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DENSITY OF INFIMUM-STABLE CONVEX CONES

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ABSTRACT. Let X be a compact Hausdorff space and let A be a linear subspace of $C(X;\mathbb{R})$ containing the constant functions, and separating points from probability measures. Then the inf-lattice generated by A is uniformly dense in $C(X;\mathbb{R})$. We show that this is a corollary of the Choquet-Deny Theorem, thus simplifying the proof and extending to the nonmetric case a result of McAfee and Reny.

Let X be a compact Hausdorff space and let $C(X;\mathbb{R})$ the space of all continuous real-valued functions be endowed with the sup-norm. Let A be a linear subspace of $C(X;\mathbb{R})$, containing the constant functions. Let

$$A_m = \{\inf(f_1, \dots, f_n);\ f_i \in A,\ 1 \le i \le n,\ n \in \mathbb{N}\},$$
$$A_M = \{\sup(f_1, \dots, f_n);\ f_i \in A,\ 1 \le i \le n,\ n \in \mathbb{N}\}.$$

Then A_m (resp. A_M) is a convex conic inf-lattice (resp. sup-lattice), and McAfee and Reny [3] proved that $\overline{A_m} = \overline{A_M} = C(X;\mathbb{R})$ if and only if A separates points from probability measures, in the case X is a metric space. Our aim is to give a simpler proof of the above result, which is valid even without the restriction of X being a metric space. The proof is based on the classical Choquet-Deny Theorem (see Choquet-Deny [1] or Nachbin [4, §21]). We present the proof of an improved version of this result (see Theorem 1). Our proof follows closely the arguments of Nachbin [4].

Let us recall that a subset S of $C(X;\mathbb{R})$ is called an *inf-lattice* (resp. *sup-lattice*) or *infimum-stable* (resp. *supremum-stable*) subset if $f, g \in S$ implies $f \wedge g \in S$ (resp. $f \vee g \in S$). Here $(f \wedge g)(x) = \inf(f(x), g(x))$ and $(f \vee g)(x) = \sup(f(x), g(x))$, for all $x \in X$. On the other hand, $S \subset C(X;\mathbb{R})$ is a *convex cone* if and only if $f, g \in S$ and $\lambda \ge 0$ imply $f + g \in S$ and $\lambda f \in S$.

Lemma 1. *Let $S \subset C(X;\mathbb{R})$ be an infimum-stable subset and, for each point $x \in X$, let $P_x = \{f \in C(X;\mathbb{R});\ f \ge 0,\ f(x) = 0\}$. Then*

$$\overline{S} = \bigcap\{\overline{S - P_x};\ x \in X\}.$$

Proof. For each $x \in X$, we have $0 \in P_x$. Hence $\overline{S} \subset \overline{S - P_x}$, for each $x \in X$. Conversely, assume that $f \in \overline{S - P_x}$, for each $x \in X$. Let $\varepsilon > 0$ be given. For each $x \in X$, there are $g_x \in S$ and $h_x \in P_x$ such that $\|g_x - h_x - f\| <$

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$\varepsilon/2$. Let $V_x = \{t \in X; h_x(t) < \varepsilon/2\}$. Then V_x is open and contains x. By compactness, there are $x_1, \dots, x_n \in X$ such that $X = V_{x_1} \cup \dots \cup V_{x_n}$. Let $g = \inf\{g_{x_1}, \dots, g_{x_n}\}$. Then $g \in S$. Let $t \in X$. Then, for each $j = 1, \dots, n$, we have

$$g_{x_j}(t) \geq g_{x_j}(t) - h_{x_j}(t) > f(t) - \varepsilon/2.$$

Hence $g(t) > f(t) - \varepsilon$. On the other hand, there is some index i such that $t \in V_{x_i}$, and then $h_{x_i}(t) < \varepsilon/2$ and $g(t) \leq g_{x_i}(t)$ imply $g(t) - \varepsilon/2 < g_{x_i}(t) - h_{x_i}(t) < f(t) + \varepsilon/2$. Hence $g(t) < f(t) + \varepsilon$. Therefore $\|f - g\| < \varepsilon$ and $f \in \overline{S}$. □

