Waveguiding in a dielectric medium varying slowly in one transverse direction

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We solve the problem of wave propagation in a medium whose dielectric constant varies quickly in one transverse direction and slowly in the other direction. This dielectric-constant profile is present in the waveguiding of double-heterostructure-junction lasers. The advantage of our solution is in its generality because we do not make other assumptions about the dielectric-constant function. From our solution we were able to correct the expression for the filling factor in TM modes that is seen in the literature.

1. INTRODUCTION AND RESULTS

We consider here the waveguiding of a medium whose complex dielectric constant has a fast variation along the transverse y direction and a slow variation along the transverse x direction. This situation is found in double-heterostructure (DH) injection lasers whose dielectric-constant profiles have the slab structure sketched in Fig. 1.[1–4] The active region has a larger dielectric constant ϵ and thus is able to guide light waves. Because of inhomogeneous injection, a variation of ϵ exists along the x direction, but this variation is slow, with a typical length much longer than the wavelength of the light.

Because of their importance to the theory of injection lasers, the waveguiding properties of the slab structure have been studied on many occasions.[1–3] In all instances that we found in the literature, the function $\epsilon(x, y)$ was made to assume special forms so as to make the solution of the vectorial wave equation simple, and no mention was made of the z component of the field and its dependence on x and y.[1–3] For this reason we decided to undertake the present study, eliminating all simplifying assumptions about the function $\epsilon(x, y)$ and retaining only the assumption that the variation along x is much slower than that along y.

Our method of solution of the wave equation is perturbative. We begin by solving the equation in a hypothetical medium whose dielectric constant $\tilde{\epsilon}(y)$ depends only on y. For a given angular frequency ω, we find the wave number $\tilde{k}$ of a solution with no variation along x:

$$\frac{\partial E}{\partial x} = 0.$$

Then we consider the effect of the perturbation

$$\epsilon(x, y) - \tilde{\epsilon}(y)$$

on this solution. It turns out that this effect is a modulation of the unperturbed solution by a function $\psi(x)$. We postpone to Section 2 many of the details of the solution, and here we present just the results.

For the TE solution we find the following expression for the electric field:

$$\mathbf{E} = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \begin{bmatrix} \psi(x) X^{(0)}(y) \\ 0 \\ -\dfrac{i}{k}\dfrac{\mathrm{d}\psi(x)}{\mathrm{d}x} X^{(0)}(y) \end{bmatrix} \exp(i\omega t - ikz), \quad (1)$$

and for the TM solution:

$$\mathbf{E} = \begin{bmatrix} \dfrac{i}{k}\dfrac{\mathrm{d}\psi(x)}{\mathrm{d}x} Z^{(0)}(y) \\ \psi(x) Y^{(0)}(y) \\ \psi(x) Z^{(0)}(y) \end{bmatrix} \exp(i\omega t - ikz), \quad (2)$$

where $X^{(0)}(y)$, $Y^{(0)}(y)$, and $Z^{(0)}(y)$ are the unperturbed solutions of the wave equation for the medium $\tilde{\epsilon}(y)$, that is,

$$\frac{\mathrm{d}^2 X^{(0)}(y)}{\mathrm{d}y^2} + \frac{\omega^2}{c^2}\tilde{\epsilon}(y) X^{(0)}(y) = \tilde{k}^2 X^{(0)}(y) \quad (3)$$

and

$$\left[\tilde{k}^2 - \frac{\omega^2}{c^2}\tilde{\epsilon}(y)\right] Y^{(0)}(y) - i\tilde{k}\frac{\mathrm{d}Z^{(0)}(y)}{\mathrm{d}y} = 0, \quad (4a)$$

$$-i\tilde{k}\frac{\mathrm{d}Y^{(0)}(y)}{\mathrm{d}y} - \frac{\mathrm{d}^2 Z^{(0)}(y)}{\mathrm{d}y^2} - \frac{\omega^2}{c^2}\tilde{\epsilon}(y) Z^{(0)}(y) = 0. \quad (4b)$$

The magnetic field follows from Eqs. (1) and (2):

$$\mathbf{H} = i\frac{c}{\omega}\nabla \times \mathbf{E}. \quad (5)$$

We notice that, because of the modulating function $\psi(x)$, the electric and magnetic fields are not exactly transverse in the TE and TM solutions. Since the modulating function $\psi(x)$ is slowly varying, the longitudinal components of these fields are small with respect to the transverse components.

