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COMBINATORICS OF A CERTAIN IDEAL IN THE SEGRE COORDINATE RING

PAULO BRUMATTI, PHILIPPE GIMENEZ, AND ARON SIMIS

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ABSTRACT. We focus on a “fat” model of an ideal in the class of the canonical ideal of the Segre coordinate ring, looking at its Rees algebra and related arithmetical questions.

1. INTRODUCTION

Let \mathfrak{S} be the image of the Segre map

$$\sigma = \sigma_{n-1, m-1} : \mathbb{P}^{n-1} \times \mathbb{P}^{m-1} \longrightarrow \mathbb{P}^{nm-1},$$

the so-called Segre variety. As a toric variety, \mathfrak{S} admits $k[t_{ij}]$ ($1 \leq i \leq n$, $1 \leq j \leq m$) as coordinate ring. This ring can be presented over the polynomial ring $k[\mathbf{X}] = k[X_{ij}]$ ($1 \leq i \leq n$, $1 \leq j \leq m$) by the ideal $I_2(X_{ij})$ generated by the 2×2 minors of the generic $n \times m$ matrix (X_{ij}) . It is well known that the canonical class of the latter is $(m-n)[\mathfrak{K}]$, where $\mathfrak{K} \subset S = k[\mathbf{X}]/I_2(X_{ij})$ is the ideal generated by (the residues of) the entries in the first column of the matrix (X_{ij}) (cf. [BV], (8.4)).

Now, given an integer $d \geq 1$, let $\mathfrak{K}^{[d]}$ denote the ideal generated by the d th powers of the generators of \mathfrak{K} . The main purpose of this paper is to investigate the algebraic-combinatorics of the blowup of \mathfrak{S} along the locus of $\mathfrak{K}^{[d]}$. Algebraically, we are therefore looking at the Rees algebra of the ideal $\mathfrak{K}^{[d]}$. Using the toric representation, this algebra is simply the k -subalgebra

$$k[t_{ij} s_j, (t_1 s_1)^d T, \dots, (t_n s_1)^d T] \subset k[\mathbf{t}, \mathbf{s}][T],$$

where $1 \leq i \leq n$, $1 \leq j \leq m$. Since s_1 is fixed in the d th powers, it is not difficult to see that this algebra is isomorphic to the k -algebra $R^{[d]} = k[t_{ij} s_j, t_1^d, \dots, t_n^d] \subset k[\mathbf{t}, \mathbf{s}]$.

As it turns out, $R^{[d]}$ is presented over a polynomial ring $A = k[\mathbf{X}, \mathbf{U}]$, with $\mathbf{X} = \{X_{ij}\}$, $\mathbf{U} = \{U_1, \dots, U_n\}$, by a sum of determinantal ideals, each generated by certain 2×2 minors, so our toric variety is a sort of determinantal locus lacking the generic codimension. It can be looked at as the generic version of a few classes of ideals appearing in the recent literature (cf. [Hu], [HuHu], [Sch] and [MoSi]), obtained thereof by specialization and by taking suitable free ring extensions.

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2. A PSEUDO-DETERMINANTAL LOCUS

We will fix the following notation:

- $I_r(L)$ the ideal generated by $r \times r$ minors of the matrix L .
- \mathbf{t}, \mathbf{s} sets of (toric) variables $t_1, \dots, t_n, s_1, \dots, s_m$ over a field k .
- S the coordinate ring $k[x_{ij}] = k[X_{ij}]/I_2(X_{ij})$ of the Segre embedding.
- $\mathfrak{R}^{[d]}$ the ideal (row-matrix) in S generated by the d th powers of x_{11}, \dots, x_{n1} .
- $R^{[d]}$ the toric ring $k[t_i s_j, t_1^d, \dots, t_n^d]$.
- $M(\mathbf{Y})$ a monomial in the variables \mathbf{Y} .
- $M(\mathbf{y})$ the residue of the monomial $M(\mathbf{Y})$ modulo some ideal.
- $\mathbb{M}(d, \mathbf{Y})$ the set (row, ideal) of all monomials of degree d in the variables \mathbf{Y} .
- $\mathbb{M}(d, \mathbf{y})$ the set of residues of $\mathbb{M}(d, \mathbf{Y})$.

2.1. The defining equations. One needs the following lemmata. In order to save on notation, we set sometimes $\mathbf{X}_i = X_{i1}, \dots, X_{im}$ and, correspondingly, $\mathbf{x}_i = x_{i1}, \dots, x_{im}$.