Lemma 2. *Let φ be a nonzero continuous linear form on $C(X; \mathbb{R})$ and let $x \in X$. If φ is positive on $P_x = \{f \in C(X; \mathbb{R}); f \geq 0, \ f(x) = 0\}$, there is $r \in \mathbb{R}$ such that $r < 0$ and $\varphi \geq r\delta_x$.*

Proof (Nachbin [4, §21]). Assume that $\varphi(f) \geq 0$ for all $f \geq 0$ such that $f(x) = 0$. Let

$$B = \{f \in C(X; \mathbb{R}); f \geq 0, \ f(x) = 1\}.$$

For any $f \in B$, notice that $g = f - \inf(\mathbf{1}, f)$ belongs to P_x. Hence $\varphi(g) \geq 0$ and so $\varphi(f) \geq \varphi(\inf(\mathbf{1}, f))$. Now $0 \leq \inf(\mathbf{1}, f) \leq 1$ and therefore $|\varphi(\inf(\mathbf{1}, f))| \leq \|\varphi\|$. Hence $\varphi(f) \geq -\|\varphi\|$, for all $f \in B$. Let $r = -\|\varphi\|$.

Let $f \geq 0$ be given in $C(X; \mathbb{R})$. If $f(x) > 0$, then $\varphi(f/f(x)) \geq r$ and so $\varphi(f) \geq rf(x)$. If $f(x) = 0$, then $f \in P_x$ and so $\varphi(f) \geq 0 = rf(x)$. Hence, $\varphi \geq r\delta_x$. □

If $S \subset C(X; \mathbb{R})$ is an infimum-stable convex cone, let $\Gamma(S)$ be the set of all pairs (x, φ), where x belongs to X and φ is a positive linear form on $C(X; \mathbb{R})$, such that

$$\varphi(g) \leq g(x), \quad \text{for all } g \in S.$$

Let $\widehat{K}(S)$ be the set of all functions $f \in C(X; \mathbb{R})$ such that $\varphi(f) \leq f(x)$, for all $(x, \varphi) \in \Gamma(S)$. With this notation, the following improved version of the Choquet-Deny Theorem is true.

Theorem 1. *Let $S \subset C(X; \mathbb{R})$ be an infimum-stable convex cone. Then $\overline{S} = \widehat{K}(S)$.*

Proof. It is easy to see that $\widehat{K}(S)$ is a closed subset containing S. Hence $\overline{S} \subset \widehat{K}(S)$.

Conversely, let $f \in C(X; \mathbb{R})$ be such that $f \notin \overline{S}$. By Lemma 1, there exists some $x \in X$ such that $f \notin \overline{S - P_x}$. Since $S - P_x$ is convex, by the Hahn-Banach Theorem there is a nonzero continuous linear form ψ on $C(X; \mathbb{R})$ and $c \in \mathbb{R}$ such that $\psi(g - h) \leq c < \psi(f)$ for all $g \in S$ and $h \in P_x$. Since $0 \in S$, we have $\psi(-\lambda h) \leq c$, for all $\lambda > 0$. Dividing by λ and letting $\lambda \to \infty$ we get $\psi(h) \geq 0$, for all $h \in P_x$. By Lemma 2, there is $r \in \mathbb{R}$ such that $r < 0$ and $\psi \geq r\delta_x$. Then $\varphi = \delta_x - r^{-1}\psi$ is positive. Since $0 \in P_x$, we have $\psi(g) \leq c < \psi(f)$, for every $g \in S$. Hence

$$\varphi(g) - g(x) = -r^{-1}\psi(g) \leq -r^{-1}c < -r^{-1}\psi(f) = \varphi(f) = f(x).$$

Since $\lambda g \in S$, for all $\lambda > 0$, we have $\varphi(\lambda g) - \lambda g(x) \leq -r^{-1}c$. Dividing by λ and letting $\lambda \to \infty$, we get $\varphi(g) - g(x) \leq 0$, for all $g \in S$. On the other hand,

$0 \in S$ implies $0 = \varphi(0) - 0 \leq -r^{-1}c < \varphi(f) - f(x)$, and so $f(x) < \varphi(f)$. Hence $f \notin \widehat{K}(S)$. □

Corollary 1. *Let $S \subset C(X; \mathbb{R})$ be an infimum-stable convex cone. Then S is uniformly dense if and only if the following is true: for every $f \in C(X; \mathbb{R})$, one has $\varphi(f) \leq f(x)$ whenever $(x, \varphi) \in \Gamma(S)$.*

Let us recall that A is said to *separate points from probability measures* if for any probability measure μ on X, and any $x \in X$, such that $\mu(g) = g(x)$, for all $g \in A$, then necessarily $\mu = \delta_x$, the Dirac measure at x.