The modulating function satisfies the following equation:

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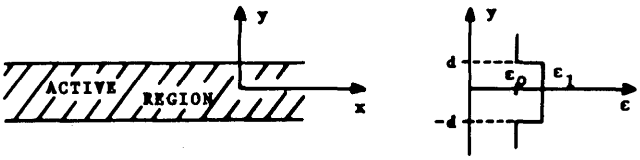

Fig. 1. Injection laser with slab waveguiding dielectric-constant profile.

$$\frac{d^2\psi(x)}{dx^2} + \frac{\omega^2}{c^2}\Delta\epsilon(x)\psi(x) = (k^2 - \tilde{k}^2)\psi(x), \qquad (6)$$

where k is the wave number being looked for, $\tilde{k}$ is the wave number of the unperturbed solution, and $\Delta\epsilon(x)$ is an effective dielectric constant of perturbation with the following definition: For the TE waves

$$\Delta\epsilon(x) = \frac{\int_{-\infty}^{+\infty} X^{(0)}(y)[\epsilon(x,y) - \tilde{\epsilon}(y)]X^{(0)}(y)\mathrm{d}y}{\int_{-\infty}^{+\infty} X^{(0)}(y)X^{(0)}(y)\mathrm{d}y}, \qquad (7)$$

and for the TM solutions

$$\Delta\epsilon(x) = \frac{\int_{-\infty}^{+\infty} [\epsilon(x,y) - \tilde{\epsilon}(y)][Y^{(0)}(y)^{+}Y^{(0)}(y) + Z^{(0)}(y)^{+}Z^{(0)}(y)]\mathrm{d}y}{\int_{-\infty}^{+\infty} Y^{(0)}(y)^{+}Y^{(0)}(y)\mathrm{d}y - \frac{i}{\tilde{k}}\int_{-\infty}^{+\infty} Y^{(0)}(y)^{+}\frac{\mathrm{d}Z^{(0)}(y)}{\mathrm{d}y}\mathrm{d}y}. \qquad (8)$$

In Eq. (8), $Y^{(0)}(y)^{+}$ and $Z^{(0)}(y)^{+}$ are the solutions of Eqs. (4) when we change $\tilde{k}$ into $-\tilde{k}$.

The effective dielectric constants of perturbation given by Eqs. (7) and (8) are the main results of our derivation. The expression for $\Delta\epsilon(x)$ for TE waves comes as no surprise and leads trivially to the definition of the filling factor Γ_{TE} that is current in injection-laser literature.[1,2] If we use the dielectric-constant profile of Streifer and Kapon[3] in our Eqs. (7) and (6), we arrive at their Eqs. (15) and (16). /In the case of TM waves, Eq. (8) leads to the following filling factor:

$$\Gamma_{TM} = \frac{\int_{-d}^{d} [Y^{(0)}(y)^{+}Y^{(0)}(y) + Z^{(0)}(y)^{+}Z^{(0)}(y)]\mathrm{d}y}{\int_{-\infty}^{+\infty} Y^{(0)}(y)^{+}Y^{(0)}(y)\mathrm{d}y - \frac{i}{\tilde{k}}\int_{-\infty}^{+\infty} Y^{(0)}(y)^{+}\frac{\mathrm{d}Z^{(0)}(y)}{\mathrm{d}y}\mathrm{d}y}, \qquad (9)$$

which is different from the expression given by Hakki and Paoli.[5] In Fig. 2 we compare Γ_{TE} and Γ_{TM} for the dielectric-constant profile sketched in Fig. 1. One notices that Γ_{TE} does not depend on the ratio $(\epsilon_1 - \epsilon_0)/\epsilon_1$ and is always greater than Γ_{TM}.