(2.1.1) Lemma. *For any pair of indices $1 \leq i_1, i_2 \leq n$, consider the involutive k -algebra automorphism $\Phi = \Phi_{i_1, i_2}$ of the polynomial ring $k[X_{ij}] = k[\mathbf{X}_1, \dots, \mathbf{X}_n]$ such that*

$$\Phi(X_{i,j}) = \begin{cases} X_{i_2,j} & \text{if } i = i_1, \\ X_{i_1,j} & \text{if } i = i_2, \\ X_{i,j} & \text{otherwise.} \end{cases}$$

Then:

- (i) Φ induces an automorphism of $S = k[X_{ij}]/I_2(X_{ij})$.
- (ii) For any two monomials $M = M(\mathbf{X}_{i_1}), N = N(\mathbf{X}_{i_1}) \in k[\mathbf{X}_{i_1}]$ of the same degree, one has $M\Phi_{i_1, i_2}(N) \equiv N\Phi_{i_1, i_2}(M) \pmod{I_2(X_{ij})}$.

Proof. (i) Clearly, the ideal $I_2(X_{ij})$ is invariant under Φ . Since Φ is an involution (i.e., $\Phi = \Phi^{-1}$), it then induces an automorphism of S .

(ii) One proceeds by induction on the common degree of M and N . The result is trivial if $M = N$, so assume these are distinct monomials. Now write $M = X_{i_1, j_1} M_1$ and $N = X_{i_1, j_2} N_1$, with $j_1 \neq j_2$. Then, with $\Phi = \Phi_{i_1, i_2}$ and by the inductive hypothesis:

$$\begin{aligned} M\Phi(N) &= X_{i_1, j_1} X_{i_2, j_2} M_1 \Phi(N_1) \equiv X_{i_1, j_2} X_{i_2, j_1} M_1 \Phi(N_1) \\ &\equiv X_{i_1, j_2} X_{i_2, j_1} N_1 \Phi(M_1) = N\Phi(M), \end{aligned}$$

as required. □

(2.1.2) Remark. Part (ii) of Lemma (2.1.1) has been used before in different forms (cf., e.g., [Gim, Lemme 5.12.1]).

(2.1.3) Lemma. *The first syzygies of the ideal $\mathfrak{R}^{[d]} \subset S$ are generated by the first syzygies of all pairs $\{x_{i_1, 1}^d, x_{i_2, 1}^d\}$, $1 \leq i_1, i_2 \leq n$ and these are generated by those syzygies whose coordinates are terms αM , $\alpha \in k$ and M a monomial.*

Proof. This is a direct consequence of the fact that S is defined by a binomial ideal [EiSt, Corollary 1.7 (b)]. □

Here is the basic technical result of this section:

(2.1.4) Proposition. *Let $d \geq 1$. The ideal $\mathfrak{R}^{[d]} \subset S$ has the following presentation as an S -module:*

$$\left(\bigwedge^2 S^n\right) \oplus^{C(m,d)} \xrightarrow{\psi^{[d]}} S^n \xrightarrow{\mathfrak{R}^{[d]}} S,$$

where $C(m, d) = \binom{m-1+d}{d}$, $\mathfrak{R}^{[d]}$ stands for the map given by the row-matrix $(x_{11}^d \dots x_{n1}^d)$ and $\psi^{[d]}$ is given by the matrix

$$\left(\begin{array}{cccc|cccc|cccc|cccc} -\mathbb{M}(d, \mathbf{x}_2) & -\mathbb{M}(d, \mathbf{x}_3) & \dots & -\mathbb{M}(d, \mathbf{x}_n) & 0 & 0 & \dots & 0 & -\mathbb{M}(d, \mathbf{x}_2) & -\mathbb{M}(d, \mathbf{x}_3) & \dots & -\mathbb{M}(d, \mathbf{x}_n) & \dots & 0 & 0 & 0 \\ \mathbb{M}(d, \mathbf{x}_1) & 0 & \dots & 0 & \mathbb{M}(d, \mathbf{x}_2) & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & 0 \\ 0 & \mathbb{M}(d, \mathbf{x}_1) & \dots & 0 & 0 & \mathbb{M}(d, \mathbf{x}_2) & \dots & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \vdots & \vdots & \dots & \vdots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \vdots & \vdots & \dots & \vdots & 0 & 0 & \dots & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \dots & \dots & \dots & \dots & \dots & -\mathbb{M}(d, \mathbf{x}_n) & \dots & \dots \\ 0 & 0 & \dots & \mathbb{M}(d, \mathbf{x}_1) & 0 & 0 & \dots & \mathbb{M}(d, \mathbf{x}_2) & \dots & \dots & \dots & \dots & \dots & \mathbb{M}(d, \mathbf{x}_{n-1}) & \dots & \dots \end{array} \right).$$

Proof. The containment $\text{Im } \psi^{[d]} \subset \ker \mathfrak{R}^{[d]}$ is a straightforward consequence of Lemma (2.1.1), (ii).

