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Group classification and conservation laws for a two-dimensional generalized Kuramoto–Sivashinsky equation

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ABSTRACT

The two-dimensional anisotropic Kuramoto–Sivashinsky equation is a fourth-order nonlinear evolution equation in two spatial dimensions that arises in sputter erosion and epitaxial growth on vicinal surfaces. A generalization of this equation is proposed and studied via group analysis methods. The complete group classification of this generalized Kuramoto–Sivashinsky equation is carried out; it is classified according to the property of the self-adjointness and the corresponding conservation laws are established.

1. Introduction

The celebrated Kuramoto–Sivashinsky Equation (KSE)

$$u_t + u_{xxxx} + u_{xx} + \frac{1}{2}u_x^2 = 0, \tag{1}$$

where $u = u(x, t)$, is an equation that for nearly half a century has attracted the attention of many researchers from various areas due to its simple and yet rich dynamics [1]. It first appeared in the mid-1970s by Kuramoto in the study of angular-phase turbulence for a system of reaction–diffusion equations modeling the Belousov–Zhabotinskii reaction in three spatial dimensions [2–4] and independently by Sivashinsky in the study of hydrodynamic instabilities in laminar flame fronts [5–7].

In a physical context Eq. (1) is used to model continuous media that exhibits chaotic behavior such as weak turbulence on interfaces between complex flows (quasi-planar flame front and the fluctuation of the positions of a flame front, fluctuations in thin viscous fluid films flowing over inclined planes or vertical walls, dendritic phase change fronts in binary alloy mixtures), small perturbations of a metastable planar front or interface (spatially uniform oscillating chemical reaction in a homogeneous medium) and physical systems driven far from the equilibrium due to intrinsic instabilities (instabilities of dissipative trapped ion modes in plasmas and phase dynamics in reaction–diffusion systems) [8–13].

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As a dynamical system the KSE is known for its chaotic solutions and complicated behavior due to the terms that appear. Namely, the u_{xx} term acts as an energy source and has a destabilizing effect at large scale, the dissipative u_{xxxx} term provides dumping in small scales and, finally, the nonlinear term provides stabilization by transferring energy between large and small scales. Because of this fact, Eq. (1) was studied extensively as a paradigm of finite dynamics in a partial differential equation (PDE). Its multi-modal, oscillatory and chaotic solutions have been investigated [14–18]; its non-integrability was established via its Painlevé analysis [19–21] and due to its bifurcation behavior a connection to low finite-dimensional dynamical systems is established [22,23].

The generalization of KSE to two dimensions comes naturally, the two-dimensional Kuramoto–Sivashinsky equation,

$$u_t + \nabla^4 u + \nabla^2 u + (\nabla u)\cdot(\nabla u) = 0, \tag{2}$$

where now $u = u(x, y, t)$ and $\nabla^2 = \nabla\cdot\nabla$, $\nabla^4 = \nabla\cdot\nabla(\nabla\cdot\nabla)$. Eq. (2) has equally attracted much attention because of the same spatiotemporal chaos properties that exhibits and its applications in modeling complex dynamics in hydrodynamics [24–27]. Nevertheless, due to the additional spatial dimension equation (2) is very challenging and even its well-posedness is still an open problem [28,29].

One generalization of Eq. (2) of much interest is the *anisotropic* two-dimensional Kuramoto–Sivashinsky equation,

$$u_t = \frac{1}{2}u_x^2 + \frac{\beta}{2}u_y^2 - u_{xx} - \alpha u_{yy} - u_{xxxx} - 2u_{xxyy} - u_{yyyy}, \tag{3}$$

where the two real parameters α, β control the anisotropy of the linear and the nonlinear term, respectively, in other words, the stability of the solutions of Eq. (3). The anisotropic two-dimensional Kuramoto–Sivashinsky equation, due to the fact that it describes linearly unstable surface dynamics in the presence of in-plane anisotropy, has a wide range of applications, for instance, as a model for the nonlinear evolution of sputter-eroded surfaces and describing the epitaxial growth of a vicinal surface destabilized by step edge barriers; for further details see [30] and the references therein, in particular [31].

This paper focuses on the following generalization of the anisotropic KSE (3):

$$u_t = \frac{1}{2}u_x^2 + h(u)u_y^2 + r(u)u_{xx} + g(u)u_{yy} - u_{xxxx} - 2u_{xxyy} - u_{yyyy} + f(u) \tag{4}$$

and its study under the prism of Lie point symmetries and conservation laws. (Here, f, h, g and r are considered smooth functions of $u = u(x, y, t)$.)

The symmetries of a differential equation are of fundamental importance since they are a structural property of the equation. In addition, finding the symmetries of a differential equation is an analytic method that can be applied to integrable and non-integrable equations alike. Nevertheless the symmetry analysis is constrained to rudimentary generalizations of Eq. (2) [32,33]. Things are worse when one looks for conservation laws. For the complexity of the calculations involved, the research is constrained to generalizations of the one dimensional Kuramoto–Sivashinsky equation [34].

In this frame two different classifications are performed: the complete group classification and a classification with respect to the property of self-adjointness. Having the symmetries for each possible case and the self-adjoint cases at hand, the conservations laws for that system by using the Noether operator $\mathcal{N}$ are obtained; see also [35–37].

Calculating the symmetries of the system (3), obtaining its adjoint system and applying the Noether operator to obtain the conserved vectors are well-defined algorithmic procedures. Nevertheless, the calculations involved are usually very difficult and extensive even for the simplest equations. Thus, it may become very tedious and error prone. For that reason the use of computer algebra systems like *Mathematica, Maple, Reduce*, etc. and of special symbolic packages that are build based on them is very crucial. For this work the *Mathematica* package SYM [38–40] was extensively used for all the results that follow, namely, for obtaining the symmetries of the system, to get and simplify the adjoint system and the conserved vectors that emerge from the use of the Noether operator.

In Section 2 the definitions and the analytical tools used are introduced. Section 3 explores the self-adjointness of Eq. (4). Then, the complete group classification of Eq. (4) is carried out in Section 4 followed by Section 5 where the conservation laws are established. Finally in Section 6 some comments and concluding remarks are presented.

2. Notation and methodology

We shall employ in this work the two analytical tools: the symmetry analysis and the use of the Noether operator identity for the explicit construction of conservation laws. For both the necessary definitions of the notions that will be encountered in the main body of this work are illustrated below, adapted accordingly to the needs of the present paper.

For brevity, we denote:

$$\Delta(x, y, t, u, u_x, \ldots, u_{yyyy}) = \frac{1}{2}u_x^2 + h(u)u_y^2 + r(u)u_{xx} + g(u)u_{yy} - u_{xxxx} - 2u_{xxyy} - u_{yyyy} - u_t + f(u).$$

2.1. Modern group analysis

The symmetry or modern group analysis is a valuable analytic tool for the investigation of differential equations. For a full treatise of the subject there is a wealth of classical texts that encompass all aspects of the theory [41–47].

Definition 2.1. Let the differential operator,

$$X = \xi^1(x, y, t, u)\partial_x + \xi^2(x, y, t, u)\partial_y + \xi^3(x, y, t, u)\partial_t + \eta(x, y, t, u)\partial_u. \tag{5}$$

This operator, from now on called the *infinitesimal generator*, determines a *Lie point symmetry* of Eq. (4), if and only if, its action on the equation will be, modulo the equation itself, identically zero, that is:

$$X^{(4)}\left[\Delta(x, y, t, u, u_x, \dots, u_{yyyy})\right]\Big|_{\Delta(x,y,t,u,u_x,\dots,u_{yyyy})=0} \equiv 0, \tag{6}$$

where $X^{(4)}$ is the fourth order prolongation of the operator X given by

$$X^{(4)} = X + \sum_{s=1}^{4} \eta^{(s)}_{i_1\dots i_s} \frac{\partial}{\partial u_{i_1\dots i_s}}, \quad i_n = 1, 2, 3 \tag{7}$$

with

$$\eta^{(1)}_i = D_i\eta - (D_i\xi^j)u_j, \qquad \eta^{(s)}_{i_1\dots i_s} = D_{i_s}\eta^{(s-1)}_{i_1\dots i_{s-1}} - (D_{i_s}\xi^j)u_{i_1\dots i_{s-1}j}$$

and the partial derivatives denoted by

$$u_i = \frac{\partial u}{\partial x^i}, \qquad (x^1, x^2, x^3) = (x, y, t).$$

From condition (6), called the linearized symmetry condition, an overdetermined system of linear partial differential equations emerges. By solving this system, called the determining equations, we determine the coefficients ξ^i, η of the infinitesimal generator, hence, the point symmetries of the equation. The group classification occurs in that phase: the determining equations contain also the functions f, g, h, r. The group classification is performed by investigating each case where specific relations among the unknown elements remove equations from the set of determining equations, and by doing that enlarging the set of solutions.

2.2. The adjoint and self-adjoint concepts and conservation laws

In accordance with [35–37] we introduce the required notions that will enable us to construct conservation laws for Eq. (4). The significance of conservation laws is considerable; they are used for investigating integrability and linearization mappings, for establishing the existence and uniqueness as well as for analyzing the stability and global behavior of solutions. Last but not least, they have a pivotal role in the development of numerical methods and for finding nonlocally related systems and potential variables.

Definition 2.2. The *adjoint equation* to Eq. (4) is

$$\frac{\delta\mathcal{L}}{\delta u} = 0, \tag{8}$$

where $\mathcal{L}$ is the *formal Lagrangian* given by

$$\mathcal{L} = \upsilon(x, y, t)\Delta(x, y, t, u, u_x, \dots, u_{yyyy}).$$

Here υ is a new dependent variable, called also the *nonlocal variable*, and $\delta/\delta u$ is the Euler–Lagrange operator

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \sum_{i_x+i_y+i_t=1}^{\infty} (-1)^{i_x+i_y+i_t} D_x^{i_x} D_y^{i_y} D_t^{i_t} \frac{\partial}{\partial u_{i_x x\, i_y y\, i_t t}},$$

with $u_{i_x x\, i_y y\, i_t t}$ denoting $\partial^{i_x+i_y+i_t}u/\partial x^{i_x}\partial y^{i_y}\partial t^{i_t}$, $i_x, i_y, i_t \geq 0$.

Definition 2.3. We say that Eq. (4) is *strictly self-adjoint* if the adjoint equation (8) becomes equivalent to Eq. (4) after the substitution $\upsilon = u$:

$$\frac{\delta\mathcal{L}}{\delta u} = \lambda\Delta(x, y, t, u, u_x, \dots, u_{yyyy}),$$

with λ a generic coefficient.

Definition 2.4. We say that Eq. (4) is *quasi self-adjoint* if the adjoint equation (8) becomes equivalent to Eq. (4) after the substitution $\upsilon = \phi(u)$, where $\phi(u) \neq 0$.

Definition 2.5. We say that Eq. (4) is *nonlinearly self-adjoint* if the adjoint equation (8) becomes equivalent to Eq. (4) after the substitution $\upsilon = \phi(x, y, t, u)$, where $\phi(x, y, t, u) \neq 0$.

Remark 2.6. The concept of the nonlinear self-adjointness can be further extended by considering differential substitutions of the form

$$\upsilon = \phi(x, y, t, u, u_{(1)}, \ldots, u_{(r)}),$$

where $u_{(r)}$ are the derivatives of u of order r.

Remark 2.7. From the above definitions it is obvious that if an equation is strictly or quasi self-adjoint then it is also nonlinearly self-adjoint.

Theorem 2.1 (*Explicit Formula for Conserved Vectors*). *Let Eq. (4) be nonlinearly self-adjoint and (5) its Lie point symmetry. A conserved vector can be constructed by the following formula:*

$$C^i = \xi^i \mathcal{L} + \sum_{i_x+i_y+i_t=0}^{\infty} D_x^{i_x} D_y^{i_y} D_t^{i_t}(W) \frac{\delta^* \mathcal{L}}{\delta^* u_{i\, i_x x\, i_y y\, i_t t}}, \quad i = x, y, t,\ i_i \geq 0 \tag{9}$$

where

$$W = \eta - \xi^1 u_x - \xi^2 u_y - \xi^3 u_t,$$

$\delta^*/\delta^* u$ is the "weighted" Euler–Lagrange operator

$$\frac{\delta^* \mathcal{L}}{\delta^* u_{i_x x\, i_y y\, i_t t}} = \frac{\partial}{\partial u} + \sum_{s=j_x+j_y+j_t=1}^{\infty} (-1)^s \frac{\binom{s}{j_x, j_y, j_t}}{\binom{s+i_x+i_y+i_t}{i_x+j_x,\, i_y+j_y,\, i_t+j_t}} D_x^{j_x} D_y^{j_y} D_t^{j_t} \frac{\partial}{\partial u_{(i_x+j_x)x\,(i_y+j_y)y\,(i_t+j_t)t}}$$

with $\binom{N}{i_1, i_2, \ldots, i_r} = \frac{N}{i_1! i_2! \ldots i_r!}$, $N = i_1 + i_2 + \cdots + i_r$, *the multinomial and $\mathcal{L}$ the formal Lagrangian after substituting with* $\upsilon = \phi(x, y, t, u)$.

Proof. For a proof see [35]. □

Remark 2.8. From a conserved vector (C^x, C^y, C^t) the conservation law has the form

$$D_x(C^x) + D_y(C^y) + D_t(C^t) = 0$$

satisfied on the solutions of Eq. (4).

Definition 2.9. A conserved vector is called *trivial* if,

- its divergence is identically zero and,
- the components of the vector vanish on the solutions of Eq. (4).

Remark 2.10. Only the nontrivial conserved vectors will be considered.

3. The self-adjointness classification

As it can be seen by the definitions of self-adjointness the first step is to obtain the adjoint equation. For Eq. (4) the adjoint equation is

$$\begin{aligned} &2(h(u) - g')u_y \upsilon_y - g(u)\upsilon_{yy} - r(u)\upsilon_{xx} + (1 - 2r')u_x \upsilon_x \\ &\quad - \upsilon \left(f' + (g'' - h' u_y)u_y^2 + 2(g' - h(u))u_{yy} + r'' u_x^2 + (2r' - 1)u_{xx}\right) + \upsilon_{yyyy} + 2\upsilon_{xxyy} + \upsilon_{xxxx} - \upsilon_t = 0 \end{aligned} \tag{10}$$

where $\upsilon = \upsilon(x, y, t)$ is the new dependent variable.

