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Kelvin functions for determination of magnetic susceptibility in nonmagnetic metals

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A method to calculate the real and imaginary parts of the magnetic permeability and susceptibility of nonmagnetic metals by using Kelvin functions is presented. The exact treatment is shown for the massive cylindrical geometry. An expression for a hollow circular cylinder is discussed and expanded to the thin-shell limit.

I. INTRODUCTION

It is well known that the electrical resistivity of a metal can be measured by the eddy currents induced in the samples.[1] Most of the methods applied in that determination require a thorough knowledge of the analytical expression for a relative magnetic permeability[2] or of the relative magnetic susceptibility[3,4] in accordance with the magnetic excitation used. In the particular case in which the excitation is an alternating magnetic field and the related geometrical symmetries are cylindrical, the expressions obtained for the real and imaginary parts of the magnetic susceptibility as a function of the frequency of the external field, the geometry, and the electrical conductivity of the sample involve Bessel functions with imaginary arguments.[5] For samples with solid circular cylindrical geometry, the magnetic susceptibility was deduced in an approximate form[6] using expansions developed by Fraser and Shoenberg.[7] In this paper we used the so-called Kelvin functions[8] to get the magnetic susceptibility and the behavior of its real and imaginary parts for two different cylindrical geometries submitted to an alternating magnetic field type excitation. In Sec. II we give an exact treatment for the solid circular cylindrical geometry, the same analyzed by Chambers and Park.[6] A similar treatment is developed for hollow cylindrical geometries in Sec. III. The limits for the massive cylinder and the metallic cylindrical shells are analyzed.

II. MAGNETIC SUSCEPTIBILITY OF A MASSIVE CYLINDER

Let us consider the response of an infinite, nonmagnetic solid circular cylinder of radius a and conductivity σ to an external alternating magnetic field oriented along the axis of the cylindrical sample.

The magnetic field is of the form

$$\mathbf{H}(t) = H_0 \exp(i\omega t)\hat{\mathbf{Z}}. \tag{1}$$

The relative magnetic permability of the system is defined by

$$\mu(\omega) = \langle H_i \rangle / H_0, \tag{2}$$

where $\langle H_i \rangle$ is the mean value of the magnetic field over the transversal section of the sample.

To find the permeability one has to solve Maxwell's equations for the inner cylindrical regions, i.e.,

$$\nabla\times\mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}, \tag{3}$$

$$\nabla\times\mathbf{H} = \sigma\mathbf{E} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{4}$$

Assuming a time dependence of the form (1) and considering the symmetry requirements $[\mathbf{H} = H(r)\hat{z}$ and $\mathbf{E} = E(r)\hat{\phi}]$, we can combine Eqs. (3) and (4) to get

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dH}{dr}\right) - i\omega\mu_0 H - \omega^2\epsilon_0\mu_0 H = 0, \tag{5}$$

or, neglecting displacement currents,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dH}{dr}\right) - i\omega\mu_0\sigma H = 0. \tag{6}$$

The above equation is known as Kelvin's equation of zero order, and has the following general solution:

$$H(r) = A_0[\mathrm{ber}_0(Kr) + i\,\mathrm{bei}_0(Kr)] + B_0[\mathrm{ker}_0(Kr) + i\,\mathrm{kei}_0(Kr)], \tag{7}$$

where

$$K = (\omega\mu_0\sigma)^{1/2}, \tag{8}$$

and A_0 and B_0 are constants.

The functions ber_0 and bei_0 are Kelvin functions of zero order and first kind, defined by

$$\mathrm{ber}_n(x) + i\,\mathrm{bei}_n(x) = J_n\left[x \exp\left(\frac{3\pi i}{4}\right)\right], \tag{9}$$

where n is a real number, x is a non-negative real value, and J_n is the Bessel function of order n and first kind.

Equivalently, the functions ker_0 and kei_0 are Kelvin functions of zero order and second kind, defined by

$$\mathrm{ker}_n(x) + i\,\mathrm{kei}_n(x) = e^{-(1/2)n\pi i} K_n[x \exp(i\pi/4)], \tag{10}$$

where K_n is the modified Bessel function of order n and second kind.