For the reverse inclusion, by Lemma (2.1.3) and by an obvious symmetrical argument we may assume that we are given a relation of the form

$$(2-1) \quad X_{11}^d M + X_{21}^d N \equiv 0 \pmod{I_2(X_{ij})},$$

where M and N are terms in $k[\mathbf{X}]$.

The crucial point is to establish that $\deg_{\mathbf{x}_2} M \geq d$. At any rate, one has $\deg_{\mathbf{x}_2} M \geq 1$, otherwise by setting to zero all the variables in \mathbf{X}_2 , it would follow that $X_{11}^d M \in I_2(X_{ij}) (i \neq 2)$ which is absurd.

We proceed by induction on d , the assertion for $d = 1$ having just been shown. Thus, let $d \geq 2$ and assume that $\deg_{\mathbf{x}_2} M = d_0 < d$. By the preceding, $d_0 \geq 1$, hence $d - d_0 \leq d - 1$. Write $M = \widetilde{M}M_1$, with $\widetilde{M} \in \mathbb{M}(d_0, \mathbf{X}_2)$. By Lemma (2.1.1), (ii), one has $X_{11}^{d_0} \widetilde{M} \equiv X_{21}^{d_0} \Phi_{12}(\widetilde{M})$, hence (2-1) yields

$$-X_{21}^d N \equiv X_{11}^{d_0} X_{11}^{d-d_0} \widetilde{M} M_1 \equiv X_{21}^{d_0} X_{11}^{d-d_0} \Phi_{12}(\widetilde{M}) M_1,$$

from which it follows that $X_{11}^{d-d_0} \Phi_{12}(\widetilde{M}) M_1 + X_{21}^{d-d_0} N \equiv 0$. Then, by the inductive hypothesis we know that $\deg_{\mathbf{x}_2} \Phi_{12}(\widetilde{M}) M_1 \geq d - d_0 > 0$. But since $\Phi_{12}(\widetilde{M}) \in \mathbb{M}(d_0, \mathbf{X}_1)$, we see that $\deg_{\mathbf{x}_2} \Phi_{12}(\widetilde{M}) M_1 = \deg_{\mathbf{x}_2} M_1 = 0$, a contradiction.

Thus, we can write $M = \widetilde{M}M_1$, where $\widetilde{M} \in \mathbb{M}(d, \mathbf{X}_2)$. By Lemma (2.1.1), (ii), we have $X_{11}^d \widetilde{M} \equiv X_{21}^d \Phi_{12}(\widetilde{M})$, from which it follows that $X_{11}^d M \equiv X_{21}^d \Phi_{12}(\widetilde{M}) M_1$. Using (2-1), one then obtains $X_{21}^d (\Phi_{12}(\widetilde{M}) M_1 + N) \equiv 0$, hence $N \equiv -\Phi_{12}(\widetilde{M}) M_1$ because $I_2(X_{ij})$ is a prime ideal. Since $\widetilde{M} \in \mathbb{M}(d, \mathbf{X}_2)$, it follows that $\Phi_{12}(\widetilde{M}) \in \mathbb{M}(d, \mathbf{X}_1)$.

Altogether, one gets

$$\begin{pmatrix} M(\mathbf{x}) \\ N(\mathbf{x}) \end{pmatrix} = M_1(\mathbf{x}) \begin{pmatrix} \widetilde{M}(\mathbf{x}_2) \\ -(\widetilde{M}(\mathbf{x}_1)) \end{pmatrix} \in \text{Im } \psi^{[d]},$$

as was to be shown. □

Here is the main result of this section.