Theorem 3.1. *Eq. (4) has no strictly self-adjoint subcase.*

Proof. By making the substitution $\upsilon = u$ in the adjoint equation (10) and then substituting u_t from Eq. (4), we get

$$\begin{aligned} &\left(h(u) - 2g'\right) u_y^2 - 2g(u)u_{yy} + 2u_{yyyy} + \frac{1}{2}\left(1 - 4r'\right) u_x^2 - 2r(u)u_{xx} \\ &\quad - u\left(f' + \left(g^{''-h'}\right) u_y^2 + 2\left(g' - h(u)\right) u_{yy} + r'' u_x^2 + \left(-1 + 2r'\right) u_{xx}\right) + 4u_{xxyy} + 2u_{xxxx} - f(u) \equiv 0. \end{aligned}$$

It is obvious that there is no choice of the functions f, h, g, r satisfying the above condition. In other words there is no choice of the functions f, h, g, r that will make Eq. (4) strictly self-adjoint. □

Theorem 3.2. *Eq. (4) is quasi self-adjoint when* $f = c_1$, $r = u/2 + c_2$, $h = g'$.

Proof. After making the substitution $\upsilon = \phi(u)$ in the adjoint equation (10) and then substituting u_t from Eq. (4), we get

$$\begin{aligned}
&\left(\left(h(u) - 2g'\right)\phi' + \phi(u)\left(h' - g''\right) - g(u)\phi'' + 6\phi'''u_{yy} + 2\phi'''u_{xx}\right)u_y^2 \\
&+ \left(\phi(u)\left(1 - 2r'\right) - 2r(u)\phi' + 2\phi''u_{yy}\right)u_{xx} + 4\phi'u_{xxyy} + 2\phi''''u_y^2u_x^2 \\
&+ \left(\frac{1}{2}\left(\left(1 - 4r'\right)\phi' - 2\left(\phi(u)r'' + r(u)\phi''\right)\right) + 2\phi'''(u_{yy} + 3u_{xx})\right)u_x^2 \\
&+ 3\phi''u_{yy}^2 + 2\phi'u_{yyyy} + \phi''''u_x^4 + 4\phi''u_{xy}^2 + 3\phi''u_{xx}^2 + 2\phi'u_{xxxx} \\
&- (\phi(u)f' + f(u)\phi') + \phi''''u_y^4 + \left(2\phi(u)\left(h(u) - g'\right) - 2g(u)\phi'\right)u_{yy} \\
&+ 4\phi''\left(u_{yyy} + u_{xxy}\right)u_y + 4\left(2\phi'''u_yu_{xy} + \phi''u_{xyy} + \phi''u_{xxx}\right)u_x \equiv 0.
\end{aligned}$$

This condition must vanish for every solution u of Eq. (4). Hence we arrive at the system:

$$\begin{aligned}
&\phi(u)f' + f(u)\phi' = 0, \\
&\phi(u)\left(1 - 2r'\right) - 2r(u)\phi' = 0, \\
&\frac{\partial}{\partial x}(u)\phi(u) = \phi(u)g' + g(u)\phi', \\
&\phi' = 0, \qquad \phi'' = 0, \qquad \phi''' = 0, \qquad \phi'''' = 0, \\
&\left(1 - 4r'\right)\phi' - 2\left(\phi(u)r'' + r(u)\phi''\right) = 0, \\
&h(u)\phi' - 2g'\phi' + \phi(u)\left(h' - g''\right) - g(u)\phi'' = 0.
\end{aligned}$$

Solving the above system we get that $\phi(u) = c \neq 0$ and as stated, $f = c_1, r = u/2 + c_2, h = g'$. □

Theorem 3.3. *Eq. (4) is nonlinear self-adjoint if and only if:*

1. $r = u/2 + \alpha, g = \beta u + \gamma, h = \beta, f = \delta u^2 + \epsilon u + \zeta$,
2. $r = u/2 + \alpha, h = g', f = \beta u^2 + \gamma u + \delta + c\int g(u)\,du,\ \beta, c, g'' \neq 0$,
3. $r = u/2 + \alpha, h = g', f = \beta u + \gamma + c\int g(u)\,du,\ c, g'' \neq 0$,
4. $r = u/2 + \alpha, h = g', f = \beta u^2 + \gamma u + \delta,\ \beta, g'' \neq 0$,
5. $r = u/2 + \alpha, h = g', f = \beta u + \gamma,\ g'' \neq 0$.

Proof. After making the substitution $\upsilon = \phi(x, y, t, u)$ in the adjoint equation (10) and then substituting u_t from Eq. (4), we get

$$\begin{aligned}
&-\phi(x, y, t, u)f' - f(u)\phi_u + 2u_{yyyy}\phi_u + 4u_{xxyy}\phi_u + 2u_{xxxx}\phi_u + 3u_{yy}^2\phi_{uu} \\
&+ 4u_{xy}^2\phi_{uu} + 3u_{xx}^2\phi_{uu} + u_y^4\phi_{uuuu} + u_x^4\phi_{uuuu} - \phi_t + 4u_{yyy}\phi_{yu} + 4u_{xxy}\phi_{yu} \\
&+ 4u_y^3\phi_{yuuu} - g(u)\phi_{yy} + \phi_{yyyy} + 4u_{xyy}\phi_{xu} + 4u_{xxx}\phi_{xu} + 4u_x^3\phi_{xuuu} \\
&+ u_{xx}\left(\phi(x, y, t, u)\left(1 - 2r'\right) - 2r(u)\phi_u + 2\phi_{yyu} + 6\phi_{xxu}\right) + 2\phi_{xxyy} + \phi_{xxxx} \\
&+ u_{yy}\left(2u_{xx}\phi_{uu} + 2u_x^2\phi_{uuu} + 4u_x\phi_{xuu} + 2\left(\phi(x, y, t, u)\left(h(u) - g'\right)\right.\right. \\
&- \left.\left. g(u)\phi_u + 3\phi_{yyu} + \phi_{xxu}\right)\right) + 8u_{xy}\phi_{xyu} - r(u)\phi_{xx} \\
&+ u_y^2\left(\phi(x, y, t, u)\left(h' - g''\right) + \left(h(u) - 2g'\right)\phi_u - g(u)\phi_{uu} + 6u_{yy}\phi_{uuu}\right. \\
&+ \left. 2u_{xx}\phi_{uuu} + 2u_x^2\phi_{uuuu} + 6\phi_{yyuu} + 4u_x\phi_{xuuu} + 2\phi_{xxuu}\right) \\
&+ u_x^2\left(-\phi(x, y, t, u)r'' + \left(\frac{1}{2} - 2r'\right)\phi_u - r(u)\phi_{uu} + 6u_{xx}\phi_{uuu} + 2\phi_{yyuu} + 6\phi_{xxuu}\right) \\
&+ u_y\left(4u_{yyy}\phi_{uu} + 4u_{xxy}\phi_{uu} + 2\left(h(u) - g'\right)\phi_y - 2g(u)\phi_{yu} + 12u_{yy}\phi_{yuu}\right. \\
&+ \left. 4u_{xx}\phi_{yuu} + 4u_x^2\phi_{yuuu} + 8u_{xy}\phi_{xuu} + u_x\left(8u_{xy}\phi_{uuu} + 8\phi_{xyuu}\right) + 4\left(\phi_{yyyu} + \phi_{xxyu}\right)\right) \\
&+ u_x\left(4u_{xyy}\phi_{uu} + 4u_{xxx}\phi_{uu} + 8u_{xy}\phi_{yuu} + \left(1 - 2r'\right)\phi_x - 2r(u)\phi_{xu} + 12u_{xx}\phi_{xuu} + 4\left(\phi_{xyyu} + \phi_{xxxu}\right)\right) \equiv 0.
\end{aligned}$$

This condition must vanish for every solution u of Eq. (4). Hence we attain the system:

$$\begin{aligned}
&\left(1 - 2r'\right)\phi_x - 2r(u)\phi_{xu} + 4\left(\phi_{xyyu} + \phi_{xxxu}\right) = 0, \\
&2(h(u) - g')\phi_y - 2g(u)\phi_{yu} + 4\phi_{yyyu} + 4\phi_{xxyu} = 0, \\
&(h(u) - g')\phi(x, y, t, u) + 3\phi_{yyu} + \phi_{xxu} - g(u)\phi_u = 0, \\
&\left(1 - 2r'\right)\phi(x, y, t, u) - 2r(u)\phi_u + 2\phi_{yyu} + 6\phi_{xxu} = 0,
\end{aligned}$$

$$\phi_{yuu} = 0, \quad \phi_{uuu} = 0, \quad \phi_{xuuu} = 0, \quad \phi_{yuuu} = 0, \quad \phi_{uuuu} = 0,$$
$$\phi_u = 0, \quad \phi_{xu} = 0, \quad \phi_{yu} = 0, \quad \phi_{uu} = 0, \quad \phi_{xuu} = 0, \quad \phi_{xyu} = 0,$$
$$\left(1 - 4r'\right)\phi_u - 2r(u)\phi_{uu} - 2r''\phi(x, y, t, u) + 4\phi_{yyuu} + 12\phi_{xxuu} = 0,$$
$$f'\phi + f(u)\phi_u + \phi_t + g(u)\phi_{yy} - \phi_{yyyy} + r(u)\phi_{xx} - 2\phi_{xxyy} - \phi_{xxxx} = 0,$$
$$\left(h' - g''\right)\phi(x, y, t, u) + h(u)\phi_u - 2g'\phi_u - g(u)\phi_{uu} + 6\phi_{yyuu} + 2\phi_{xxuu} = 0.$$

Looking at the above system it is obvious that

$$\phi = \phi(x, y, t), \qquad r(u) = u/2 + c, \qquad h = g'. \tag{11}$$

By substituting (11) in the system, we arrive at the following equation:

$$\phi f' + \phi_{,t} + g(u)\phi_{,yy} - \phi_{,yyyy} + \frac{1}{2}(u + 2\mathbf{c})\phi_{,xx} - 2\phi_{,xxyy} - \phi_{,xxxx} = 0. \tag{12}$$

By differentiating equation (12) two times with respect to u we have in addition

$$\phi f''' + g''\phi_{,yy} = 0,$$
$$\phi f'' + g'\phi_{,yy} + \frac{\phi_{,xx}}{2} = 0.$$

By using these two equations in combination with Eq. (12) we arrive at the five cases stated. □

Therefore having obtained the specific self-adjoint subcases of Eq. (4) it remains to know their symmetries in order to construct the corresponding conserved vectors. In the next section the complete group classification for Eq. (4) is given.

4. The group classification

From the invariant surface condition (6) for Eq. (4) the system of the determining equations is obtained; see Appendix. By solving the subsystem not containing any of the functions f, g, h, r we obtain:

$$\xi^1(x, y, t, u) = \mathcal{F}_{11}(t) - y\mathcal{F}_{13}(t) + \frac{x}{4}\mathcal{F}_1'(t),$$
$$\xi^2(x, y, t, u) = \mathcal{F}_{12}(t) + x\mathcal{F}_{13}(t) + \frac{y}{4}\mathcal{F}_1'(t),$$
$$\xi^3(x, y, t, u) = \mathcal{F}_1(t),$$
$$\eta(x, y, t, u) = \mathcal{F}_{15}(y, t) - \frac{u}{2}\mathcal{F}_1'(t) - \frac{1}{8}x\left(8\mathcal{F}_{11}'(t) - 8y\mathcal{F}_{13}'(t) + x\mathcal{F}_1''(t)\right),$$

and the remaining determining equations are:

$$r'\mathcal{F}_1'' = 0,$$
$$h'\mathcal{F}_1'' = 0,$$
$$g'\mathcal{F}_1'' = 0,$$
$$r'\mathcal{F}_{13}' = 0,$$
$$h'\mathcal{F}_{13}' = 0,$$
$$g'\mathcal{F}_{13}' = 0,$$
$$r'\mathcal{F}_{11}' = 0,$$
$$h'\mathcal{F}_{11}' = 0,$$
$$g'\mathcal{F}_{11}' = 0,$$
$$f'\mathcal{F}_1'' - \mathcal{F}_1''' = 0,$$
$$f'\mathcal{F}_{11}' - \mathcal{F}_{11}'' = 0,$$
$$f'\mathcal{F}_{13}' - \mathcal{F}_{13}'' = 0,$$
$$(1 + 2h(u))\mathcal{F}_{13}' = 0,$$
$$(2h(u) - 1)\mathcal{F}_{13} = 0,$$
$$h'\left(2\mathcal{F}_{15} - u\mathcal{F}_1'\right) = 0,$$
$$(g(u) - r(u))\mathcal{F}_{13} = 0,$$
$$4\mathcal{F}_{12}' + y\mathcal{F}_1'' + 8h(u)\mathcal{F}_{15y} = 0,$$

Table 1
Complete group classification for Eq. (4).