2. MATHEMATICAL DERIVATION

Unperturbed Solution

The E field of the wave is written as

$$\mathbf{E} = \begin{bmatrix} X(x,y) \\ Y(x,y) \\ Z(x,y) \end{bmatrix} \times \exp(i\omega t - ikz) \qquad (10)$$

and satisfies the following equation:

$$\nabla(\nabla\cdot\mathbf{E}) - \nabla\mathbf{E} - \frac{\omega^2}{c^2}\epsilon\mathbf{E} = 0. \qquad (11)$$

In many previous works the effects of the divergence of E have been neglected,[6] but in what follows we make no such approximation.

The three components of E may be expanded in a complete set of orthonormal functions (Hermite–Gaussian, for instance):

$$X(x,y) = a_{jJ}h_j(x)H_J(y), \qquad (12a)$$

$$Y(x,y) = b_{jJ}h_j(x)H_J(y), \qquad (12b)$$

$$Z(x,y) = c_{jJ}h_j(x)H_J(y), \qquad (12c)$$

where we sum in the repeated indices. We assume that the expansions above converge rapidly if the functions $h_j(x)$ and $H_j(x)$ have conveniently chosen widths. For instance, in the case of Hermite–Gaussian functions, we would choose the width parameters according to the typical lengths of variation of the fields in the x and y directions. Naturally, $h_j(x)$ will have widths much larger than $H_J(y)$. Then, by making scalar products of Eq. (11) with the vectors

$$\begin{bmatrix} h_i(x)H_I(y) \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ h_i(x)H_I(y) \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ h_i(x)H_I(y) \end{bmatrix}$$

and by integrating in the xy plane, we obtain the following set of equations:

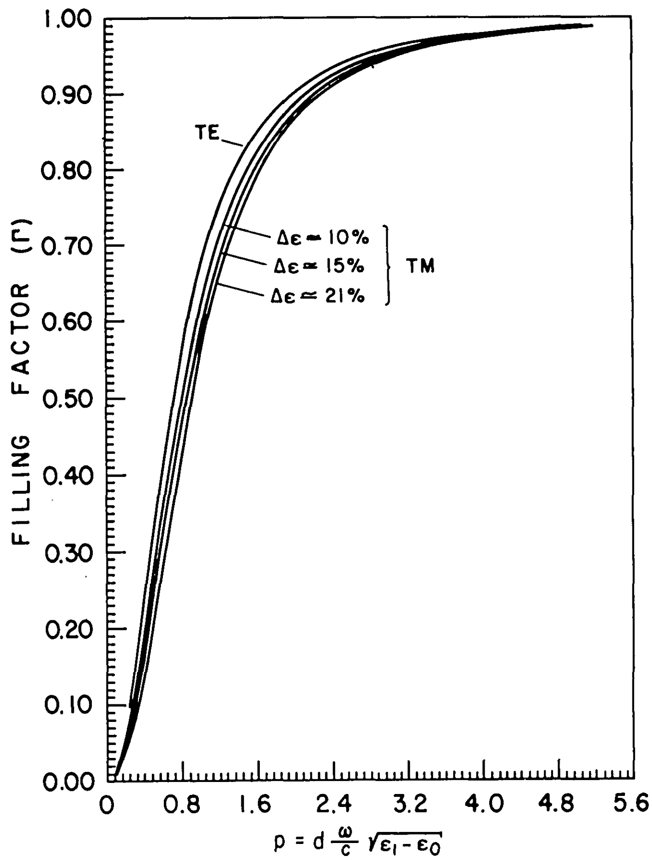

Fig. 2. Filling factor Γ as function of frequency and thickness for TE waves and TM waves and different values of the ratio $\Delta\epsilon = (\epsilon_1 - \epsilon_0)/\epsilon_1$.