(2.1.5) Theorem. *Let $d \geq 1$ be an integer and let $\mathfrak{R}^{[d]} \subset S = k[X_{ij}]/I_2(X_{ij})$ as before stand for the ideal generated by the d th powers of the generators of the ideal*

\mathfrak{R} of S . Also let $R^{[d]} = k[t_1 s_j, t_1^d, \dots, t_n^d] \subset k[\mathbf{t}, \mathbf{s}]$ ($1 \leq i \leq n, 1 \leq j \leq m$). Then:

- (i) The ideal $\mathfrak{R}^{[d]}$ is of linear type.
- (ii) There is a presentation

$$R^{[d]} \simeq k[\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n, \mathbf{U}] / \sum_{1 \leq i_1 < i_2 \leq n} I_2(L_{i_1, i_2})$$

where

$$L_{i_1, i_2} = \begin{pmatrix} X_{i_1, 1} \cdots X_{i_1, m} & U_{i_1} \cdot \mathbb{M}(d-1, \mathbf{X}_{i_2}) \\ X_{i_2, 1} \cdots X_{i_2, m} & U_{i_2} \cdot \mathbb{M}(d-1, \mathbf{X}_{i_1}) \end{pmatrix},$$

with $U_{i_i} \cdot \mathbb{M}(d-1, \mathbf{X}_{i_i})$ designating the row whose entries are the entries of $\mathbb{M}(d-1, \mathbf{X}_{i_i})$ multiplied by the variable U_{i_i} .

Proof. (i) We show that the generators $x_{11}^d, \dots, x_{n1}^d$ of $\mathfrak{R}^{[d]}$ form a d -sequence. For that, we use the characterization of such sequences as given in [HSV, Section 6] to the effect that

$$((x_{11}^d, \dots, x_{s1}^d) : x_{s+1,1}^d) \cap \mathfrak{R}^{[d]} = (x_{11}^d, \dots, x_{s1}^d) \quad \text{for } 0 \leq s \leq n-1.$$

By Proposition (2.1.4), one sees that

$$((x_{11}^d, \dots, x_{s1}^d) : x_{s+1,1}^d) = (\mathbb{M}(d, \mathbf{x}_1), \mathbb{M}(d, \mathbf{x}_2), \dots, \mathbb{M}(d, \mathbf{x}_s)),$$

hence we are to prove that

$$(\mathbb{M}(d, \mathbf{x}_1), \mathbb{M}(d, \mathbf{x}_2), \dots, \mathbb{M}(d, \mathbf{x}_s)) \cap (x_{11}^d, \dots, x_{n1}^d) \subset (x_{11}^d, \dots, x_{s1}^d).$$

Set $J_1 = (\mathbb{M}(d, \mathbf{x}_1), \mathbb{M}(d, \mathbf{x}_2), \dots, \mathbb{M}(d, \mathbf{x}_s))$ and $J_2 = (x_{11}^d, \dots, x_{n1}^d)$.

To compute the above intersection of monomial ideals modulo the binomial ideal $I_2(X_{ij})$ we follow the prescription given in [EiSt, Proof of Corollary 1.6]: choose a monomial order on the polynomial ring $A = k[X_{ij}]$ and take the standard monomials mod $I_2(X_{ij})$; then, $\mathcal{M}(\mathbf{x}) \subset A/I_2(X_{ij})$, the set of residues of the standard monomials, is a vector space basis of $A/I_2(X_{ij})$; next, one takes a vector space basis \mathcal{J}_1 (resp \mathcal{J}_2) of J_1 (resp. J_2) mod $I_2(X_{ij})$ which is contained in $\mathcal{M}(\mathbf{x})$; at the outset, $\mathcal{J}_1 \cap \mathcal{J}_2$ is a vector space basis of the ideal $J_1 \cap J_2$.

Now, in the present case, choosing a suitable order, the 2×2 minors already form a Gröbner basis of the ideal $I_2(X_{ij})$ (cf., e.g., [Stu]). Therefore, a monomial in $\mathcal{M}(\mathbf{x})$ is characterized by the property that it involves indeterminates belonging to one and only one row or to one and only one column of the matrix (x_{ij}) . It follows from this that

$$\mathcal{J}_1 = \left(\bigcup_{\substack{1 \leq i \leq s \\ r \geq d}} \mathbb{M}(r, \mathbf{x}_i) \right) \cup \{x_{ij}^d M(\mathbf{x}^j) \mid 1 \leq i \leq s, 1 \leq j \leq m\}$$

is a vector basis of J_1 , where $M(\mathbf{x}^j)$ designates a monomial involving only variables along the j th column.