f	g	h	r	Symmetries
$\forall$	$\forall$	$\forall$	$\forall$	$\mathfrak{X}_1=\partial_t,\ \mathfrak{X}_2=\partial_x,\ \mathfrak{X}_3=\partial_y$
$\forall$	r	$\frac{1}{2}$	$\forall$	$\mathfrak{X}_4=x\partial_y-y\partial_x$
$\alpha u+\beta,\alpha\neq 0$	r	$\frac{1}{2}$	c	$\mathfrak{X}_4=e^{\alpha t}\partial_u,\ \mathfrak{X}_5=x\partial_y-y\partial_x,$ $\mathfrak{X}_6=\frac{e^{\alpha t}}{\alpha}\partial_y-e^{\alpha t}y\partial_u,$ $\mathfrak{X}_7=\frac{e^{\alpha t}}{\alpha}\partial_x-e^{\alpha t}x\partial_u$
β	r	$\frac{1}{2}$	$c\neq 0$	$\mathfrak{X}_4=\partial_u,\ \mathfrak{X}_5=t\partial_x-x\partial_u,$ $\mathfrak{X}_6=t\partial_y-y\partial_u,$ $\mathfrak{X}_7=x\partial_y-y\partial_x$
β	r	$\frac{1}{2}$	0	$\mathfrak{X}_4=\partial_u,\ \mathfrak{X}_5=t\partial_x-x\partial_u,$ $\mathfrak{X}_6=t\partial_y-y\partial_u,\ \mathfrak{X}_7=x\partial_y-y\partial_x,$ $\mathfrak{X}_8=2(3\beta t-u)\partial_u+4t\partial_t+x\partial_x+y\partial_y$
γr^3	r	$\frac{1}{2}$	$\alpha u+\beta,\alpha\neq 0$	$\mathfrak{X}_4=x\partial_y-y\partial_x,$ $\mathfrak{X}_5=4\alpha t\partial_t+\alpha x\partial_x+\alpha y\partial_y-2(\alpha u+\beta)\partial_u$
$\zeta(\alpha u+\beta)^3,\ \alpha\neq 0$	$\gamma(\alpha u+\beta)$	$\frac{1}{2}$	$\delta(\alpha u+\beta),\delta\neq\gamma$	$\mathfrak{X}_4=4\alpha t\partial_t+\alpha x\partial_x+\alpha y\partial_y-2(\alpha u+\beta)\partial_u$
δ	α	$\gamma\neq 0$	$\beta\neq\alpha$	$\mathfrak{X}_4=\partial_u,\ \mathfrak{X}_5=t\partial_x-x\partial_u,$ $\mathfrak{X}_6=2t\partial_y-\frac{y}{\gamma}\partial_u$
γ	α	0	$\beta\neq\alpha$	$\mathfrak{X}_4=t\partial_x-x\partial_u,$ $\mathfrak{X}_5=\mathcal{F}(y,t)\partial_u,\ \mathcal{F}_t-\alpha\mathcal{F}_{yy}+\mathcal{F}_{yyyy}=0$
$\zeta(\alpha u+\beta)^3$ [a]	$\gamma(\alpha u+\beta)$ [a]	$\epsilon\neq\frac{1}{2}$	$\delta(\alpha u+\beta)$ [a]	$\mathfrak{X}_4=4\alpha t\partial_t+\alpha x\partial_x+\alpha y\partial_y-2(\alpha u+\beta)\partial_u$
$\delta u+\epsilon,\ \delta\neq 0$	α [b]	$\gamma\neq 0,\ \frac{1}{2}$	β [b]	$\mathfrak{X}_4=e^{t\delta}\partial_u,\ \mathfrak{X}_5=\frac{e^{t\delta}}{\delta}\partial_x-e^{t\delta}x\partial_u$ $\mathfrak{X}_6=\frac{2e^{t\delta}}{\delta}\partial_y-\frac{e^{t\delta}y}{\gamma}\partial_u$
$\gamma u+\delta,\ \gamma\neq 0$	α [b]	0	β [b]	$\mathfrak{X}_4=\frac{e^{t\gamma}}{\gamma}\partial_x-e^{t\gamma}x\partial_u$ $\mathfrak{X}_5=\mathcal{F}(y,t)\partial_u,\ \mathcal{F}_t-\alpha\mathcal{F}_{yy}-\gamma\mathcal{F}+\mathcal{F}_{yyyy}=0$
δ	α [b]	$\gamma\neq 0,\ \frac{1}{2}$	β [b]	$\mathfrak{X}_4=t\partial_x-x\partial_u,\ \mathfrak{X}_5=2t\partial_y-\frac{y}{\gamma}\partial_u$
γ	α [b]	0	β [b]	$\mathfrak{X}_4=t\partial_x-x\partial_u$ $\mathfrak{X}_5=\mathcal{F}(y,t)\partial_u,\ \mathcal{F}_t-\alpha\mathcal{F}_{yy}+\mathcal{F}_{yyyy}=0$
$\beta u+\gamma,\ \beta\neq 0$	0	$\alpha\neq 0,\ \frac{1}{2}$	0	$\mathfrak{X}_4=e^{t\alpha}\partial_u,\ \mathfrak{X}_5=\frac{e^{t\beta}}{\beta}\partial_x-e^{t\beta}x\partial_u$ $\mathfrak{X}_6=\frac{2e^{t\beta}}{\beta}\partial_y-\frac{e^{t\beta}y}{\alpha}\partial_u$
$\alpha u+\beta,\ \alpha\neq 0$	0	0	0	$\mathfrak{X}_4=\frac{e^{t\alpha}}{\alpha}\partial_x-e^{t\alpha}x\partial_u$ $\mathfrak{X}_5=\mathcal{F}(y,t)\partial_u,\ \mathcal{F}_t-\alpha\mathcal{F}+\mathcal{F}_{yyyy}=0$
β	0	$\alpha\neq 0,\ \frac{1}{2}$	0	$\mathfrak{X}_4=\partial_u,\ \mathfrak{X}_5=t\partial_x-x\partial_u$ $\mathfrak{X}_6=2t\partial_y-\frac{y}{\alpha}\partial_u,$ $\mathfrak{X}_7=4t\partial_t+x\partial_x+y\partial_y-2(u-3\beta t)\partial_u$
α	0	0	0	$\mathfrak{X}_4=t\partial_x-x\partial_u,\ \mathfrak{X}_5=4t\partial_t+x\partial_x+y\partial_y-2(u-3\beta t)\partial_u$ $\mathfrak{X}_6=\mathcal{F}(y,t)\partial_u,\ \mathcal{F}_t-\alpha\mathcal{F}+\mathcal{F}_{yyyy}=0$

[a] $\alpha(\gamma^2+\delta^2+\zeta^2)\neq 0$.
[b] $\alpha^2+\beta^2\neq 0$.

$$8\mathcal{F}_{15}r'+4\left(r(u)-ur'\right)\mathcal{F}_1'=0,$$
$$8\mathcal{F}_{15}g'+4\left(g(u)-ug'\right)\mathcal{F}_1'=0,$$
$$4\mathcal{F}_{15}f'+6f(u)\mathcal{F}_1'-2uf'\mathcal{F}_1'+2u\mathcal{F}_1''-r(u)\mathcal{F}_1''-4\mathcal{F}_{15t}+4g(u)\mathcal{F}_{15yy}-4\mathcal{F}_{15yyyy}=0.$$

The resulted group classification is summarized in Table 1. The first row gives the symmetries that occur for every possible choice of the functions f, g, h and r. In each subsequent row a special case appears along with the additional symmetries it has.

5. Conservation laws

By consulting Table 1 one can observe that the only point symmetries admitted by the self-adjoint classification given in Section 3 are $\mathfrak{X}_1=\partial_t,\ \mathfrak{X}_2=\partial_x,\ \mathfrak{X}_3=\partial_y$. The only exception is the nonlinear self-adjoint case,

$$u_t=\frac{1}{2}u_x^2+\frac{1}{2}u_y^2+\left(\frac{u}{2}+\alpha\right)(u_{xx}+u_{yy})-u_{xxxx}-2u_{xxyy}-u_{yyyy}+\delta u^2+\epsilon u+\zeta$$

which also admits the symmetry $\mathfrak{X}_4=x\partial_y-y\partial_x$. The nontrivial conserved vectors for each case follow.

Case 1 (Quasi Self-Adjoint). According to Theorem 3.2 the quasi self-adjoint case is

$$u_t = \alpha + g' u_y^2 + g(u)u_{yy} - u_{yyyy} + \frac{u_x^2}{2} + \left(\beta + \frac{1}{2}u\right) u_{xx} - 2u_{xxyy} - u_{xxxx},$$

with formal Lagrangian $\mathcal{L} = c\Delta(x, y, t, u, u_x, \ldots, u_{yyyy})$. Using the formula (9), for each one of the above mentioned three point symmetries, three trivial conserved vectors are obtained.

Case 2 (Nonlinear Self-Adjoint). Following Theorem 3.3, to each one of the five possible nonlinear self-adjoint cases formula (9) is applied using each of its admitted point symmetries. The conserved vector is given in the form (C^1, C^2, C^3) yielding the conservation law $D_xC^1 + D_yC^2 + D_tC^3 = 0$ for every solution of the case of Eq. (4) studied.

• Subcase 2.1 ($r = u/2 + \alpha, g = \beta u + \gamma, h = \beta, f = \delta u^2 + \epsilon u + \zeta$). For this case Eq. (4) takes the form:

$$u_t = \zeta + \epsilon u + \delta u^2 + \beta u_y^2 + (\gamma + \beta u)\, u_{yy} - u_{yyyy} + \frac{u_x^2}{2} + \left(\alpha + \frac{u}{2}\right) u_{xx} - 2u_{xxyy} - u_{xxxx}. \quad (13)$$

The new dependent variable $\upsilon = \mathcal{F}_1(x, y, t)$, where $\mathcal{F}_1(x, y, t)$ is any solution of the system

$$2\delta\mathcal{F}_1 + \beta\mathcal{F}_{1yy} + \frac{\mathcal{F}_{1xx}}{2} = 0,$$
$$\epsilon\mathcal{F}_1 + \mathcal{F}_{1t} + \gamma\mathcal{F}_{1yy} - \mathcal{F}_{1yyyy} + \alpha\mathcal{F}_{1xx} - 2\mathcal{F}_{1xxyy} - \mathcal{F}_{1xxxx} = 0.$$

$\boldsymbol{\partial_t}$:

$$\begin{aligned}
C^1 = {}& \frac{1}{12}\big(4(1-6\beta)\mathcal{F}_{1yyt}u_x + 3u^2\mathcal{F}_{1xt} - 8\mathcal{F}_{1yt}u_{xy} + 8u_y\mathcal{F}_{1xyt} \\
&- 6\mathcal{F}_{1t}\left((2\alpha + 8\delta + u)u_x - 2\left(u_{xyy} + u_{xxx}\right)\right) - 4\mathcal{F}_{1xt}\left(u_{yy} + 3u_{xx}\right)\big) \\
&+ u\left((\alpha + 4\delta)\mathcal{F}_{1xt} - \mathcal{F}_{1xyyt}\right) + 2\beta u\mathcal{F}_{1xyyt},
\end{aligned}$$

$$\begin{aligned}
C^2 = {}& \frac{1}{6}\big(3\beta u^2\mathcal{F}_{1yt} + \mathcal{F}_{1t}\left(-2(3\gamma + 4\delta + 3\beta u)u_y + 6\left(u_{yyy} + u_{xxy}\right)\right) \\
&- 2\left((-3 + 2\beta)u_y\mathcal{F}_{1yyt} + 2\mathcal{F}_{1xt}u_{xy} - 2u_x\mathcal{F}_{1xyt} + \mathcal{F}_{1yt}\left(3u_{yy} + u_{xx}\right)\right)\big) \\
&+ u\left((\gamma + 4\delta)\mathcal{F}_{1yt} + (-1 + 2\beta)\mathcal{F}_{1yyyt}\right),
\end{aligned}$$

$$C^3 = -\zeta\mathcal{F}_1(x, y, t) + u\mathcal{F}_{1t}.$$

This conserved vector will be nontrivial if and only if $\mathcal{F}_{1t} \neq 0$.

$\boldsymbol{\partial_x}$:

$$\begin{aligned}
C^1 = {}& \frac{1}{6}\big(3u_{,x}\left(2(1-2\beta)\mathcal{F}_{1,xyy} - (2\alpha + 8\delta + u)\mathcal{F}_{1,x}\right) - 2\mathcal{F}_{1,y}u_{,xxy} \\
&+ 2\left((6\beta - 1)\mathcal{F}_{1,yy}u_{,xx} + \mathcal{F}_{1,x}\left(u_{,xyy} + 3u_{,xxx}\right) - 2u_{,xy}\mathcal{F}_{1,xy}\right)\big) - \mathcal{F}_{1,y}u_{,yyy}, \\
&- \mathcal{F}_1(x, y, t)\left(\zeta - u_{,t} + \beta u_{,y}^2 + \gamma u_{,yy} + u\left(\epsilon + \delta u + \beta u_{,yy}\right) - 4\delta u_{,xx}\right),
\end{aligned}$$

$$\begin{aligned}
C^2 = {}& \frac{1}{3}\big((-3 + 2\beta)\mathcal{F}_{1yy}u_{xy} + \mathcal{F}_1(x, y, t)\left(3\beta u_y u_x + (3\gamma + 4\delta + 3\beta u)u_{xy}\right) \\
&- 2\mathcal{F}_{1xy}u_{xx} + \mathcal{F}_{1x}\left(3u_{yyy} + 5u_{xxy}\right) + \mathcal{F}_{1y}u_{xxx}\big) + (1 - 2\beta)\mathcal{F}_{1yyy}u_x + \mathcal{F}_{1y}\left(u_{xyy} - (\gamma + 4\delta + \beta u)u_x\right),
\end{aligned}$$

$$C^3 = -\mathcal{F}_1(x, y, t)u_x.$$

This conserved vector will be nontrivial if and only if $\mathcal{F}_{1x} \neq 0$.

$\boldsymbol{\partial_y}$:

$$\begin{aligned}
C^1 = {}& \frac{1}{6}\big(2\left(-2u_{xy}\mathcal{F}_{1xy} + (-1 + 6\beta)\mathcal{F}_{1yy}u_{xx} + \mathcal{F}_{1x}\left(u_{xyy} + 3u_{xxx}\right)\right) \\
&- 3(2\alpha + 8\delta + u)u_x\mathcal{F}_{1x} - 2\mathcal{F}_{1y}\left(3u_{yyy} + u_{xxy}\right)\big) + (1 - 2\beta)\mathcal{F}_{1xyy}u_x \\
&- \mathcal{F}_1(x, y, t)\left(\zeta - u_t + \beta u_y^2 + \gamma u_{yy} + u\left(\epsilon + \delta u + \beta u_{yy}\right) - 4\delta u_{xx}\right),
\end{aligned}$$

$$\begin{aligned}
C^2 = {}& \frac{1}{3}\big((2\beta - 3)\mathcal{F}_{1yy}u_{xy} + (3\gamma + 4\delta + 3\beta u)u_{xy}\mathcal{F}_1(x, y, t) - 2\mathcal{F}_{1xy}u_{xx} + 5u_{xxy}\mathcal{F}_{1x} + u_{xxx}\mathcal{F}_{1y}\big) \\
&+ (1 - 2\beta)\mathcal{F}_{1yyy}u_x + \beta u_y u_x\mathcal{F}_1(x, y, t) + u_{yyy}\mathcal{F}_{1x} + \mathcal{F}_{1y}\left(u_{xyy} - (\gamma + 4\delta + \beta u)u_x\right),
\end{aligned}$$

$$C^3 = -\mathcal{F}_1(x, y, t)u_x.$$

This conserved vector will be nontrivial if and only if $\mathcal{F}_{1y} \neq 0$.