$$\begin{bmatrix} (P^2)_{IJ}\delta_{ij} + k^2\delta_{IJ}\delta_{ij} - \frac{\omega^2}{c^2}\epsilon_{iI,jJ} & -P_{IJ}p_{ij} & k\delta_{IJ}p_{ij} \\ -P_{IJ}p_{ij} & \delta_{IJ}(p^2)_{ij} + k^2\delta_{IJ}\delta_{ij} - \frac{\omega^2}{c^2}\epsilon_{iI,jJ} & kP_{IJ}\delta_{ij} \\ k\delta_{IJ}p_{ij} & kP_{IJ}\delta_{ij} & \delta_{IJ}(p^2)_{ij} + (P^2)_{IJ}\delta_{ij} - \frac{\omega^2}{c^2}\epsilon_{iI,jJ} \end{bmatrix} \times \begin{pmatrix} a_{jJ} \\ b_{jJ} \\ c_{jJ} \end{pmatrix} = 0, \tag{13}$$

where

$$p_{ij} = -i \int h_i(x) \frac{\mathrm{d}}{\mathrm{d}x} h_j(x)\mathrm{d}x, \tag{14a}$$

$$P_{IJ} = -i \int H_I(y) \frac{\mathrm{d}}{\mathrm{d}y} H_J(y)\mathrm{d}y, \tag{14b}$$

$$\epsilon_{iI,jJ} = \iint h_i(x)H_I(y)\epsilon(x,y)h_j(x)H_J(y)\mathrm{d}x\mathrm{d}y, \tag{15}$$

$$(P^2)_{IJ} = \sum_K P_{IK}P_{KJ}, \tag{16a}$$

$$(p^2)_{ij} = \sum_k p_{ik}p_{kj}. \tag{16b}$$

In Eq. (13) we have a matrix multiplication followed by sums in the repeated indices. Equation (13) can be solved to determine k for any dielectric waveguide when ω is given. In our case, we begin the solution by considering a guide whose dielectric constant $\tilde{\epsilon}(y)$ does not depend on x. For a wave with no variation along x, we set

$$p_{ij} = 0, \tag{17a}$$

$$k \rightarrow \tilde{k}, \tag{17b}$$

$$\epsilon_{iI,jJ} \rightarrow \tilde{\epsilon}_{IJ}\delta_{ij}, \tag{17c}$$

and from Eq. (13) we obtain

$$\begin{bmatrix} (P^2)_{IJ} + \tilde{k}^2\delta_{IJ} - \frac{\omega^2}{c^2}\tilde{\epsilon}_{IJ} & 0 & 0 \\ 0 & \tilde{k}^2\delta_{IJ} - \frac{\omega^2}{c^2}\tilde{\epsilon}_{IJ} & \tilde{k}P_{IJ} \\ 0 & \tilde{k}P_{IJ} & (P^2)_{IJ} - \frac{\omega^2}{c^2}\tilde{\epsilon}_{IJ} \end{bmatrix} \times \begin{bmatrix} a_J{}^{(0)} \\ b_J{}^{(0)} \\ c_J{}^{(0)} \end{bmatrix} = 0. \tag{18}$$

These equations have two solutions: The TE field corresponds to the solution of the upper-left-hand submatrix, and the TM field corresponds to the lower-right-hand submatrix. We can verify readily that the upper-left-hand submatrix is the matrix version of the differential equation [Eq. (3)] for

$$X^{(0)}(y) = \sum_J a_J{}^{(0)}H_J(y). \tag{19}$$

Analogously, the lower-right-hand submatrix is the matrix version of the coupled Eqs. (4), where

$$Y^{(0)}(y) = \sum_J b_J{}^{(0)}H_J(y), \tag{20a}$$

$$Z^{(0)}(y) = \sum_J c_J{}^{(0)}H_J(y). \tag{20b}$$

Perturbed TE Solution

We consider now the dielectric-constant perturbation that is due to the difference