By a similar token,

$$\mathcal{J}_2 = \{x_{i1}^d M(\mathbf{x}^j), x_{i1}^d M(\mathbf{x}_i) \mid 1 \leq i \leq n\}$$

is a vector basis of J_2 , where $M(\mathbf{x}_i)$ designates a monomial involving only variables along the i th row. One clearly has $\mathcal{J}_1 \cap \mathcal{J}_2 = \{x_{i1}^d M(\mathbf{x}^j), x_{i1}^d M(\mathbf{x}_i) \mid 1 \leq i \leq s\}$. Therefore, the ideal $J_1 \cap J_2$ is generated by $\{x_{i1}^d, \mid 1 \leq i \leq s\}$, as was to be shown.

(ii) By part (i), the canonical surjection $\mathcal{S}(\mathfrak{R}^{[d]}) \rightarrow \mathcal{R}(\mathfrak{R}^{[d]})$ is an isomorphism, where $\mathcal{S}(\mathfrak{R}^{[d]})$ and $\mathcal{R}(\mathfrak{R}^{[d]})$ denote the symmetric and the Rees algebra of the ideal $\mathfrak{R}^{[d]}$, respectively. On the other hand, by Proposition (2.1.4), $\mathcal{S}(\mathfrak{R}^{[d]})$ admits the presentation that is being proposed for $R^{[d]}$. Therefore, it suffices to show that $R^{[d]}$ is isomorphic to $\mathcal{R}(\mathfrak{R}^{[d]})$. Clearly,

$$\mathcal{R}(\mathfrak{R}^{[d]}) \simeq S[\mathfrak{R}^{[d]}T] \simeq k[t_i s_j, (t_1 s_1)^d T, \dots, (t_n s_1)^d T] \subset k[\mathbf{t}, \mathbf{s}][T].$$

Since s_1^d is a common factor throughout the terms $t_i^d s_1^d T$ and these have a fixed degree, we see that there is an isomorphism $k[t_i s_j, t_1^d, \dots, t_n^d] \simeq k[t_i s_j, (t_1 s_1)^d T, \dots, (t_n s_1)^d T]$. \square

2.2. Hilbert function data of $R^{[d]}$. The reader is referred to [HUT] and [STV] for the background needed in this portion. Again, one considers the Segre ring $S = k[\mathbf{X}]/I_2(\mathbf{X})$, which will be thought of as the current base ring. By Theorem (2.1.5), $R^{[d]}$ is isomorphic to the Rees algebra of the ideal $\mathfrak{R}^{[d]} \subset S$ and, moreover, as such, it has a natural structure of standard bigraded k -algebra, its presentation ideal over S being bihomogeneous with respect to the two sets of variables $\mathbf{X} = \{X_{ij}\}$ and $\mathbf{U} = \{U_1, \dots, U_n\}$.

Consider an \mathbb{N}^{n+1} -gradation on $S[\mathbf{U}]$ by setting

$$S[\mathbf{U}]_{(a_0, a_1, \dots, a_n)} := S_{a_0} U_1^{a_1} \dots U_n^{a_n}.$$

Let \succeq be the graded lexicographic order on the monoid \mathbb{N}^{n+1} . It induces a filtration \mathcal{F} on $S[\mathbf{U}]$, with $\mathcal{F}_{\mathbf{a}} := \bigoplus_{\mathbf{b} \succeq \mathbf{a}} S[\mathbf{U}]_{\mathbf{b}}$, hence also on the residue ring $R^{[d]} \simeq S[\mathbf{U}]/\mathcal{J}$ which we still denote by \mathcal{F} . Letting \mathcal{J}^* denote the ideal generated by the initial forms of \mathcal{J} , one has $\text{gr}_{\mathcal{F}}((R^{[d]})) \simeq S[\mathbf{U}]/\mathcal{J}^*$ as bigraded k -algebras.