$\boldsymbol{y\partial_x - x\partial_y}$:

If, in addition, $\beta = 1/2$ and $\gamma = \alpha$ then Eq. (13) admits also the symmetry $x\partial_y - y\partial_x$. Employing it we have:

$$\begin{aligned}
C^1 = \frac{1}{6}\Big(&2xu_{yyy}\mathcal{F}_{1x} + 3yuu_x\mathcal{F}_{1x} + u_y\left(4\mathcal{F}_{1yy} - 3x(2\alpha + 8\delta + u)\mathcal{F}_{1x}\right)\\
&+ 4x\mathcal{F}_{1yy}u_{xy} + 8\mathcal{F}_{1x}u_{xy} - 4xu_{yy}\mathcal{F}_{1xy} + 4u_x\mathcal{F}_{1xy} + 4yu_{xy}\mathcal{F}_{1xy}\\
&- 2y\mathcal{F}_{1x}u_{xyy} - 4y\mathcal{F}_{1yy}u_{xx} + 2\mathcal{F}_{1y}\left(2u_{yy} + 2xu_{xyy} - 2u_{xx} + yu_{xxy}\right)\\
&+ 3\mathcal{F}_1(x,y,t)\left(yu_y^2 + xu_yu_x + u\left(u_y + yu_{yy} + xu_{xy}\right)\right)\Big) + y\alpha u_x\mathcal{F}_{1x}\\
&+ 4y\delta u_x\mathcal{F}_{1x} + x\mathcal{F}_{1x}u_{xxy} - y\mathcal{F}_{1x}u_{xxx} + \mathcal{F}_{1y}\left(yu_{yyy} + xu_{xxx}\right)\\
&+ \mathcal{F}_1(x,y,t)\left(y\zeta + y\delta u^2 - yu_t + (\alpha + 4\delta)u_y + y\alpha u_{yy} + x\alpha u_{xy} + 4x\delta u_{xy} + \epsilon yu - xu_{xyyy} - 4y\delta u_{xx}\right),\\
C^2 = \frac{1}{6}\Big(&-4xu_{yy}\mathcal{F}_{1yy} + 3yu\mathcal{F}_{1y}u_x + 4\mathcal{F}_{1yy}u_x + 4u_{yy}\mathcal{F}_{1x} - 8\mathcal{F}_{1y}u_{xy}\\
&+ 4y\mathcal{F}_{1yy}u_{xy} - u_y\left(3x(2\alpha + 8\delta + u)\mathcal{F}_{1y} + 4\mathcal{F}_{1xy}\right) - 4xu_{xy}\mathcal{F}_{1xy}\\
&+ 4x\mathcal{F}_{1x}u_{xyy} - 4\mathcal{F}_{1x}u_{xx} + 4y\mathcal{F}_{1xy}u_{xx} + 2x\mathcal{F}_{1y}u_{xxy} - 10y\mathcal{F}_{1x}u_{xxy}\\
&- 2y\mathcal{F}_{1y}u_{xxx} - \mathcal{F}_1(x,y,t)\left(8\delta u_x - 8x\delta u_{yy} + 3yu_yu_x + 3xu_x^2\right.\\
&\left.+ 8y\delta u_{xy} + 3u\left(u_x + yu_{xy} + xu_{xx}\right)\right)\Big) + x\mathcal{F}_{1y}u_{yyy} + y\alpha\mathcal{F}_{1y}u_x + 4y\delta\mathcal{F}_{1y}u_x - yu_{yyy}\mathcal{F}_{1x} - y\mathcal{F}_{1y}u_{xyy} - x\mathcal{F}_{1x}u_{xxx}\\
&- \mathcal{F}_1(x,y,t)\left(x\zeta + x\delta u^2 - xu_t + \alpha u_x + y\alpha u_{xy} - u_{xyy} + x\alpha u_{xx} + \epsilon xu - xu_{xxyy}\right),\\
C^3 = \ &\mathcal{F}_1(x,y,t)\left(yu_x - xu_y\right).
\end{aligned}$$

This conserved vector will be nontrivial if and only if $\mathcal{F}_1 \neq \mathcal{F}(x^2 + y^2, t)$.

- Subcase 2.2 ($r = u/2 + \delta, h = g', f = \frac{\alpha u^2}{2} + \beta u + \gamma + c\int g(u)\,du,\ \alpha, c, g'' \neq 0$). For this case Eq. (4) takes the form:

$$u_t = \frac{\alpha u^2}{2} + \beta u + \gamma + \mathbf{c}\int g(u)\,du + g'u_y^2 + g(u)u_{yy} - u_{yyyy} + \frac{u_x^2}{2} + \left(\frac{u}{2} + \delta\right)u_{xx} - 2u_{xxyy} - u_{xxxx}.$$

The new dependent variable is

$$\upsilon = e^{t\left(4\alpha^2 - \beta + \mathbf{c}^2 + 2\alpha(\delta + 2\mathbf{c})\right)}\left(\cos(\sqrt{2\alpha}x)\left(\mathbf{c}_1\cos(\sqrt{\mathbf{c}}y) + \mathbf{c}_3\sin(\sqrt{\mathbf{c}}y)\right) + \sin(\sqrt{2\alpha}x)\left(\mathbf{c}_2\cos(\sqrt{\mathbf{c}}y) + \mathbf{c}_4\sin(\sqrt{\mathbf{c}}y)\right)\right).$$

$\boldsymbol{\partial_t}$:

$$\begin{aligned}
C^1 &= \frac{1}{4}e^{\mathcal{D}t}\mathcal{D}\left(\mathcal{A}\sqrt{2\alpha}\,(8\alpha + 4\delta + 8\mathbf{c} + u)\,u - 2\mathcal{B}\,(4\alpha + 2\delta + 4\mathbf{c} + u)\,u_x - 4\mathcal{A}\sqrt{2\alpha}\mathcal{A}u_{xx} + 4\mathcal{B}u_{xxx}\right),\\
C^2 &= e^{\mathcal{D}t}\mathcal{D}\left(\mathcal{C}\sqrt{\mathbf{c}}\left(\int g(u)\,du + \mathbf{c}u\right) - \mathcal{B}\,(\mathbf{c} + g(u))\,u_y - \mathcal{C}\sqrt{\mathbf{c}}u_{yy} + \mathcal{B}\mathcal{D}u_{yyy} - 2\mathcal{C}\mathcal{D}\sqrt{\mathbf{c}}u_{xx} + 2\mathcal{B}\mathcal{D}u_{xxy}\right),\\
C^3 &= -e^{\mathcal{D}t}\mathcal{B}(\gamma - \mathcal{D}u).
\end{aligned}$$

This conserved vector will be nontrivial if and only if $\mathcal{D} \neq 0$.

$\boldsymbol{\partial_x}$:

$$\begin{aligned}
C^1 &= \frac{1}{2}e^{\mathcal{D}t}\left(4\mathcal{B}\alpha u_{xx} + 2\sqrt{2\alpha}\mathcal{A}u_{xxx} - 2\mathcal{B}\gamma - 4\mathcal{B}\alpha\,(2\alpha + \delta + 2\mathbf{c})\,u - \mathcal{B}\alpha u^2 - \sqrt{2\alpha}\mathcal{A}\,(4\alpha + 2\delta + 4\mathbf{c} + u)\,u_x\right),\\
C^2 &= \sqrt{2\alpha}e^{\mathcal{D}t}\left(\mathcal{E}\sqrt{\mathbf{c}}\int g(u)\,du + \mathcal{E}\mathbf{c}^{3/2}u - (\mathcal{A}\mathbf{c} + \mathcal{A}g(u))u_y - \mathcal{E}\sqrt{\mathbf{c}}u_{yy} + \mathcal{A}u_{yyy} - 2\mathcal{E}\sqrt{\mathbf{c}}u_{xx} + 2\mathcal{A}u_{xxy}\right),\\
C^3 &= \sqrt{2\alpha}e^{\mathcal{D}t}\mathcal{A}u.
\end{aligned}$$

$\boldsymbol{\partial_y}$:

$$\begin{aligned}
C^1 &= \frac{1}{4}\sqrt{\mathbf{c}}e^{\mathcal{D}t}\left(4\mathcal{C}u_{xxx} + \sqrt{2}\mathcal{E}\sqrt{\alpha}\,(u\,(8\alpha + 4\delta + 8\mathbf{c} + u) - 4u_{xx}) - 2\mathcal{C}\,(4\alpha + 2\delta + 4\mathbf{c} + u)\,u_x\right),\\
C^2 &= -e^{\mathcal{D}t}\left(\mathcal{B}\gamma + \mathcal{B}\mathbf{c}\int g(u)\,du + \mathcal{B}\mathbf{c}^2u + \mathcal{C}\sqrt{\mathbf{c}}\,(\mathbf{c} + g(u))\,u_y - \sqrt{\mathbf{c}}\left(\mathcal{B}\sqrt{\mathbf{c}}\left(u_{yy} + 2u_{xx}\right) + \mathcal{C}\left(u_{yyy} + 2u_{xxy}\right)\right)\right),\\
C^3 &= \mathcal{C}\sqrt{\mathbf{c}}e^{\mathcal{D}t}u.
\end{aligned}$$

In the above formulas

$$\mathcal{A} = \sin(\sqrt{2\alpha}x)\left(\mathbf{c}_1\cos(\sqrt{\mathbf{c}}y)+\mathbf{c}_3\sin(\sqrt{\mathbf{c}}y)\right)-\cos(\sqrt{2\alpha}x)\left(\mathbf{c}_2\cos(\sqrt{\mathbf{c}}y)+\mathbf{c}_4\sin(\sqrt{\mathbf{c}}y)\right),$$
$$\mathcal{B} = \cos(\sqrt{2\alpha}x)\left(\mathbf{c}_1\cos(\sqrt{\mathbf{c}}y)+\mathbf{c}_3\sin(\sqrt{\mathbf{c}}y)\right)+\sin(\sqrt{2\alpha}x)\left(\mathbf{c}_2\cos(\sqrt{\mathbf{c}}y)+\mathbf{c}_4\sin(\sqrt{\mathbf{c}}y)\right),$$
$$\mathcal{C} = \cos(\sqrt{2\alpha}x)\left(\mathbf{c}_3\cos(\sqrt{\mathbf{c}}y)-\mathbf{c}_1\sin(\sqrt{\mathbf{c}}y)\right)+\sin(\sqrt{2\alpha}x)\left(\mathbf{c}_4\cos(\sqrt{\mathbf{c}}y)-\mathbf{c}_2\sin(\sqrt{\mathbf{c}}y)\right),$$
$$\mathcal{D} = 2\alpha(2\alpha+\delta)+4\alpha\mathbf{c}+\mathbf{c}^2-\beta$$

and

$$\mathcal{E} = \sin(\sqrt{2\alpha}x)\left(\mathbf{c}_1\sin(\sqrt{\mathbf{c}}y)-\mathbf{c}_3\cos(\sqrt{\mathbf{c}}y)\right)+\cos(\sqrt{2\alpha}x)\left(\mathbf{c}_4\cos(\sqrt{\mathbf{c}}y)-\mathbf{c}_2\sin(\sqrt{\mathbf{c}}y)\right).$$

• Subcase 2.3 ($r = u/2+\gamma, h = g', f = \alpha u+\beta+c\int g(u)\,du,\ c, g''\neq 0$). For this case Eq. (4) assumes the form:

$$u_t = \beta u+\gamma+c\int g(u)\,du+g'u_y^2+g(u)u_{yy}-u_{yyyy}+\frac{u_x^2}{2}+\frac{2\alpha+u}{2}u_{xx}-2u_{xxyy}-u_{xxxx}.$$

The new dependent variable υ is,

$$\upsilon = e^{(\mathbf{c}^2-\alpha)t}\left((\mathbf{c}_1+\mathbf{c}_2x)\cos(\sqrt{\mathbf{c}}y)+(\mathbf{c}_3+\mathbf{c}_4x)\sin(\sqrt{\mathbf{c}}y)\right).$$

$\boldsymbol{\partial_t}$:

$$C^1 = \frac{1}{4}e^{\mathcal{D}t}\mathcal{D}\left(\mathcal{B}u\,(4\gamma+8\mathbf{c}+u)-2\mathcal{A}\,(2\gamma+4\mathbf{c}+u)\,u_x-4\mathcal{B}u_{xx}+4\mathcal{A}u_{xxx}\right),$$
$$C^2 = e^{\mathcal{D}t}\mathcal{D}\left(\mathcal{C}\sqrt{\mathbf{c}}\int g(u)\,du+\mathcal{C}\mathbf{c}^{3/2}u-\mathcal{A}\mathbf{c}u_y-\mathcal{A}g(u)u_y-\mathcal{C}\sqrt{\mathbf{c}}u_{yy}+\mathcal{A}u_{yyy}-2\mathcal{C}\sqrt{\mathbf{c}}u_{xx}+2\mathcal{A}u_{xxy}\right),$$
$$C^3 = -e^{\mathcal{D}t}\mathcal{A}(\beta-\mathcal{D}u).$$

This conserved vector will be nontrivial if and only if $\mathcal{D}\neq 0$.

$\boldsymbol{\partial_x}$:

$$C^1 = \frac{1}{2}e^{\mathcal{D}t}\mathcal{B}\left(u\,(2\mathcal{D}x-u_x)+2\left(xu_t-\beta x-\gamma u_x+2u_{xyy}+u_{xxx}\right)\right),$$
$$C^2 = e^{\mathcal{D}t}\left(\mathcal{E}\sqrt{\mathbf{c}}\left(\int g(u)\,du+\mathbf{c}u\right)-\mathcal{B}\,(\mathbf{c}+g(u))\,u_y-\mathcal{E}\sqrt{\mathbf{c}}u_{yy}+\mathcal{B}u_{yyy}\right),$$
$$C^3 = -e^{\mathcal{D}t}\mathcal{B}xu_x.$$

This conserved vector will be nontrivial if and only if $\mathbf{c}_2^2+\mathbf{c}_4^2\neq 0$.