(i) $\epsilon(x,y) - \tilde{\epsilon}(y)$

between the true dielectric constant $\epsilon(x,y)$ and the approximative one $\tilde{\epsilon}(y)$. Following this perturbation, there results a difference

(ii) $k^2 - \tilde{k}^2$

between the wave numbers, and the momentum matrix elements

(iii) p_{ij}

cease to be null. Then we compare Eqs. (13) and (18) and use the following formula of perturbation theory:

(1) Perturbations i, ii, and iii are taken to the second order in the TE–TE matrix elements (upper-left-hand side).

(2) The perturbations are taken to the zeroth order in the TM–TM matrix elements (lower-right-hand side).

(3) The perturbations are taken to the first order in the TE–TM matrix elements (off-diagonal side).

We arrive at

$$\begin{bmatrix} (\mathrm{TE})_{IJ}\delta_{ij} + (k^2 - \tilde{k}^2)\delta_{IJ}\delta_{IJ} - \frac{\omega^2}{c^2}(\epsilon_{iI,jJ} - \tilde{\epsilon}_{IJ}\delta_{ij}) & -P_{IJ}p_{ij} & k\delta_{IJ}p_{ij} \\ -P_{IJ}p_{ij} & \multicolumn{2}{c}{[\mathrm{TM}]_{IJ}\delta_{ij}} \\ \tilde{k}\delta_{IJ}p_{ij} & & \end{bmatrix} \times \begin{pmatrix} a_{jJ} \\ b_{jJ} \\ c_{jJ} \end{pmatrix}, \tag{21}$$

where $(\mathrm{TE})_{IJ}$ and $[\mathrm{TM}]_{IJ}$ mean the submatrices in Eq. (18). Then, by using the second and third rows of Eq. (21) to eliminate b_{jJ} and c_{jJ}, we obtain

$$\Big[(\mathrm{TE})_{IJ}\delta_{ij} + (k^2 - \tilde{k}^2)\delta_{IJ}\delta_{ij} - \frac{\omega^2}{c^2}(\epsilon_{iI,jJ} - \tilde{\epsilon}_{IJ}\delta_{ij}) - p_{ik}p_{kj}(-P_{IK}\ \tilde{k}\delta_{IK})[\mathrm{TM}]_{KL}{}^{-1}\begin{pmatrix} -P_{LJ} \\ \tilde{k}\delta_{LJ} \end{pmatrix}\Big] a_{jJ} = 0. \tag{22}$$

When taking the inverse of the matrix [TM], the reader must not be misled by our compact notation: This matrix is not 2 × 2 but is much larger because of the indices K and L.

We look for a field whose x component is

$$X(x,y) = X^{(0)}(y)\psi(x) = \sum_J a_j{}^{(0)}H_J(y) \sum_j \xi_j h_j(x), \tag{23}$$

namely, the unperturbed component $X^{(0)}(y)$ times a modulating function $\psi(x)$. Thus

$$a_{jJ} = a_J{}^{(0)}\xi_j, \tag{24}$$

where the unperturbed coefficients $a_J{}^{(0)}$ are normalized according to

$$\sum_I a_I{}^{(0)}a_I{}^{(0)} = 1. \tag{25}$$

Now, by inserting Eq. (24) into Eq. (22), multiplying by $a_I{}^{(0)}$, summing in I, and using the fact that

$$[\mathrm{TM}]_{KL}{}^{-1}\begin{bmatrix} -P_{LJ}a_J{}^{(0)} \\ \tilde{k}a_L{}^{(0)} \end{bmatrix} = \begin{bmatrix} 0 \\ -\dfrac{1}{\tilde{k}}\,a_J{}^{(0)} \end{bmatrix}, \tag{26}$$

which follows from Eq. (18), we obtain

$$\sum_j \left[(k^2 - \tilde{k}^2)\delta_{ij} - \frac{\omega^2}{c^2}\int \Delta\epsilon(x)h_i(x)h_j(x)\mathrm{d}x - \int h_i(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}h_j(x)\mathrm{d}x\right]\xi_j = 0, \tag{27}$$

where $\Delta\epsilon(x)$ has been defined by Eq. (7).