By Proposition (2.1.4) (or by the proof of Theorem (2.1.5), (i)) and [HUT, Lemma 1.1], one obtains

$$\mathcal{J}^* = (\mathbb{M}(d, \mathbf{x}_1) U_2, (\mathbb{M}(d, \mathbf{x}_1), \mathbb{M}(d, \mathbf{x}_2)) U_3, \dots, (\mathbb{M}(d, \mathbf{x}_1), \dots, \mathbb{M}(d, \mathbf{x}_{n-1})) U_n).$$

(2.2.1) Proposition. *With the preceding notation and considering $R^{[d]}$ and $\text{gr}_{\mathcal{F}}((R^{[d]}))$ as \mathbb{N} -graded rings (via the homomorphism $\mathbb{N}^2 \rightarrow \mathbb{N}, (a, b) \mapsto a + b$), one has:*

- (i) $R^{[d]}$ and $\text{gr}_{\mathcal{F}}((R^{[d]}))$ admit the same Hilbert function.
- (ii) The multiplicity of $R^{[d]}$ is

$$e(R^{[d]}) = \sum_{j=0}^{n-1} d^j \binom{m+n-j-2}{n-j-1}.$$

Proof. (i) This is easy and holds quite generally.

(ii) We apply [HUT, Theorem 1.4] (or rather, its recipe), for which we first check its hypotheses. In the present situation, they boil down to the equalities

$$\dim S/I_j = \dim S - j, \quad 1 \leq j \leq n - 1,$$

where $I_j = (\mathbb{M}(d, \mathbf{x}_1), \dots, \mathbb{M}(d, \mathbf{x}_j))$. To verify these, we show that $\text{ht } I_j = j$ for $1 \leq j \leq n - 1$ (recalling that S is Cohen–Macaulay). For every such j , consider the prime ideal

$$\begin{aligned} P_j &= (\{X_{kl} \mid 1 \leq k \leq j, 1 \leq l \leq m\}) + I_2(\{X_{k'l} \mid j+1 \leq k' \leq n, 1 \leq l \leq m\}) \\ &= (\{\mathbf{X}_1, \dots, \mathbf{X}_j\}) + I_2(\mathbf{X} \setminus \{\mathbf{X}_1, \dots, \mathbf{X}_j\}) \subset k[\mathbf{X}]. \end{aligned}$$

Clearly, $P_j S$ is a prime as well and contains I_j . It follows that

$$\begin{aligned} \text{ht } I_j &\leq \text{ht } P_j S = \text{ht } P_j - \text{ht } I_2(\mathbf{X}) \\ &= \text{ht}(\{\mathbf{X}_1, \dots, \mathbf{X}_j\}) + \text{ht } I_2(\mathbf{X} \setminus \{\mathbf{X}_1, \dots, \mathbf{X}_j\}) - (n - 1)(m - 1) \\ &= jm + (n - j - 1)(m - 1) - (n - 1)(m - 1) = j \end{aligned}$$

On the other hand, it is easy to see that every prime ideal of S containing I_j already contains $P_j S$. This leads to $\text{ht } I_j = j$, as required.

We now compute the multiplicity $e(S/I_j)$ by the associativity formula. By the above calculation, this formula reduces to

$$e(S/I_j) = \ell(S_{P_j S}/I_{j P_j S})e(S/P_j S).$$

To simplify the notation, set $P = P_j, I = I_j$. Observe that the ideal PS_{PS}/I_{PS} is generated by the images of the variables X_{11}, \dots, X_{j1} . Indeed, typically, $X_{k1}X_{nl} - X_{kl}X_{n1} \equiv 0 \pmod{I_2(\mathbf{X})}$. Since X_{n1} is invertible, the image of X_{kl} belongs to the ideal generated by the image of X_{k1} , for $1 \leq k \leq j$. The above length is then given by the number of monomials $\{X_{11}^{a_1} \cdots X_{j1}^{a_j} \mid 0 \leq a_k \leq d - 1, 1 \leq k \leq j\}$. This number is clearly d^j .

Next, one has $S/P_j S = k[\mathbf{X}]/P \simeq k[\mathbf{X} \setminus \{\mathbf{X}_1, \dots, \mathbf{X}_j\}]/I_2(\mathbf{X} \setminus \{\mathbf{X}_1, \dots, \mathbf{X}_j\})$, which is a Segre ring of size $(n - j) \times m$. Therefore, $e(S/P) = \binom{m+n-j-2}{n-j-1}$ by a well-known formula (cf., e.g., [STV, Remark 2.5]).

To piece everything together, [HUT, Theorem 1.4] tells us that $e(R^{[d]}) = \sum_{j=0}^{n-1} e(S/I_j)$, hence we are through. □

(2.2.2) *Remark.* By Proposition (2.2.1), (i), it is in principle possible to compute the Hilbert function of $R^{[d]}$, but it is hardly the case that it may be of any usefulness here. Thus, for example, $\dim R^{[d]} = m + n$ follows directly from the fact that $R^{[d]}$ is a Rees algebra of an ideal in the $m + n - 1$ -dimensional domain S .