$\boldsymbol{\partial_y}$:

$$\begin{aligned}C^1 = {} & \frac{1}{4}e^{\mathcal{D}t}\sqrt{\mathbf{c}}\left(2\beta x\,(2\mathbf{c}_1+\mathbf{c}_2x)\sin(\sqrt{\mathbf{c}}y)+\mathcal{E}u^2-4\mathcal{E}\left(2u_{yy}+u_{xx}\right)\right.\\ & \left.-2\beta x\,(2\mathbf{c}_3+\mathbf{c}_4x)\cos(\sqrt{\mathbf{c}}y)+u\,(4\mathcal{E}\gamma-2\mathcal{C}u_x)+4\mathcal{C}\left(u_{xxx}-\gamma u_x+2u_{xyy}\right)\right),\end{aligned}$$
$$C^2 = \sqrt{\mathbf{c}}e^{\mathcal{D}t}\left(\mathcal{A}\sqrt{\mathbf{c}}u_{yy}+\mathcal{C}u_{yyy}-\mathcal{A}\sqrt{\mathbf{c}}\left(\int g(u)\,du+\mathbf{c}u\right)-\mathcal{C}\,(\mathbf{c}+g(u))\,u_y\right),$$
$$C^3 = \mathcal{C}\sqrt{\mathbf{c}}e^{\mathcal{D}t}.$$

For the above conserved vectors we have used the following notation:

$$\mathcal{A} = (\mathbf{c}_1+\mathbf{c}_2x)\cos(\sqrt{\mathbf{c}}y)+(\mathbf{c}_3+\mathbf{c}_4x)\sin(\sqrt{\mathbf{c}}y),$$
$$\mathcal{B} = \mathbf{c}_2\cos(\sqrt{\mathbf{c}}y)+\mathbf{c}_4\sin(\sqrt{\mathbf{c}}y),$$
$$\mathcal{C} = (\mathbf{c}_3+\mathbf{c}_4x)\cos(\sqrt{\mathbf{c}}y)-(\mathbf{c}_1+\mathbf{c}_2x)\sin(\sqrt{\mathbf{c}}y),$$
$$\mathcal{D} = \mathbf{c}^2-\alpha$$

and

$$\mathcal{E} = \mathbf{c}_4\cos(\sqrt{\mathbf{c}}y)-\mathbf{c}_2\sin(\sqrt{\mathbf{c}}y).$$

• Subcase 2.4 ($r = u/2+\delta, h = g', f = \alpha u^2+\beta u+\gamma,\ \alpha, g''\neq 0$). For this case Eq. (4) takes the form:

$$u_t = u_t = \alpha u^2+\beta u+\gamma+g'u_y^2+g(u)u_{yy}-u_{yyyy}+\frac{u_x^2}{2}+\left(\delta+\frac{1}{2}u\right)u_{xx}-2u_{xxyy}-u_{xxxx}.$$

The new dependent variable υ is,

$$\upsilon = e^{\left(16\alpha^2-\beta+4\alpha\delta\right)t}\left(\left(\mathbf{c}_1 y+\mathbf{c}_2\right)\cos(2\sqrt{\alpha}x)+\left(\mathbf{c}_3 y+\mathbf{c}_4\right)\sin(2\sqrt{\alpha}x)\right).$$

$\boldsymbol{\partial_t}$:

$$C^1 = \frac{1}{2}e^{\mathcal{D}t}\mathcal{D}\left(\mathcal{B}\sqrt{\alpha}u^2 - 2\left(\mathcal{A}(4\alpha+\delta)u_x + 2\mathcal{B}\sqrt{\alpha}u_{xx} - \mathcal{A}u_{xxx}\right) + \left(4\mathcal{B}\sqrt{\alpha}(4\alpha+\delta)u - \mathcal{A}u_x\right)\right).$$
$$C^2 = -e^{\mathcal{D}t}\left(g(u)\left(\mathcal{C}u_t + \mathcal{A}\mathcal{D}u_y\right) + \mathcal{D}\left(\mathcal{C}u_{yy} - \mathcal{A}u_{yyy} + 2\left(\mathcal{C}u_{xx} - \mathcal{A}u_{xxy}\right)\right)\right).$$
$$C^3 = e^{\mathcal{D}t}\left(\mathcal{A}\mathcal{D}u + \mathcal{C}g(u)u_y - \mathcal{A}\gamma\right);$$

This conserved vector will be nontrivial if and only if $\mathcal{D}\neq 0$.

$\boldsymbol{\partial_x}$:

$$C^1 = e^{\mathcal{D}t}\left(\mathcal{C}g(u)u_y - \mathcal{A}\alpha u^2 - u\left(4\mathcal{A}\alpha(4\alpha+\delta) + \mathcal{B}\sqrt{\alpha}u_x\right) - 2\sqrt{\alpha}\left(\mathcal{B}(4\alpha+\delta)u_x - 2\mathcal{A}\sqrt{\alpha}u_{xx} - \mathcal{B}u_{xxx}\right)\right),$$
$$\begin{aligned}C^2 = {}& e^{\mathcal{D}t}\left(\sqrt{\alpha}\left(2\mathcal{B}u_{yyy} + \sin(2\sqrt{\alpha}x)\left(\gamma\left(\mathbf{c}_1 y + 2\mathbf{c}_2\right)y - 4\left(\mathbf{c}_1 y+\mathbf{c}_2\right)u_{xxy} + 2\mathbf{c}_1\left(u_{yy}+2u_{xx}\right)\right)\right.\right.\\ &\left.\left. - \cos(2\sqrt{\alpha}x)\left(2\mathbf{c}_3\left(u_{yy}+2u_{xx}\right) + \gamma\left(\mathbf{c}_3 y+2\mathbf{c}_4\right)y - 4\left(\mathbf{c}_3 y+\mathbf{c}_4\right)u_{xxy}\right)\right) - g(u)\left(2\mathcal{B}\sqrt{\alpha}u_y + \mathcal{C}u_x\right)\right),\end{aligned}$$
$$C^3 = 2e^{\mathcal{D}t}\mathcal{B}\sqrt{\alpha}u.$$

$\boldsymbol{\partial_y}$:

$$C^1 = \frac{1}{2}e^{\mathcal{D}t}\left(\mathcal{E}\sqrt{\alpha}u^2 + \left(4\mathcal{E}\sqrt{\alpha}(4\alpha+\delta) - \mathcal{C}u_x\right)u - 2\left(\mathcal{C}(4\alpha+\delta)u_x + 2\mathcal{E}\sqrt{\alpha}u_{xx} - \mathcal{C}u_{xxx}\right)\right),$$
$$C^2 = \mathcal{C}e^{\mathcal{D}t}\left(u_{yyy} + 2u_{xxy} - \gamma y - g(u)u_y\right),$$
$$C^3 = e^{\mathcal{D}t}\mathcal{C}u;$$

where

$$\mathcal{A} = \left(\mathbf{c}_1 y+\mathbf{c}_2\right)\cos(2\sqrt{\alpha}x) + \left(\mathbf{c}_3 y+\mathbf{c}_4\right)\sin(2\sqrt{\alpha}x),$$
$$\mathcal{B} = \left(\mathbf{c}_3 y+\mathbf{c}_4\right)\cos(2\sqrt{\alpha}x) - \left(\mathbf{c}_1 y+\mathbf{c}_2\right)\sin(2\sqrt{\alpha}x),$$
$$\mathcal{C} = \mathbf{c}_1\cos(2\sqrt{\alpha}x) + \mathbf{c}_3\sin(2\sqrt{\alpha}x),$$
$$\mathcal{D} = 16\alpha^2 - \beta + 4\alpha\delta$$

and

$$\mathcal{E} = \mathbf{c}_3\cos(2\sqrt{\alpha}x) - \mathbf{c}_1\sin(2\sqrt{\alpha}x).$$

This conserved vector will be nontrivial if and only if $\mathbf{c}_1^2+\mathbf{c}_3^2\neq 0$.

• Subcase 2.5 ($r = u/2+\gamma, h = g', f = \alpha u+\beta,\ g''\neq 0$). For this case Eq. (4) assumes the form:

$$u_t = u_t = \alpha u + \beta + g' u_y^2 + g(u)u_{yy} - u_{yyyy} + \frac{u_x^2}{2} + \left(\gamma + \frac{1}{2}u\right)u_{xx} - 2u_{xxyy} - u_{xxxx}.$$

The new dependent variable υ is,

$$\upsilon = e^{-\alpha t}\left(\left(\mathbf{c}_1 + \mathbf{c}_2 x\right)y + \mathbf{c}_3 + \mathbf{c}_4 x\right).$$

$\boldsymbol{\partial_t}$:

$$C^1 = \frac{\alpha}{4}e^{-\alpha t}\left(2\left(\mathcal{A}uu_x - 2\gamma\mathcal{B}\right) + 4\gamma\mathcal{A}u_x + 4\mathcal{B}u_{xx} - \mathcal{B}u^2 - 4\mathcal{A}u_{xxx}\right),$$
$$C^2 = e^{-\alpha t}\left(g(u)\left(\alpha\mathcal{A}u_y - \mathcal{C}u_t\right) + \alpha\left(\mathcal{C}u_{yy} - \mathcal{A}u_{yyy} + 2\mathcal{C}u_{xx} - 2\mathcal{A}u_{xxy}\right)\right),$$
$$C^3 = e^{-\alpha t}\left(\mathcal{C}g(u)u_y - \mathcal{A}(\beta+\alpha u)\right).$$

This conserved vector will be nontrivial if and only if $\alpha\neq 0$.

$\boldsymbol{\partial_x}$:

$$C^1 = \frac{1}{2}e^{-\alpha t}\left(2\mathbf{c}_2 x g(u)u_y - \mathcal{B}\left(2\beta x + (2\gamma+u)u_x - 4u_{xyy} - 2u_{xxx}\right)\right),$$
$$C^2 = e^{-\alpha t}\left(\mathcal{B}u_{yyy} - \mathbf{c}_2 u_{yy} - g(u)\left(\mathcal{B}u_y + \mathbf{c}_2 x u_x\right)\right),$$
$$C^3 = e^{-\alpha t}\mathcal{B}u.$$

This conserved vector will be nontrivial if and only if $\mathbf{c}_2^2+\mathbf{c}_4^2\neq 0$.

$\boldsymbol{\partial_y}$:

$$C^1 = \frac{1}{4}e^{-\alpha t}\left(\boldsymbol{c}_2 u^2 - 4\gamma \mathcal{C} u_x + 2u\left(2\gamma \boldsymbol{c}_2 - \mathcal{C} u_x\right) - 4\boldsymbol{c}_2 u_{xx} + 4\mathcal{C} u_{xxx}\right),$$
$$C^2 = e^{-\alpha t}\mathcal{C}\left(u_{yyy} + 2u_{xxy} - \beta y - g(u)u_y\right),$$
$$C^3 = e^{-\alpha t}\mathcal{C} u.$$

This conserved vector will be nontrivial if and only if $\boldsymbol{c}_1^2 + \boldsymbol{c}_2^2 \neq 0$.

In the last three conserved vectors

$$\mathcal{A} = (\boldsymbol{c}_1 + \boldsymbol{c}_2 x)\, y + \boldsymbol{c}_3 + \boldsymbol{c}_4 x,$$
$$\mathcal{B} = \boldsymbol{c}_2 y + \boldsymbol{c}_4,$$

and

$$\mathcal{C} = \boldsymbol{c}_1 + \boldsymbol{c}_2 x.$$

6. Conclusions

In the present work a generalization of the anisotropic two-dimensional Kuramoto–Sivashinsky equation was studied under the prism of the modern group analysis. Specifically, two distinct classifications were performed; one with respect to the Lie point symmetries and a second with respect to the property of self-adjointness. The wealth of information obtained by those two classifications not only shed light to the structure of the generalization studied, by highlighting the interesting subcases, but provides us also with ways for attaining analytical nontrivial solutions as well as a starting point for the numerical analysis of these equations, a fact that will be the subject of future work.

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Appendix. The determining equations

$$\xi^3_x = 0,$$
$$\xi^3_y = 0,$$
$$\xi^3_u = 0,$$
$$\xi^3_{xy} = 0,$$
$$\xi^3_{xu} = 0,$$
$$\xi^3_{yu} = 0,$$
$$\xi^3_{uu} = 0,$$
$$\xi^3_{xuu} = 0,$$
$$\xi^3_{xyu} = 0,$$
$$\xi^3_{yuu} = 0,$$
$$\xi^3_{uuu} = 0,$$
$$\xi^3_{xuuu} = 0,$$
$$\xi^3_{xyuu} = 0,$$
$$\xi^3_{yuuu} = 0,$$
$$\xi^3_{uuuu} = 0,$$
$$g(u)\xi^3_u = 0,$$
$$r(u)\xi^3_u = 0,$$
$$3\xi^3_{yy} + \xi^3_{xx} = 0,$$
$$\xi^3_{yy} + 3\xi^3_{xx} = 0,$$

$$7\xi^3_{yy} + 5\xi^3_{xx} = 0,$$
$$5\xi^3_{yy} + 7\xi^3_{xx} = 0,$$
$$\xi^2_{uuu} + 2\xi^3_{yuu} = 0,$$
$$\xi^2_{uuuu} + 2\xi^3_{yuuu} = 0,$$
$$\xi^1_{uuuu} + 2\xi^3_{xuuu} = 0,$$
$$(g(u) + r(u))\xi^3_u = 0,$$
$$\xi^1_{yuu} + \xi^2_{xuu} + \xi^3_{xyu} = 0,$$
$$\xi^1_y + \xi^2_x + 2g(u)\xi^3_{xy} = 0,$$
$$\xi^1_y + \xi^2_x + 2r(u)\xi^3_{xy} = 0,$$
$$2h'\xi^3_u + (1 + h(u))\xi^3_{uu} = 0,$$
$$2\left(1 + r'\right)\xi^3_u + r(u)\xi^3_{uu} = 0,$$
$$\xi^1_{yuuu} + \xi^2_{xuuu} + \xi^3_{xyuu} = 0,$$
$$8h'\xi^3_u + (1 + 4h(u))\xi^3_{uu} = 0,$$
$$\left(11 + 16r'\right)\xi^3_u + 8r(u)\xi^3_{uu} = 0,$$
$$6\xi^3_{yyu} + 2\xi^3_{xxu} - 6g(u)\xi^3_u = 0,$$
$$6\xi^3_{yyu} + 2\xi^3_{xxu} - 2r(u)\xi^3_u = 0,$$
$$2\xi^3_{yyu} + 6\xi^3_{xxu} - 2g(u)\xi^3_u = 0,$$
$$2\xi^3_{yyu} + 6\xi^3_{xxu} - 6r(u)\xi^3_u = 0,$$
$$6g(u)\xi^3_y - 4\left(\xi^3_{yyy} + \xi^3_{xxy}\right) = 0,$$
$$6r(u)\xi^3_x - 4\left(\xi^3_{xyy} + \xi^3_{xxx}\right) = 0,$$
$$6g(u)\xi^3_u - 4\left(3\xi^3_{yyu} + \xi^3_{xxu}\right) = 0,$$
$$6r(u)\xi^3_u - 4\left(\xi^3_{yyu} + 3\xi^3_{xxu}\right) = 0,$$
$$4h(u)\xi^3_u + 2g'\xi^3_u + g(u)\xi^3_{uu} = 0,$$
$$\xi^2_u + \left(1 + 2g'\right)\xi^3_y + 2g(u)\xi^3_{yu} = 0,$$
$$\xi^2_u + \left(1 + 2r'\right)\xi^3_y + 2r(u)\xi^3_{yu} = 0,$$
$$\xi^2_{uu} + 2h'\xi^3_y + (1 + 2h(u))\xi^3_{yu} = 0,$$
$$\xi^1_{uu} + 2h'\xi^3_x + (1 + 2h(u))\xi^3_{xu} = 0,$$
$$11h(u)\xi^3_u + 8g'\xi^3_u + 4g(u)\xi^3_{uu} = 0,$$
$$5\xi^1_u + \left(6 + 8r'\right)\xi^3_x + 8r(u)\xi^3_{xu} = 0,$$
$$\left(3 + 8h(u) + 8g'\right)\xi^3_u + 4g(u)\xi^3_{uu} = 0,$$
$$\left(1 + 2h(u) + 4r'\right)\xi^3_u + 2r(u)\xi^3_{uu} = 0,$$
$$\left(2 + 3h(u) + 4r'\right)\xi^3_u + 2r(u)\xi^3_{uu} = 0,$$
$$\left(5 + 6h(u) + 4r'\right)\xi^3_u + 2r(u)\xi^3_{uu} = 0,$$
$$\left(1 + 2h(u) + 4g'\right)\xi^3_u + 2g(u)\xi^3_{uu} = 0,$$
$$3\xi^2_u + 4\left(\left(1 + r'\right)\xi^3_y + r(u)\xi^3_{yu}\right) = 0,$$
$$\xi^1_{yu} + \xi^2_{xu} + 2g'\xi^3_{xy} + 2g(u)\xi^3_{xyu} = 0,$$
$$\xi^1_{yu} + \xi^2_{xu} + 2r'\xi^3_{xy} + 2r(u)\xi^3_{xyu} = 0,$$
$$\left(3 + 10h(u) + 4g'\right)\xi^3_u + 2g(u)\xi^3_{uu} = 0,$$
$$3h''\xi^3_u + 3h'\xi^3_{uu} + (1 + h(u))\xi^3_{uuu} = 0,$$
$$3r''\xi^3_u + \left(2 + 3r'\right)\xi^3_{uu} + r(u)\xi^3_{uuu} = 0,$$