Theorem 2. *Let A be a linear subspace of $C(X; \mathbb{R})$ such that $\mathbf{1} \in A$. Then A_m is uniformly dense if, and only if, A separates points from probability measures.*

Proof. Let $S = A_m$. Then S is an infimum-stable convex cone.

($\Rightarrow$) Assume S is dense, and let $x \in X$ and μ a probability measure on X be given such that $f(x) = \mu(f)$, for all $f \in A$. Then $g(x) \geq \mu(g)$, for all $g \in S$. Let $h \in C(X; \mathbb{R})$. By Theorem 1, $h(x) \geq \mu(h)$ and $-h(x) \geq \mu(-h)$. Hence $h(x) = \mu(h)$. This shows that A separates points from probability measures.

($\Leftarrow$) Conversely, assume that the subspace A separates points from probability measures. Let $(x, \varphi) \in \Gamma(S)$. For each $g \in A$, both g and $-g$ belong to $\widehat{K}(S)$, since $A \subset S$, and therefore $g(x) = \varphi(g)$, for all $g \in A$. The fact that $\mathbf{1} \in A$, implies $1 = \mathbf{1}(x) = \varphi(\mathbf{1})$. Therefore φ is a probability measure on X, and then $f(x) = \varphi(f)$, for all $f \in C(X; \mathbb{R})$. Hence $\widehat{K}(S) = C(X; \mathbb{R})$ and by Corollary 1, S is uniformly dense. □

Corollary 2. *Let A be a linear subspace of $C(X; \mathbb{R})$ such that $\mathbf{1} \in A$. Then $\overline{A_m} = \overline{A_M} = C(X; \mathbb{R})$ if, and only if, A separates points from probability measures.*

Proof. This follows from Theorem 2 and the fact that $A_M = -(-A)_m = -A_m$. □

Remark 1. Let us recall the notion of the *Choquet boundary* of a linear subspace A of $C(X; \mathbb{R})$, denoted by $\partial_A X$. By definition,

$$\partial_A X = \{x \in X; A(x) = \{\delta_x\}\}$$

where $A(x) = \{\mu \in \Delta; \mu(g) = g(x), \text{ for all } g \in A\}$, and Δ is the set of all probability measures on X.

Theorem 3. *Let A be a linear subspace of $C(X; \mathbb{R})$ such that $\mathbf{1} \in A$. The following are equivalent:*

(1) $\overline{A_m} = \overline{A_M} = C(X; \mathbb{R})$.
(2) *A separates points from probability measures.*
(3) $\partial_A X = X$.

Proof. By Corollary 2, (1) $\Leftrightarrow$ (2).

(2) $\Rightarrow$ (3) Let $x \in X$ be given. Let $\mu \in A(x)$. Then $\mu(g) = g(x)$, for all $g \in A$. Since A separates points from probability measures, this implies that $\mu(f) = f(x)$, for all $f \in C(X; \mathbb{R})$, i.e., $\mu = \delta_x$. Hence $x \in \partial_A X$, and so $X = \partial_A X$.

(3) $\Rightarrow$ (2) Let $f \in C(X; \mathbb{R})$ be given. Let $x \in X$ and $\varphi \in \Delta$ be such that $\varphi(g) = g(x)$, for all $g \in A$. Then $\varphi \in A(x)$. Since $x \in \partial_A X$, it follows

that $\varphi = \delta_x$. Hence $\varphi(f) = f(x)$, and A separates points from probability measures. □

Remark 2. The equivalence $(1) \Leftrightarrow (3)$ is due to Flösser, Irmisch, and Roth [2]. (See Example 4.2 of [2].) Since the equivalence $(2) \Leftrightarrow (3)$ is almost obvious, the paper [2] gives an alternative proof of the equivalence $(1) \Leftrightarrow (2)$.

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