3. THE DEFINING EQUATIONS OF THE SPECIAL ALGEBRA

As above, let $I = I^{[d]} \subset k[\mathbf{X}, \mathbf{U}]$ denote the presentation ideal of the k -algebra $R^{[d]}$ and let $\tilde{I} = IS[\mathbf{U}] \subset S[\mathbf{U}]$, an ideal generated in bidegree $(d, 1)$. We consider the Rees algebra $\mathcal{R}_{S[\mathbf{U}]}(\tilde{I})$: geometrically, one is looking at the blowup of the product $\mathfrak{S} \times \mathbb{P}_{\mathbf{U}}^{n-1}$ along the subvariety $\mathcal{B}\ell_{\mathcal{K}}(\mathfrak{S})$, where \mathcal{K} denotes the subvariety of \mathfrak{S} defined by the ideal $\mathfrak{R}^{[d]}$.

The *special algebra* (or *fiber cone algebra*) of an ideal (resp. homogeneous ideal) \mathfrak{a} in a local (resp. positively graded) ring A is the residue ring $\mathcal{F}(\mathfrak{a}) := \mathcal{R}_A(\mathfrak{a})/\mathfrak{m}\mathcal{R}_A(\mathfrak{a})$, with \mathfrak{m} standing for the maximal (resp. maximal graded) ideal of A .

We will take $A = S[\mathbf{U}]$ and $\mathfrak{a} = \tilde{I}$. As it will turn out, $\mathcal{F}(\tilde{I})$ is a nice determinantal locus which, in the case where $n = 2$, is the coordinate ring of a Veronese variety. The reason for that is a far more reaching principle which may have an independent interest outside the scope of the present work.

(3.1) Theorem. *Let \mathbf{X}, \mathbf{Y} be mutually independent sets of variables over a field k of characteristic zero, with \mathbf{X} and \mathbf{Y} having the same number of elements, and let f_1, \dots, f_r be homogeneous polynomials in the \mathbf{X} -variables, of the same degree. Let U, V be two additional variables and set $A = k[f_1 V - \Phi(f_1)U, \dots, f_r V - \Phi(f_r)U] \subset$*

$k[\mathbf{X}, \mathbf{Y}, U, V]$, where Φ as in Lemma (2.2.1) denotes the involutive k -isomorphism $X_i \mapsto Y_i$. Then

$$k[f_1, \dots, f_r] \simeq A/A \cap I_2(\mathbf{X}, \mathbf{Y}) k[\mathbf{X}, \mathbf{Y}, U, V]$$

as graded k -algebras, where $I_2(\mathbf{X}, \mathbf{Y})$ denotes the ideal of $k[\mathbf{X}, \mathbf{Y}]$ generated by the 2×2 minors of the generic matrix whose rows are \mathbf{X} and \mathbf{Y} .

Proof. Let T_1, \dots, T_r be presentation variables over k for both algebras. It will suffice to show that they have the same presentation ideal. We show, namely, that any homogeneous polynomial relation of one of the two algebras is a polynomial relation of the other. We need the notion of *polarization*.

Consider a polynomial ring $k[\mathbf{T}, \mathbf{U}]$ in two sets of indeterminates $\mathbf{T} = T_1, \dots, T_r$ and $\mathbf{U} = U_1, \dots, U_r$. Clearly, $k[\mathbf{T}, \mathbf{U}]$ is a free $k[\mathbf{U}]$ -module with basis the monomials in \mathbf{T} .

(3.2) Definition. The *polarization* of \mathbf{T} by \mathbf{U} is the (unique) $k[\mathbf{U}]$ -homomorphism P of the $k[\mathbf{U}]$ -module $k[\mathbf{T}, \mathbf{U}]$ such that $P(1) = 0$ and

$$P(\mathbf{T}^a) = \sum_{a_j \neq 0} a_j U_j T_1^{a_1} \dots T_j^{a_j-1} \dots T_r^{a_r}$$

for $\mathbf{T}^a = T_1^{a_1} \dots T_r^{a_r}$.