$$3g''\xi^3_u + \left(1 + 3g'\right)\xi^3_{uu} + g(u)\xi^3_{uuu} = 0,$$
$$\xi^1_{yuu} + \xi^2_{xuu} + 2h'\xi^3_{xy} + 2h(u)\xi^3_{xyu} = 0,$$
$$\xi^1_u + 2\left(h(u)\xi^3_x + g'\xi^3_x + g(u)\xi^3_{xu}\right) = 0,$$
$$\xi^1_u + 2\left(h(u)\xi^3_x + r'\xi^3_x + r(u)\xi^3_{xu}\right) = 0,$$
$$3\xi^1_u + 8h(u)\xi^3_x + 4g'\xi^3_x + 4g(u)\xi^3_{xu} = 0,$$
$$2g(u)\xi^3_y - 4\left(\xi^3_{yyy} + \xi^3_{xxy} - r(u)\xi^3_y\right) = 0,$$
$$12h''\xi^3_u + 12h'\xi^3_{uu} + (1 + 4h(u))\xi^3_{uuu} = 0,$$
$$\xi^1_{uuu} + 4h''\xi^3_x + 8h'\xi^3_{xu} + 4h(u)\xi^3_{xuu} = 0,$$
$$4g(u)\xi^3_x - 8\left(\xi^3_{xyy} + \xi^3_{xxx} - r(u)\xi^3_x\right) = 0,$$
$$5\xi^2_u + 12h(u)\xi^3_y + 8g'\xi^3_y + 8g(u)\xi^3_{yu} = 0,$$
$$8g(u)\xi^3_y + 4r(u)\xi^3_y - 8\left(\xi^3_{yyy} + \xi^3_{xxy}\right) = 0,$$
$$4g(u)\xi^3_x + 2r(u)\xi^3_x - 4\left(\xi^3_{xyy} + \xi^3_{xxx}\right) = 0,$$
$$\left(1 + 4g' + 4r'\right)\xi^3_u + 2(g(u) + r(u))\xi^3_{uu} = 0,$$
$$2g(u)\xi^3_u - 4\left(3\xi^3_{yyu} + \xi^3_{xxu} - r(u)\xi^3_u\right) = 0,$$
$$4g(u)\xi^3_u - 8\left(\xi^3_{yyu} + 3\xi^3_{xxu} - r(u)\xi^3_u\right) = 0,$$
$$12r''\xi^3_u + \left(13 + 12r'\right)\xi^3_{uu} + 4r(u)\xi^3_{uuu} = 0,$$
$$3\xi^2_u + \left(3 + 4h(u) + 2g'\right)\xi^3_y + 2g(u)\xi^3_{yu} = 0,$$
$$2g(u)\xi^3_u + 6r(u)\xi^3_u - 4\left(3\xi^3_{yyu} + \xi^3_{xxu}\right) = 0,$$
$$8g(u)\xi^3_u + 4r(u)\xi^3_u - 8\left(3\xi^3_{yyu} + \xi^3_{xxu}\right) = 0,$$
$$6g(u)\xi^3_u + 2r(u)\xi^3_u - 4\left(\xi^3_{yyu} + 3\xi^3_{xxu}\right) = 0,$$
$$4g(u)\xi^3_u + 2r(u)\xi^3_u - 4\left(\xi^3_{yyu} + 3\xi^3_{xxu}\right) = 0,$$
$$3\xi^1_u + 2\left(\left(1 + 3h(u) + r'\right)\xi^3_x + r(u)\xi^3_{xu}\right) = 0,$$
$$\left(5 + 12g' + 4r'\right)\xi^3_u + 2(3g(u) + r(u))\xi^3_{uu} = 0,$$
$$2\left(1 + 3g' + 2r'\right)\xi^3_u + (3g(u) + 2r(u))\xi^3_{uu} = 0,$$
$$\xi^2_u + 2\left(g'\xi^3_y + r'\xi^3_y + (g(u) + r(u))\xi^3_{yu}\right) = 0,$$
$$\xi^1_u + 2\left(g'\xi^3_x + r'\xi^3_x + (g(u) + r(u))\xi^3_{xu}\right) = 0,$$
$$4h'''\xi^3_u + 6h''\xi^3_{uu} + 4h'\xi^3_{uuu} + h(u)\xi^3_{uuuu} = 0,$$
$$\xi^2_{uu} + 2g''\xi^3_y + \xi^3_{yu} + 4g'\xi^3_{yu} + 2g(u)\xi^3_{yuu} = 0,$$
$$\xi^2_{uu} + 2r''\xi^3_y + \xi^3_{yu} + 4r'\xi^3_{yu} + 2r(u)\xi^3_{yuu} = 0,$$
$$\xi^1_{uu} + 2g''\xi^3_x + \xi^3_{xu} + 4g'\xi^3_{xu} + 2g(u)\xi^3_{xuu} = 0,$$
$$3\xi^1_{yu} + 3\xi^2_{xu} + 2\left(\left(1 + r'\right)\xi^3_{xy} + r(u)\xi^3_{xyu}\right) = 0,$$
$$3\xi^2_{uu} + 4\left(r''\xi^3_y + \left(1 + 2r'\right)\xi^3_{yu} + r(u)\xi^3_{yuu}\right) = 0,$$
$$3\xi^2_{uu} + 2r''\xi^3_y + 5\xi^3_{yu} + 4r'\xi^3_{yu} + 2r(u)\xi^3_{yuu} = 0,$$
$$5\xi^1_{uu} + 4r''\xi^3_x + 8\xi^3_{xu} + 8r'\xi^3_{xu} + 4r(u)\xi^3_{xuu} = 0,$$
$$\xi^2_{uuu} + 2h''\xi^3_y + 4h'\xi^3_{yu} + \xi^3_{yuu} + 2h(u)\xi^3_{yuu} = 0,$$
$$\xi^1_{uuu} + 2h''\xi^3_x + 4h'\xi^3_{xu} + \xi^3_{xuu} + 2h(u)\xi^3_{xuu} = 0,$$
$$5\xi^1_{uu} + 6r''\xi^3_x + 7\xi^3_{xu} + 12r'\xi^3_{xu} + 6r(u)\xi^3_{xuu} = 0,$$
$$h(u)\xi^3_u + 2g'\xi^3_u + 2r'\xi^3_u + g(u)\xi^3_{uu} + r(u)\xi^3_{uu} = 0,$$
$$3\xi^1_u + 2\left(\left(1 + 3g' + r'\right)\xi^3_x + (3g(u) + r(u))\xi^3_{xu}\right) = 0,$$
$$3\xi^1_{yu} + 3\xi^2_{xu} + 4h(u)\xi^3_{xy} + 2g'\xi^3_{xy} + 2g(u)\xi^3_{xyu} = 0,$$

$$3\left(3+8r'\right)\xi^3_u+14r(u)\xi^3_{uu}-4\left(\xi^3_{yyuu}+3\xi^3_{xxuu}\right)=0,$$
$$5h(u)\xi^3_u+2g'\xi^3_u+6r'\xi^3_u+g(u)\xi^3_{uu}+3r(u)\xi^3_{uu}=0,$$
$$4g'''\xi^3_u+6g''\xi^3_{uu}+\xi^3_{uuu}+4g'\xi^3_{uuu}+g(u)\xi^3_{uuuu}=0,$$
$$4h'\xi^3_u+3r''\xi^3_u+2h(u)\xi^3_{uu}+3r'\xi^3_{uu}+r(u)\xi^3_{uuu}=0,$$
$$8h'\xi^3_u+3g''\xi^3_u+4h(u)\xi^3_{uu}+3g'\xi^3_{uu}+g(u)\xi^3_{uuu}=0,$$
$$4r'''\xi^3_u+6r''\xi^3_{uu}+7\xi^3_{uuu}+4r'\xi^3_{uuu}+r(u)\xi^3_{uuuu}=0,$$
$$4h'''\xi^3_u+6h''\xi^3_{uu}+4h'\xi^3_{uuu}+\xi^3_{uuuu}+h(u)\xi^3_{uuuu}=0,$$
$$4h(u)\xi^3_u+4g'\xi^3_u+6r'\xi^3_u+2g(u)\xi^3_{uu}+3r(u)\xi^3_{uu}=0,$$
$$\xi^1_{yuuu}+\xi^2_{xuuu}+2h''\xi^3_{xy}+4h'\xi^3_{xyu}+2h(u)\xi^3_{xyuu}=0,$$
$$9h(u)\xi^3_u+12g'\xi^3_u+7g(u)\xi^3_{uu}-6\xi^3_{yyuu}-2\xi^3_{xxuu}=0,$$
$$\xi^2_{uuuu}+4\left(h'''\xi^3_y+3h''\xi^3_{yu}+3h'\xi^3_{yuu}+h(u)\xi^3_{yuuu}\right)=0,$$
$$\xi^1_{uuuu}+4\left(h'''\xi^3_x+3h''\xi^3_{xu}+3h'\xi^3_{xuu}+h(u)\xi^3_{xuuu}\right)=0,$$
$$26h'\xi^3_u+6g''\xi^3_u+13h(u)\xi^3_{uu}+6g'\xi^3_{uu}+2g(u)\xi^3_{uuu}=0,$$
$$4\xi^1_u+\left(5+12r'\right)\xi^3_x+14r(u)\xi^3_{xu}-4\xi^3_{xyyu}-4\xi^3_{xxxu}=0,$$
$$16h'''\xi^3_u+24h''\xi^3_{uu}+16h'\xi^3_{uuu}+\xi^3_{uuuu}+4h(u)\xi^3_{uuuu}=0,$$
$$\left(1+4g'\right)\xi^3_x+4g(u)\xi^3_{xu}+2r(u)\xi^3_{xu}-4\xi^3_{xyyu}-4\xi^3_{xxxu}=0,$$
$$2\xi^2_u+5h(u)\xi^3_y+6g'\xi^3_y+7g(u)\xi^3_{yu}-2\xi^3_{yyyu}-2\xi^3_{xxyu}=0,$$
$$\xi^2_{uu}+2\left(h'\xi^3_y+r''\xi^3_y+h(u)\xi^3_{yu}+2r'\xi^3_{yu}+r(u)\xi^3_{yuu}\right)=0,$$
$$\xi^1_{uu}+2\left(h'\xi^3_x+g''\xi^3_x+h(u)\xi^3_{xu}+2g'\xi^3_{xu}+g(u)\xi^3_{xuu}\right)=0,$$
$$\xi^1_{uu}+2\left(h'\xi^3_x+r''\xi^3_x+h(u)\xi^3_{xu}+2r'\xi^3_{xu}+r(u)\xi^3_{xuu}\right)=0,$$
$$3\xi^2_u+4h(u)\xi^3_y+2g'\xi^3_y+6r'\xi^3_y+2g(u)\xi^3_{yu}+6r(u)\xi^3_{yu}=0,$$
$$4h'\xi^3_u+6g''\xi^3_u+\xi^3_{uu}+2h(u)\xi^3_{uu}+6g'\xi^3_{uu}+2g(u)\xi^3_{uuu}=0,$$
$$4h'\xi^3_u+6r''\xi^3_u+\xi^3_{uu}+2h(u)\xi^3_{uu}+6r'\xi^3_{uu}+2r(u)\xi^3_{uuu}=0,$$
$$\xi^1_{uuu}+2g'''\xi^3_x+6g''\xi^3_{xu}+\xi^3_{xuu}+6g'\xi^3_{xuu}+2g(u)\xi^3_{xuuu}=0,$$
$$\left(1+8g'\right)\xi^3_u+4g(u)\xi^3_{uu}+2r(u)\xi^3_{uu}-4\xi^3_{yyuu}-12\xi^3_{xxuu}=0,$$
$$3\xi^1_{yuu}+3\xi^2_{xuu}+2\left(r''\xi^3_{xy}+\left(1+2r'\right)\xi^3_{xyu}+r(u)\xi^3_{xyuu}\right)=0,$$
$$6h'\xi^3_u+6r''\xi^3_u+2\xi^3_{uu}+3h(u)\xi^3_{uu}+6r'\xi^3_{uu}+2r(u)\xi^3_{uuu}=0,$$
$$h(u)\xi^3_y+2r'\xi^3_y+g(u)\xi^3_{yu}+2r(u)\xi^3_{yu}-2\xi^3_{yyyu}-2\xi^3_{xxyu}=0,$$
$$h(u)\xi^3_u+4r'\xi^3_u+g(u)\xi^3_{uu}+2r(u)\xi^3_{uu}-6\xi^3_{yyuu}-2\xi^3_{xxuu}=0,$$
$$3\xi^2_{uuu}+2r'''\xi^3_y+6r''\xi^3_{yu}+5\xi^3_{yuu}+6r'\xi^3_{yuu}+2r(u)\xi^3_{yuuu}=0,$$
$$5\xi^1_{uuu}+2r'''\xi^3_x+6r''\xi^3_{xu}+9\xi^3_{xuu}+6r'\xi^3_{xuu}+2r(u)\xi^3_{xuuu}=0,$$
$$12h'\xi^3_u+6r''\xi^3_u+5\xi^3_{uu}+6h(u)\xi^3_{uu}+6r'\xi^3_{uu}+2r(u)\xi^3_{uuu}=0,$$
$$5\xi^2_{uu}+4\left(4h'\xi^3_y+g''\xi^3_y+4h(u)\xi^3_{yu}+2g'\xi^3_{yu}+g(u)\xi^3_{yuu}\right)=0,$$
$$\xi^2_{uuuu}+2h'''\xi^3_y+6h''\xi^3_{yu}+6h'\xi^3_{yuu}+\xi^3_{yuuu}+2h(u)\xi^3_{yuuu}=0,$$
$$\xi^1_{uuuu}+2h'''\xi^3_x+6h''\xi^3_{xu}+6h'\xi^3_{xuu}+\xi^3_{xuuu}+2h(u)\xi^3_{xuuu}=0,$$
$$3\xi^1_{uu}+4\left(2h'\xi^3_x+g''\xi^3_x+2h(u)\xi^3_{xu}+2g'\xi^3_{xu}+g(u)\xi^3_{xuu}\right)=0,$$
$$3\xi^1_{uu}+2\left(5h'\xi^3_x+g''\xi^3_x+5h(u)\xi^3_{xu}+2g'\xi^3_{xu}+g(u)\xi^3_{xuu}\right)=0,$$
$$20h'\xi^3_u+6g''\xi^3_u+3\xi^3_{uu}+10h(u)\xi^3_{uu}+6g'\xi^3_{uu}+2g(u)\xi^3_{uuu}=0,$$
$$5\xi^2_{uu}+2\left(7h'\xi^3_y+3g''\xi^3_y+7h(u)\xi^3_{yu}+6g'\xi^3_{yu}+3g(u)\xi^3_{yuu}\right)=0,$$
$$16h'\xi^3_u+12g''\xi^3_u+3\xi^3_{uu}+8h(u)\xi^3_{uu}+12g'\xi^3_{uu}+4g(u)\xi^3_{uuu}=0,$$
$$\left(5+12g'+4r'\right)\xi^3_u+6g(u)\xi^3_{uu}-4\left(\xi^3_{yyuu}+3\xi^3_{xxuu}-r(u)\xi^3_{uu}\right)=0,$$