The constants A_0 and B_0 are determined from the boundary conditions. Since kei_0 is infinite at the origin, we

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must have $B_0 = 0$. The continuity of the tangential magnetic field at $r = a$ is expresed by the equation

$$H(a) = H_0 = A_0[\mathrm{ber}_0(Ka) + i\,\mathrm{bei}_0(Ka)]. \tag{11}$$

Therefore, we arrive at the following expression for the magnetic field inside the sample:

$$H(r) = H_0[\mathrm{ber}_0(Kr) + i\,\mathrm{bei}_0(Kr)]/[\mathrm{ber}_0(Ka) + i\,\mathrm{bei}_0(Ka)]. \tag{12}$$

Going back to the definition of the relative magnetic permeability, we find

$$\mu(\omega) = -\sqrt{2}\,\frac{(1+i)}{Ka}\left(\frac{\mathrm{ber}_1(Ka) + i\,\mathrm{bei}_1(Ka)}{\mathrm{ber}_0(Ka) + i\,\mathrm{bei}_0(Ka)}\right), \tag{13}$$

where ber_1 and bei_1 are the Kelvin functions of order 1 and first kind as defined in Eq. (9).

The magnetic susceptibility is then established from the magnetic permeability described in Eq. (3) obeying

$$X(\omega) = -1.0 + \mu(\omega), \tag{14}$$

such that

$$\mathrm{Re}(\omega) = -1.0 - \frac{\sqrt{2}}{x}\left(\frac{\mathrm{ber}_1(x)}{\mathrm{ber}_0(x) + \mathrm{bei}_0(x)} + \frac{\mathrm{bei}_1(x)}{\mathrm{ber}_0(x) - \mathrm{bei}_0(x)}\right) \tag{15}$$

and

$$\mathrm{Im}(\omega) = -\frac{\sqrt{2}}{x}\left(\frac{\mathrm{ber}_1(x)}{\mathrm{ber}_0(x) + \mathrm{bei}_0(x)} + \frac{\mathrm{bei}_1(x)}{\mathrm{ber}_0(x) - \mathrm{bei}_0(x)}\right), \tag{16}$$

where Re and Im are the real and imaginary part of the susceptibility, respectively, and

$$X \equiv Ka = (\omega\mu_0\sigma)^{1/2}a. \tag{17}$$

The behavior of (15) and (16) as a function of X is shown in Fig. 1(a) and agrees with the approximate treatment given by Chambers and Parker.[6]

III. MAGNETIC SUSCEPTIBILITY OF A HOLLOW CIRCULAR CYLINDER

The same procedure we developed in the last section will now be adapted to the case in which the sample has the geometrical configuration shown in Fig. 2. To find the magnetic permeability of the hollow circular cylinder, one has to solve Maxwell's equations for regions 1 and 2.

The pertinent Maxwell equations for region 1 are

$$\nabla \times \mathbf{E}_1 = -\mu_0 \frac{\partial \mathbf{H}_1}{\partial t} \tag{18}$$

and

$$\nabla \times \mathbf{H}_1 = \sigma \mathbf{E}_1 + \epsilon_0 \frac{\partial \mathbf{E}_1}{\partial t}. \tag{19}$$

Assuming the same time dependence as in Sec. II and considering that the conductivity in region I is zero, we get from the combination of Eqs. (18) and (19) the following equation:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dH_1}{dr}\right) - \omega^2\epsilon_0\mu_0 H_1 = 0, \tag{20}$$