One sets $P_0(\mathbf{T}^a) = \mathbf{T}^a$ and $P_l(\mathbf{T}^a) = P_{l-1}(P(\mathbf{T}^a))$. Next, consider the k -algebra homomorphism $\Psi' : k[\mathbf{T}, \mathbf{U}] \rightarrow k[f_1, \dots, f_r, \Phi(f_1), \dots, \Phi(f_r)]$ such that $\Psi'(T_j) = f_j$, $\Psi'(U_j) = \Phi(f_j)$, and let Ψ denote the restriction of Ψ' to $k[\mathbf{T}]$.

Let $F(\mathbf{T}) = \sum_a \alpha_a \mathbf{T}^a \in k[\mathbf{T}]$ be a homogeneous polynomial of degree t , with $a = (a_1, \dots, a_r)$, $|a| = t$ and $\mathbf{T}^a = T_1^{a_1} \dots T_r^{a_r}$, and let s denote the common degree of the f 's. We claim that $X_1^s \Psi'(P(F(\mathbf{T}))) \equiv t Y_1^s \Psi(F(\mathbf{T})) \pmod{I_2(\mathbf{X}, \mathbf{Y})}$. Indeed, it follows from Lemma (2.2.1) that, for a given term $\alpha_a \mathbf{T}^a$ of $F(\mathbf{T})$ ($\alpha_a \neq 0$), one has

$$\Psi'(P(\mathbf{T}^a)) \equiv (a_{i(a)} + \dots + a_r) \Phi(f_{i(a)}) f_{i(a)}^{a_{i(a)}-1} f_{i(a)+1}^{a_{i(a)+1}} \dots f_r^{a_r},$$

where $a_{i(a)} \neq 0, a_i = 0 \pmod{I_2(\mathbf{X}, \mathbf{Y})}$ ($i < i(a)$). By summing up over all terms of $F(\mathbf{T})$, one obtains

$$\Psi(P(F(\mathbf{T}))) \equiv t \sum_a \alpha_a \Phi(f_{i(a)}) f_{i(a)}^{a_{i(a)}-1} f_{i(a)+1}^{a_{i(a)+1}} \dots f_r^{a_r} \pmod{I_2(\mathbf{X}, \mathbf{Y})}.$$

Again by Lemma (2.2.1), one has $X_1^s \Phi(f_j) = Y_1^s f_j$. Substituting yields the desired result.

Next, by iterating the polarization, one easily gets

$$(3-1) \quad X_1^{ls} \Psi'(P_l(F(\mathbf{T}))) \equiv \frac{t!}{(t-l)!} Y_1^{ls} \Psi(F(\mathbf{T})) \pmod{I_2(\mathbf{X}, \mathbf{Y})}$$

where $P_l(F(\mathbf{T})) = 0$ if $l > t$.

On the other hand, a computation yields

$$F(f_1 V - \Phi(f_1) U, \dots, f_r V - \Phi(f_r) U) = \sum_{l=0}^t (-1)^l \Psi'(P_l(F(\mathbf{T}))) V^{t-l} U^l.$$

Using (3-1) with $l = t$, one gets

$$X_1^{ls} F(f_1 V - \Phi(f_1) U, \dots, f_r V - \Phi(f_r) U) \equiv g \Psi(F(\mathbf{T})) \pmod{I_2(\mathbf{X}, \mathbf{Y})},$$

where

$$g = \sum_{l=0}^t (-1)^l \frac{t!}{(t-l)!} X_1^{(t-l)s} Y_1^{ls} V^{t-l} U^l \notin I_2(\mathbf{X}, \mathbf{Y})[\mathbf{X}, \mathbf{Y}, U, V].$$

One concludes that $F(f_1V - \Phi(f_1)U, \dots, f_rV - \Phi(f_r)U) \in I_2(\mathbf{X}, \mathbf{Y})[\mathbf{X}, \mathbf{Y}, U, V]$ if and only if $\Psi(F(\mathbf{T})) \in I_2(\mathbf{X}, \mathbf{Y})[\mathbf{X}, \mathbf{Y}, U, V] \cap k[\mathbf{X}] = (0)$.

This finishes the proof. \square

(3.3) Corollary. *Notation as in the beginning of the section. Moreover, let $n = 2$. Then $\mathcal{F}(\tilde{I})$ is isomorphic to the homogeneous coordinate ring of the duple Veronese model of \mathbb{P}^{m-1} . In particular, $\mathcal{F}(\tilde{I})$ is normal and Cohen–Macaulay.*

Proof. By Proposition (2.2.3), $\mathcal{F}(\tilde{I}) \simeq k[M_\alpha]$, where M_α runs through the monomials of degree d in the variables \mathbf{X} . \square

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