Equation (27) is equivalent to Eq. (6), where

$$\psi(x) = \sum_j \xi_j h_j(x). \tag{28}$$

Further, using Eqs. (26) and (21) we obtain

$$b_{jJ} = 0, \tag{29a}$$

$$c_{jJ} = \frac{1}{\tilde{k}}\,a_J{}^{(0)}p_{jl}\xi_l, \tag{29b}$$

from which follows the expression for the electric field given by Eq. (1).

Perturbed TM Solution

We look for x-confined solutions of the form

$$Y(x,y) = \psi(x)Y^{(0)}(y) = \sum_{jJ}\xi_j b_J{}^{(0)}h_j(x)H_J(y), \tag{30a}$$

$$Z(x,y) = \psi(x)Z^{(0)}(y) = \sum_{jJ}\xi_j c_J{}^{(0)}h_j(x)H_J(y), \tag{30b}$$

where the modulating function $\psi(x)$ has expansion coefficients ξ_j. Thus

$$b_{jJ} = \xi_j b_J{}^{(0)}, \tag{31a}$$

$$c_{jJ} = \xi_j c_J{}^{(0)}. \tag{31b}$$

When the formulas of perturbation theory are applied, Eq. (13) becomes

$$\left\{\begin{array}{lll} (\mathrm{TE})_{IJ}\delta_{ij} & -P_{IJ}p_{ij} & \tilde{k}\delta_{IJ}p_{ij} \\ -P_{IJ}p_{ij} & [\mathrm{TM}]_{IJ}\delta_{ij} + \Big[\delta_{IJ}(p^2)_{ij} + (k^2 - \tilde{k}^2)\delta_{IJ}\delta_{ij} - \dfrac{\omega^2}{c^2}\Delta\epsilon_{iIjJ} & (k - \tilde{k})P_{IJ}\delta_{ij} \\ \tilde{k}\delta_{IJ}p_{ij} & (k - \tilde{k})P_{IJ}\delta_{ij} & \delta_{IJ}(p^2)_{ij} - \dfrac{\omega^2}{c^2}\Delta\epsilon_{iIjJ}\Big] \end{array}\right\} \times \begin{pmatrix} a_{jJ} \\ b_{jJ} \\ c_{jJ} \end{pmatrix} = 0. \tag{32}$$

The first line of this equation is used to solve for a_{jJ}:

$$a_{jJ} = -p_{jl}\xi_l(\mathrm{TE})_{JK}{}^{-1}[-P_{KL}b_L{}^{(0)} + \tilde{k}c_K{}^{(0)}], \tag{33}$$

where we have used Eqs. (31). By inserting Eq. (33) into the second and third lines of Eq. (32) and multiplying by the row matrix

$$[b_I{}^{(0)+}c_I{}^{(0)+}],$$

where $b_I{}^{(0)+}$ and $c_I{}^{(0)+}$ are the solutions of the equations dual to the lower-right-hand block of Eq. (18), namely,

$$\sum_I [b_I{}^{(0)+}c_I{}^{(0)+}][\mathrm{TM}]_{IJ} = 0, \tag{34}$$

we obtain a mathematical development similar to the one of the perturbed TE solutions. The results are embodied by Eq. (2), which gives the field, Eq. (6) for the modulating function, and Eq. (8) for the effective dielectric constant of perturbation.

3. SUMMARY

We have solved, in general terms, the problem of wave propagation in a medium with poor confinement ability in one transverse direction. The advantage of our solution is in its generality. For TM modes we were able to correct the expression for the filling factor that is presented in the literature.

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