shared. The concept behind LII flow when sequential measurements are made is an interesting way to understand the meaning of the correlation distribution when measurements are carried out. In that sense, not only the way a quantum system is affected by local measurements can be quantified but also the symmetry of such a system, under local measurements. An example of the usefulness of these new relations was given in the investigation of the yet not well understood entanglement sudden death phenomenon. Also a deep discussion on the distinction between EOF and QD in terms of residual flow of LII in and out of a purifying ancilla is made possible. The relation of the QD to the conditional entropy is quite important for understanding when the negative signal of the conditional entropy occurs. That is ruled by the QD balance between the environment and the system as well. We believe that the discussion presented here may contribute further to the understanding of the distribution of entanglement and quantum correlation in general for multipartite systems.

Acknowledgments

This work was supported by FAPESP and CNPq through the National Institute for Science and Technology of Quantum Information. MCO acknowledges support from iCORE.

References

- [1] Ollivier H and Zurek W H 2002 *Phys. Rev. Lett.* **88** 017901
- [2] Henderson L and Vedral V 2001 *J. Phys. A: Math. Gen.* **34** 6899
- [3] Terhal B M *et al* 2002 *J. Math. Phys.* **43** 4286
- DiVincenzo D *et al* 2004 *Phys. Rev. Lett.* **92** 067902
- Luo S 2008 *Phys. Rev. A* **77** 022301
- Piani M *et al* 2008 *Phys. Rev. Lett.* **100** 090502
- Wu S *et al* 2009 *Phys. Rev. A* **80** 032319
- Modi K *et al* 2010 *Phys. Rev. Lett.* **104** 080501
- Girolami D, Paternostro M and Adesso G 2011 *J. Phys. A: Math. Theor.* **44** 352002
- [4] Datta A, Shaji A and Caves C M 2008 *Phys. Rev. Lett.* **100** 050502
- Werlang T, Souza S, Fanchini F F and Villas Boas C J 2009 *Phys. Rev. A* **80** 024103
- Fanchini F F, Werlang T, Brasil C A, Arruda L G E and Caldeira A O 2010 *Phys. Rev. A* **81** 052107
- Xu J-S, Xu X-Y, Li C-F, Zhang C-J, Zou X-B and Guo G-C 2010 *Nat. Commun.* **1** 7
- Fanchini F F, Castelano L K and Caldeira A O 2010 *New J. Phys.* **12** 073009
- Mazzola L, Piilo J and Maniscalco S 2010 *Phys. Rev. Lett.* **104** 200401
- Soares-Pinto D O, Celeri L C, Auccaise R, Fanchini F F, deAzevedo E R, Maziero J, Bonagamba T J and Serra R M 2010 *Phys. Rev. A* **81** 062118
- [5] Maziero J, Werlang T, Fanchini F F, Céleri L C and Serra R M 2010 *Phys. Rev. A* **81** 022116

- [6] Maziero J, Celeri L C, Serra R M and Vedral V 2009 *Phys. Rev. A* **80** 044102
- [7] Fanchini F F, Cornelio M F, de Oliveira M C and Caldeira A O 2011 *Phys. Rev. A* **84** 012313
- [8] Cornélio M F, de Oliveira M C and Fanchini F F 2011 *Phys. Rev. Lett.* **107** 020502
- [9] Madhok V and Datta A 2011 *Phys. Rev. A* **83** 032323
Cavalcanti D, Aolita L, Boixo S, Modi K, Piani M and Winter A 2011 *Phys. Rev. A* **83** 032324
- [10] Zurek W H 2003 *Phys. Rev. A* **67** 012320
- [11] Cover T M and Thomas J A 2006 *Elements of Information Theory* (New York: Wiley)
- [12] Bennett C H, DiVincenzo D P, Smolin J A and Wootters W K 1996 *Phys. Rev. A* **54** 3824–3851
- [13] Deutsch D and Hayden P 2000 *Proc. R. Soc. A* **456** 1759
- [14] Koashi M and Winter A 2004 *Phys. Rev. A* **69** 022309
- [15] López C E, Romero G, Lastra F, Solano E and Retamal J C 2008 *Phys. Rev. Lett.* **101** 080503
- [16] Zyczkowski K, Horodecki P, Horodecki M and Horodecki R 2001 *Phys. Rev. A* **65** 012101
Diósi L 2003 *Lec. Notes Phys.* **622** 157
Dodd P J and Halliwell J J 2004 *Phys. Rev. A* **69** 052105
Yu T and Eberly J H 2004 *Phys. Rev. Lett.* **93** 140404
Almeida M P, de Melo F, Hor-Meyll M, Salles A, Walborn S P, Souto Ribeiro P H and Davidovich L 2007
Science **316** 579
- [17] Lastra F, López C E, Roa L and Retamal J C 2011 arXiv:1105.6321
- [18] Horodecki M, Oppenheim J and Winter A 2005 *Nature* **436** 673