$$3\xi^1_{uu} + 2\left(3h'\xi^3_x + r''\xi^3_x + \xi^3_{xu} + 3h(u)\xi^3_{xu} + 2r'\xi^3_{xu} + r(u)\xi^3_{xuu}\right) = 0,$$
$$3\xi^2_{uu} + 4h'\xi^3_y + 2g''\xi^3_y + 3\xi^3_{yu} + 4h(u)\xi^3_{yu} + 4g'\xi^3_{yu} + 2g(u)\xi^3_{yuu} = 0,$$
$$2\xi^1_u + \left(3 + 6g' + 2r'\right)\xi^3_x + 6g(u)\xi^3_{xu} + 4r(u)\xi^3_{xu} - 4\xi^3_{xyyu} - 4\xi^3_{xxxu} = 0,$$
$$5h(u)\xi^3_u + 2g'\xi^3_u + 6r'\xi^3_u + 2g(u)\xi^3_{uu} + 3r(u)\xi^3_{uu} - 6\xi^3_{yyuu} - 2\xi^3_{xxuu} = 0,$$
$$18g''\xi^3_u + 6r''\xi^3_u + 5\xi^3_{uu} + 18g'\xi^3_{uu} + 6r'\xi^3_{uu} + 6g(u)\xi^3_{uuu} + 2r(u)\xi^3_{uuu} = 0,$$
$$3\xi^1_{yuu} + 3\xi^2_{xuu} + 4h'\xi^3_{xy} + 2g''\xi^3_{xy} + 4h(u)\xi^3_{xyu} + 4g'\xi^3_{xyu} + 2g(u)\xi^3_{xyuu} = 0,$$
$$6h''\xi^3_u + 4r'''\xi^3_u + 6h'\xi^3_{uu} + 6r''\xi^3_{uu} + 2h(u)\xi^3_{uuu} + 4r'\xi^3_{uuu} + r(u)\xi^3_{uuuu} = 0,$$
$$\xi^2_u + 3h(u)\xi^3_y + g'\xi^3_y + 3r'\xi^3_y + 2g(u)\xi^3_{yu} + 3r(u)\xi^3_{yu} - 2\xi^3_{yyyu} - 2\xi^3_{xxyu} = 0,$$
$$g(u)^2\xi^3_y - 2f'\xi^3_y + 2\eta_{yu} - 2f(u)\xi^3_{yu} - 3\xi^2_{yy} - \xi^2_{xx} - 2g(u)\left(\xi^3_{yyy} - \xi^3_{xxy}\right) = 0,$$
$$\xi^1_{yy} - r(u)^2\xi^3_x + 2f'\xi^3_x - 2\eta_{xu} + 2f(u)\xi^3_{xu} + 3\xi^1_{xx} + 2r(u)\left(\xi^3_{xyy} + \xi^3_{xxx}\right) = 0,$$
$$2\left(3g''\xi^3_x + r''\xi^3_x + \xi^3_{xu} + 6g'\xi^3_{xu} + 2r'\xi^3_{xu} + 3g(u)\xi^3_{xuu} + r(u)\xi^3_{xuu}\right) + 3\xi^1_{uu} = 0,$$
$$g(u)\xi^1_y - 2\xi^1_{yyy} + r(u)\xi^2_x - 4f'\xi^3_{xy} + 4\eta_{xyu} - 4f(u)\xi^3_{xyu} - 2\xi^2_{xyy} - 2\xi^1_{xxy} - 2\xi^2_{xxx} = 0,$$
$$f(u)\xi^3_u + \xi^3_t - g(u)\xi^3_{yy} - 2r(u)\xi^3_{yy} + \xi^3_{yyyy} - 4\xi^1_x - 7r(u)\xi^3_{xx} + 2\xi^3_{xxyy} + \xi^3_{xxxx} = 0,$$
$$f(u)\xi^3_u + \xi^3_t - 4\xi^2_y - 7g(u)\xi^3_{yy} + \xi^3_{yyyy} - 2g(u)\xi^3_{xx} - r(u)\xi^3_{xx} + 2\xi^3_{xxyy} + \xi^3_{xxxx} = 0,$$
$$9h''\xi^3_y + g'''\xi^3_y + 18h'\xi^3_{yu} + 3g''\xi^3_{yu} + 9h(u)\xi^3_{yuu} + 3g'\xi^3_{yuu} + g(u)\xi^3_{yuuu} + \frac{5}{2}\xi^2_{uuu} = 0,$$
$$5h''\xi^3_x + g'''\xi^3_x + 10h'\xi^3_{xu} + 3g''\xi^3_{xu} + 5h(u)\xi^3_{xuu} + 3g'\xi^3_{xuu} + g(u)\xi^3_{xuuu} + \frac{3}{2}\xi^1_{uuu} = 0,$$
$$42h''\xi^3_u + 4g'''\xi^3_u + 42h'\xi^3_{uu} + 6g''\xi^3_{uu} + 14h(u)\xi^3_{uuu} + 4g'\xi^3_{uuu} + g(u)\xi^3_{uuuu} = 0,$$
$$4f'\xi^3_u - 2\eta_{uu} + 2f(u)\xi^3_{uu} + 4\xi^2_{yu} + 3\xi^3_{yy} + 2h(u)\xi^3_{yy} + 4\xi^1_{xu} + \xi^3_{xx} + 6h(u)\xi^3_{xx} = 0,$$
$$10h'\xi^3_u + 3g''\xi^3_u + 9r''\xi^3_u + 5h(u)\xi^3_{uu} + 3g'\xi^3_{uu} + 9r'\xi^3_{uu} + g(u)\xi^3_{uuu} + 3r(u)\xi^3_{uuu} = 0,$$
$$g(u)r(u)\xi^3_y - 2f'\xi^3_y + 2\eta_{yu} - 2f(u)\xi^3_{yu} - \xi^2_{yy} - 2r(u)\xi^3_{yyy} - 4\xi^1_{xy} - 3\xi^2_{xx} - 2r(u)\xi^3_{xxy} = 0,$$
$$\frac{3}{2}\xi^1_{uuu} + 3h''\xi^3_x + r'''\xi^3_x + 6h'\xi^3_{xu} + 3r''\xi^3_{xu} + \xi^3_{xuu} + 3h(u)\xi^3_{xuu} + 3r'\xi^3_{xuu} + r(u)\xi^3_{xuuu} = 0,$$
$$30h''\xi^3_u + 8g'''\xi^3_u + 30h'\xi^3_{uu} + 12g''\xi^3_{uu} + 3\xi^3_{uuu} + 10h(u)\xi^3_{uuu} + 8g'\xi^3_{uuu} + 2g(u)\xi^3_{uuuu} = 0,$$
$$3\xi^2_{uuu} + 4h''\xi^3_y + 2g'''\xi^3_y + 8h'\xi^3_{yu} + 6g''\xi^3_{yu} + 3\xi^3_{yuu} + 4h(u)\xi^3_{yuu} + 6g'\xi^3_{yuu} + 2g(u)\xi^3_{yuuu} = 0,$$
$$18h''\xi^3_u + 8r'''\xi^3_u + 18h'\xi^3_{uu} + 12r''\xi^3_{uu} + 5\xi^3_{uuu} + 6h(u)\xi^3_{uuu} + 8r'\xi^3_{uuu} + 2r(u)\xi^3_{uuuu} = 0,$$
$$3\xi^1_{yy} + 2f'\xi^3_x + 4\xi^2_{xy} + \xi^1_{xx} + g(u)\left(-r(u)\xi^3_x + 2\left(\xi^3_{xyy} + \xi^3_{xxx}\right)\right) - 2\eta_{xu} + 2f(u)\xi^3_{xu} = 0,$$
$$2\left(h''\xi^3_y + r'''\xi^3_y + 2h'\xi^3_{yu} + 3r''\xi^3_{yu} + h(u)\xi^3_{yuu} + 3r'\xi^3_{yuu} + r(u)\xi^3_{yuuu}\right) + \xi^2_{uuu} = 0,$$
$$2h'\xi^3_y + g''\xi^3_y + 3r''\xi^3_y + 2h(u)\xi^3_{yu} + 2g'\xi^3_{yu} + 6r'\xi^3_{yu} + g(u)\xi^3_{yuu} + 3r(u)\xi^3_{yuu} + \frac{3}{2}\xi^2_{uu} = 0,$$
$$2f''\xi^3_y + 4f'\xi^3_{yu} - 2\eta_{yuu} + 2f(u)\xi^3_{yuu} + \xi^2_{yyu} + \xi^3_{yyy} + 4\xi^1_{xyu} + 3\xi^2_{xxu} + \xi^3_{xxy} - \frac{1}{2}r(u)\xi^2_u - \frac{1}{2}g(u)\xi^3_y = 0,$$
$$4f'\xi^3_u - r(u)^2\xi^3_u - 2\eta_{uu} + 2f(u)\xi^3_{uu} + \xi^3_{yy} + 2r'\xi^3_{yy} + 8\xi^1_{xu} + 3\xi^3_{xx} + 6r'\xi^3_{xx} + 2r(u)\left(\xi^3_{yyu} + 3\xi^3_{xxu}\right) = 0,$$
$$3\eta_{uu} - 6f'\xi^3_u - 3f(u)\xi^3_{uu} - 2\xi^3_{yy} - 2r'\xi^3_{yy} - 2r(u)\xi^3_{yyu} - 12\xi^1_{xu} - 6\xi^3_{xx} - 6r'\xi^3_{xx} - 6r(u)\xi^3_{xxu} = 0,$$
$$f(u)\xi^3_u + \xi^3_t - 2\xi^2_y - 2g(u)\xi^3_{yy} - 3r(u)\xi^3_{yy} + \xi^3_{yyyy} - 2\xi^1_x - 3g(u)\xi^3_{xx} - 2r(u)\xi^3_{xx} + 2\xi^3_{xxyy} + \xi^3_{xxxx} = 0,$$
$$6f''\xi^3_{uu} + 4f'\xi^3_{uuu} - \eta_{uuuu} + f(u)\xi^3_{uuuu} + \xi^3_{yyuu} + 4\xi^1_{xuuu} + 3\xi^3_{xxuu} - \frac{1}{4}\left(1 - 16f'''\right)\xi^3_u - \frac{1}{2}r(u)\xi^3_{uu} = 0,$$
$$3\eta_{uu} - 6f'\xi^3_u - 3f(u)\xi^3_{uu} - 12\xi^2_{yu} - 12h(u)\xi^3_{yy} - 6g'\xi^3_{yy} - 6g(u)\xi^3_{yyu} - 4h(u)\xi^3_{xx} - 2g'\xi^3_{xx} - 2g(u)\xi^3_{xxu} = 0,$$
$$4f'\xi^3_u - 2\eta_{uu} + 2f(u)\xi^3_{uu} + 4\xi^2_{yu} + 3\xi^3_{yy} + 2g'\xi^3_{yy} + 4\xi^1_{xu} + \xi^3_{xx} + 6g'\xi^3_{xx} + g(u)\left(-r(u)\xi^3_u + 2\xi^3_{yyu} + 6\xi^3_{xxu}\right) = 0,$$
$$2f'\xi^3_u - \eta_{uu} + f(u)\xi^3_{uu} + 4\xi^2_{yu} + 3h(u)\xi^3_{yy} + 3g'\xi^3_{yy} + h(u)\xi^3_{xx} + g'\xi^3_{xx} + g(u)\left(3\xi^3_{yyu} + \xi^3_{xxu}\right) - \frac{1}{2}g(u)^2\xi^3_u = 0,$$
$$6\xi^1_{yyu} - g(u)\xi^1_u - 2h(u)r(u)\xi^3_x + 4f''\xi^3_x + 8f'\xi^3_{xu} - 4\eta_{xuu} + 4f(u)\xi^3_{xuu}$$

$$+8\xi^2_{xyu}+4h(u)\xi^3_{xyy}+2\xi^1_{xxu}+4h(u)\xi^3_{xxx}=0,$$

$$2f'\xi^3_u-\eta_{uu}+f(u)\xi^3_{uu}+2\xi^2_{yu}+h(u)\xi^3_{yy}+3r'\xi^3_{yy}+3r(u)\xi^3_{yyu}+2\xi^1_{xu}$$
$$+3h(u)\xi^3_{xx}+r'\xi^3_{xx}+r(u)\xi^3_{xxu}-\frac{1}{2}g(u)r(u)\xi^3_u=0,$$

$$2f'\xi^3_u-\eta_{uu}+f(u)\xi^3_{uu}+2\xi^2_{yu}+g'\xi^3_{yy}+3r'\xi^3_{yy}+g(u)\xi^3_{yyu}+3r(u)\xi^3_{yyu}$$
$$+2\xi^1_{xu}+3g'\xi^3_{xx}+r'\xi^3_{xx}+3g(u)\xi^3_{xxu}+r(u)\xi^3_{xxu}=0,$$

$$2h(u)\xi^1_y+2g(u)\xi^1_{yu}-4\xi^1_{yyyu}+\xi^2_x+2r(u)\xi^2_{xu}-8f''\xi^3_{xy}-16f'\xi^3_{xyu}$$
$$+8\eta_{xyuu}-8f(u)\xi^3_{xyuu}-4\xi^2_{xyyu}-4\xi^1_{xxyu}-4\xi^2_{xxxu}=0,$$

$$\xi^1_u-4\xi^1_{yyuu}+\xi^3_x-8f'''\xi^3_x-24f''\xi^3_{xu}+2r(u)\left(\xi^1_{uu}+\xi^3_{xu}\right)-24f'\xi^3_{xuu}$$
$$+8\eta_{xuuu}-8f(u)\xi^3_{xuuu}-4\xi^3_{xyyu}-12\xi^1_{xxuu}-4\xi^3_{xxxu}=0,$$

$$\eta_t-\eta(x,y,t,u)f'-f(u)^2\xi^3_u-g(u)\eta_{yy}+\eta_{yyyy}-r(u)\eta_{xx}+2\eta_{xxyy}+\eta_{xxxx}$$
$$+f(u)\left(\eta_u-\xi^3_t+g(u)\xi^3_{yy}-\xi^3_{yyyy}+r(u)\xi^3_{xx}-2\xi^3_{xxyy}-\xi^3_{xxxx}\right)=0,$$

$$4f'''\xi^3_y-r(u)\xi^2_{uu}-h(u)\xi^3_y-g(u)\xi^3_{yu}+12f''\xi^3_{yu}+12f'\xi^3_{yuu}-4\eta_{yuuu}$$
$$+4f(u)\xi^3_{yuuu}+2\xi^2_{yyuu}+2\xi^3_{yyyu}+8\xi^1_{xyuu}+6\xi^2_{xxuu}+2\xi^3_{xxyu}-\frac{1}{2}\xi^2_u=0,$$

$$6f''\xi^3_u+6f'\xi^3_{uu}-2\eta_{uuu}+2f(u)\xi^3_{uuu}+4\xi^2_{yuu}+2h'\xi^3_{yy}+3\xi^3_{yyu}+4\xi^1_{xuu}$$
$$+6h'\xi^3_{xx}+\xi^3_{xxu}+h(u)\left(-r(u)\xi^3_u+2\xi^3_{yyu}+6\xi^3_{xxu}\right)-\frac{1}{2}g(u)\xi^3_u=0,$$

$$2f''\xi^3_x+4f'\xi^3_{xu}-2\eta_{xuu}+2f(u)\xi^3_{xuu}+4\xi^2_{xyu}+2g'\xi^3_{xyy}+\xi^1_{xxu}+2g'\xi^3_{xxx}$$
$$-r(u)g'\xi^3_x+3\xi^1_{yyu}-\frac{1}{2}g(u)\left(\xi^3_x+2r(u)\xi^3_{xu}-4\left(\xi^3_{xyyu}+\xi^3_{xxxu}\right)\right)=0,$$

$$3\xi^1_{yyu}+6f''\xi^3_x+12f'\xi^3_{xu}-6\eta_{xuu}+6f(u)\xi^3_{xuu}+2\xi^3_{xyy}+2r'\xi^3_{xyy}+9\xi^1_{xxu}$$
$$+2\xi^3_{xxx}+r(u)^2\xi^3_{xu}-\frac{1}{2}r(u)\left(2\xi^1_u+\left(3+2r'\right)\xi^3_x-4\left(\xi^3_{xyyu}+\xi^3_{xxxu}\right)\right)+2r'\xi^3_{xxx}=0,$$

$$2f''\xi^3_y+4f'\xi^3_{yu}-g(u)\left(r'\xi^3_y+r(u)\xi^3_{yu}\right)+2f(u)\xi^3_{yuu}-h(u)r(u)\xi^3_y$$
$$-2\eta_{yuu}+\xi^2_{yyu}+2r'\xi^3_{yyy}+2r(u)\xi^3_{yyyu}+4\xi^1_{xyu}+3\xi^2_{xxu}+2r'\xi^3_{xxy}+2r(u)\xi^3_{xxyu}=0,$$

$$h(u)^2\xi^3_u+2g(u)h'\xi^3_u-4f'''\xi^3_u-6f''\xi^3_{uu}-4f'\xi^3_{uuu}+\eta_{uuuu}-f(u)\xi^3_{uuuu}$$
$$-12h'\xi^3_{yyu}-2h''\xi^3_{xx}-4h'\xi^3_{xxu}+h(u)\left(g(u)\xi^3_{uu}-2\left(3\xi^3_{yyuu}+\xi^3_{xxuu}\right)\right)-6h''\xi^3_{yy}-4\xi^2_{yuuu}=0,$$

$$-\xi^1_t+g(u)\xi^1_{yy}-\xi^1_{yyyy}-\eta_x+2r(u)f'\xi^3_x-2r(u)\eta_{xu}-4f'\xi^3_{xyy}+4\eta_{xyyu}$$
$$+f(u)\left(-\xi^1_u+\xi^3_x+2r(u)\xi^3_{xu}-4\xi^3_{xyyu}-4\xi^3_{xxxu}\right)-\xi^1_{xxxx}+r(u)\xi^1_{xx}-2\xi^1_{xxyy}-4f'\xi^3_{xxx}+4\eta_{xxxu}=0,$$

$$18f''\xi^3_u+18f'\xi^3_{uu}-6\eta_{uuu}+6f(u)\xi^3_{uuu}+2r''\xi^3_{yy}+5\xi^3_{yyu}+4r'\xi^3_{yyu}$$
$$+12r'\xi^3_{xxu}-r(u)^2\xi^3_{uu}-\frac{1}{2}r(u)\left(\left(5+4r'\right)\xi^3_u-4\left(\xi^3_{yyuu}+3\xi^3_{xxuu}\right)\right)+15\xi^3_{xxu}+24\xi^1_{xuu}+6r''\xi^3_{xx}=0,$$

$$3\xi^1_{yyuu}-r(u)h'\xi^3_x+2f'''\xi^3_x+6f''\xi^3_{xu}+6f'\xi^3_{xuu}-2\eta_{xuuu}+2f(u)\xi^3_{xuuu}$$
$$+4\xi^2_{xyuu}-\frac{1}{2}g(u)\xi^1_{uu}-\frac{1}{2}h(u)\left(\xi^1_u+\xi^3_x+2r(u)\xi^3_{xu}-4\xi^3_{xyyu}-4\xi^3_{xxxu}\right)+2h'\xi^3_{xyy}+\xi^1_{xxuu}+2h'\xi^3_{xxx}=0,$$

$$6g''\xi^3_{xx}-2r(u)g'\xi^3_u+6f''\xi^3_u+6f'\xi^3_{uu}-2\eta_{uuu}+2f(u)\xi^3_{uuu}+4\xi^2_{yuu}$$
$$+2g''\xi^3_{yy}+\xi^3_{xxu}+12g'\xi^3_{xxu}-\frac{1}{2}g(u)\left(\xi^3_u+2r(u)\xi^3_{uu}-4\xi^3_{yyuu}-12\xi^3_{xxuu}\right)+3\xi^3_{yyu}+4g'\xi^3_{yyu}+4\xi^1_{xuu}=0,$$

$$2g(u)f'\xi^3_y-\xi^2_t-2h(u)\eta_y-2g(u)\eta_{yu}+g(u)\xi^2_{yy}-4f'\xi^3_{yyy}+4\eta_{yyyu}-\xi^2_{yyyy}$$
$$-f(u)\left(\xi^2_u-2h(u)\xi^3_y-2g(u)\xi^3_{yu}+4\xi^3_{yyyu}+4\xi^3_{xxyu}\right)-2\xi^2_{xxyy}-\xi^2_{xxxx}+r(u)\xi^2_{xx}-4f'\xi^3_{xxy}+4\eta_{xxyu}=0,$$

$$6f''\xi^3_y-g(u)^2\xi^3_{yu}+12f'\xi^3_{yu}-6\eta_{yuu}+6f(u)\xi^3_{yuu}+9\xi^2_{yyu}+4h(u)\xi^3_{yyy}$$
$$+2g'\xi^3_{xxy}-g(u)\left(\xi^2_u+3h(u)\xi^3_y+g'\xi^3_y-2\xi^3_{yyyu}-2\xi^3_{xxyu}\right)+2g'\xi^3_{yyy}+3\xi^2_{xxu}+4h(u)\xi^3_{xxy}=0,$$

$$2f'''\xi^3_y+6f''\xi^3_{yu}+6f'\xi^3_{yuu}-2\eta_{yuuu}+2f(u)\xi^3_{yuuu}+3\xi^2_{yyuu}+2h'\xi^3_{yyy}$$

$$+2h'\xi^3_{xxy} - h(u)^2\xi^3_y - \frac{1}{2}h(u)\left(\xi^2_u + 2g(u)\xi^3_{yu} - 4\left(\xi^3_{yyyu} + \xi^3_{xxyu}\right)\right) + \xi^2_{xxuu} - \frac{1}{2}g(u)\left(\xi^2_{uu} + 2h'\xi^3_y\right) = 0,$$

$$2r(u)\xi^1_x - \eta(x,y,t,u)r' - r(u)\xi^3_t + g(u)r(u)\xi^3_{yy} - 2f'\xi^3_{yy} + 2\eta_{yyu} - r(u)\xi^3_{yyyy} - 4\xi^1_{xyy}$$
$$+ r(u)^2\xi^3_{xx} - 6f'\xi^3_{xx} + 6\eta_{xxu} - 2r(u)\xi^3_{xxyy} - 4\xi^1_{xxx} - r(u)\xi^3_{xxxx} - f(u)\left(r(u)\xi^3_u + 2\xi^3_{yyu} + 6\xi^3_{xxu}\right) = 0,$$

$$g(u)^2\xi^3_{yy} - \eta(x,y,t,u)g' - g(u)\xi^3_t + 2g(u)\xi^2_y - 6f'\xi^3_{yy} + 6\eta_{yyu} - 4\xi^2_{yyy} + g(u)r(u)\xi^3_{xx} - 2f'\xi^3_{xx}$$
$$+ 2\eta_{xxu} - f(u)\left(g(u)\xi^3_u + 6\xi^3_{yyu} + 2\xi^3_{xxu}\right) - g(u)\xi^3_{yyyy} - 4\xi^2_{xxy} - 2g(u)\xi^3_{xxyy} - g(u)\xi^3_{xxxx} = 0,$$

$$9f''\xi^3_u + 9f'\xi^3_{uu} + 3f(u)\xi^3_{uuu} + 12\xi^2_{yuu} + 15h'\xi^3_{yy} + 3g''\xi^3_{yy} + 6g'\xi^3_{yyu} - 3\eta_{uuu} + 15h(u)\xi^3_{yyu} + 5h'\xi^3_{xx}$$
$$+ g''\xi^3_{xx} + 5h(u)\xi^3_{xxu} + 2g'\xi^3_{xxu} - \frac{1}{2}g(u)^2\xi^3_{uu} - \frac{1}{2}g(u)\left(5h(u)\xi^3_u + 2g'\xi^3_u - 6\xi^3_{yyuu} - 2\xi^3_{xxuu}\right) = 0,$$

$$4r(u)h'\xi^3_u - 16f'''\xi^3_u + g(u)\xi^3_{uu} - 24f''\xi^3_{uu} - 16f'\xi^3_{uuu} + 4\eta_{uuuu} - 8\xi^2_{yuuu} - 4f(u)\xi^3_{uuuu} - 4h''\xi^3_{yy} - 8h'\xi^3_{yyu}$$
$$- 6\xi^3_{yyuu} - 8\xi^1_{xuuu} - 12h''\xi^3_{xx} - 2\xi^3_{xxuu} - 24h'\xi^3_{xxu} + 2h(u)\left(\xi^3_u + r(u)\xi^3_{uu} - 2\xi^3_{yyuu} - 6\xi^3_{xxuu}\right) = 0,$$

$$3f''\xi^3_u + 3f'\xi^3_{uu} - \eta_{uuu} + f(u)\xi^3_{uuu} + 2\xi^2_{yuu} + h'\xi^3_{yy} + 3r''\xi^3_{yy} + 6r'\xi^3_{yyu} + 3h'\xi^3_{xx} + r''\xi^3_{xx} + 2r'\xi^3_{xxu}$$
$$+ r(u)\xi^3_{xxuu} - \frac{1}{2}g(u)\left(2r'\xi^3_u + r(u)\xi^3_{uu}\right) - \frac{1}{2}h(u)\left(r(u)\xi^3_u - 2\left(\xi^3_{yyu} + 3\xi^3_{xxu}\right)\right) + 3r(u)\xi^3_{yyuu} + 2\xi^1_{xuu} = 0,$$

$$2\xi^1_x - \eta_u - \xi^3_t + g(u)\xi^3_{yy} - 4f''\xi^3_{yy} - 8f'\xi^3_{yyu} + 4\eta_{yyuu} - 4f(u)\xi^3_{yyuu} - \xi^3_{yyyy}$$
$$- 8\xi^1_{xyyu} - 12f''\xi^3_{xx} + r(u)\left(4f'\xi^3_u - 2\eta_{uu} + 2f(u)\xi^3_{uu} + 4\xi^1_{xu} + \xi^3_{xx}\right)$$
$$- 24f'\xi^3_{xxu} + 12\eta_{xxuu} - 12f(u)\xi^3_{xxuu} - 2\xi^3_{xxyy} - 8\xi^1_{xxxu} - \xi^3_{xxxx} = 0,$$

$$2g(u)\xi^2_{yu} - \eta(x,y,t,u)h' + 2g(u)f'\xi^3_u - g(u)\eta_{uu} + f(u)g(u)\xi^3_{uu} - 6f''\xi^3_{yy}$$
$$- 12f'\xi^3_{yyu} + 6\eta_{yyuu} - 6f(u)\xi^3_{yyuu} - 4\xi^2_{yyyu} - 2f''\xi^3_{xx} - 4f'\xi^3_{xxu} + 2\eta_{xxuu}$$
$$- h(u)\left(\eta_u + \xi^3_t - 2\xi^2_y - g(u)\xi^3_{yy} + \xi^3_{yyyy} - r(u)\xi^3_{xx} + 2\xi^3_{xxyy} + \xi^3_{xxxx}\right) - 2f(u)\xi^3_{xxuu} - 4\xi^2_{xxyu} = 0,$$

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