which has the exact solution

$$H_1(r) = A_1 I_0(K_1 r) + B_1 K_0(K_1 r), \tag{21}$$

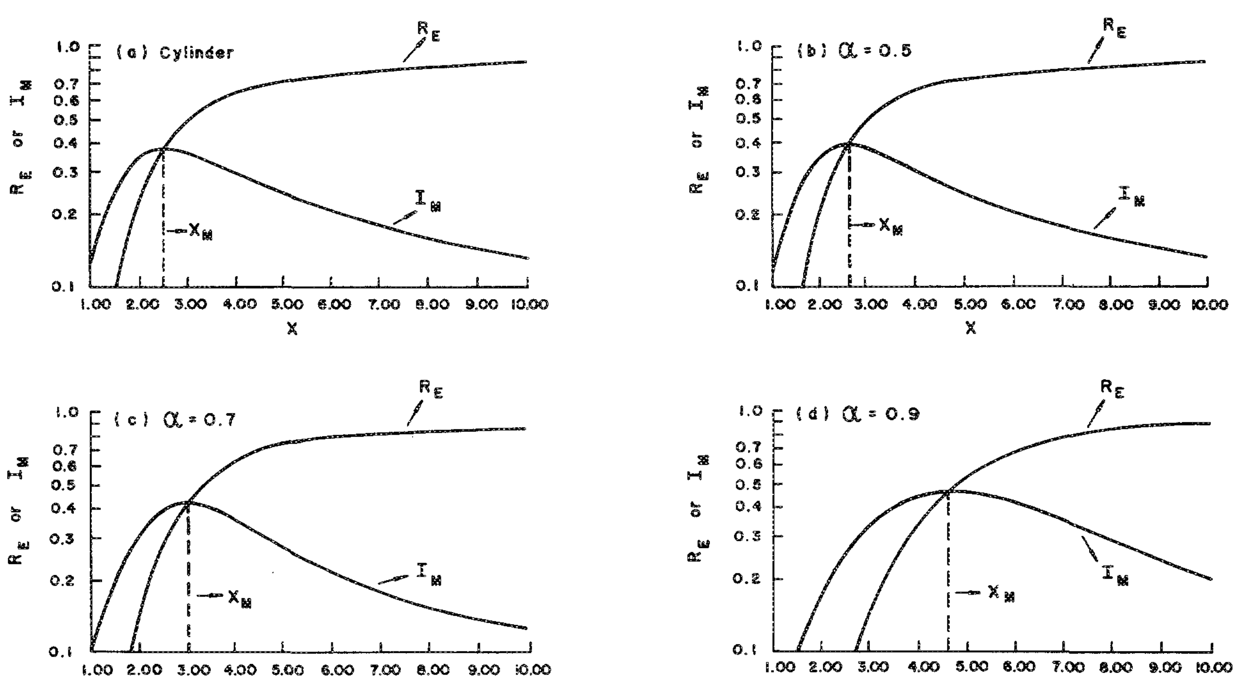

FIG. 1. Negative real and imaginary parts of the relative magnetic susceptibility as a function of x for (a) massive cylinder; (b), (c), and (d) hollow cylinders with $\alpha = 0.5$, 0.7, and 0.9, respectively. x_M denotes the crossover between the two curves.

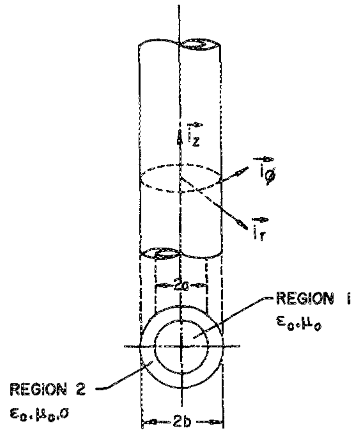

FIG. 2. Geometrical configuration of the metals. The cross section of the hollow circular cylinder shows the two different regions.

where

$$K_1 = \omega(\epsilon_0\mu_0)^{1/2}. \tag{22}$$

A_1 and B_1 are constants to be determined from the boundary conditions, and I_0 and K_0 are modified Bessel functions of order 0 and 1, respectively. Since K_0 is infinite at the origin we must have $B_1 = 0$.

It is interesting to comment on the behavior of I_0 in this region. Supposing that the samples have diameters of 1 cm and the range of frequencies for the external magnetic field does not go beyond 10^8 Hz, the maximum argument value of I_0 will be

$$K_1 a = \omega(\epsilon_0\mu_0)^{1/2} a \sim 0.33\times 10^{-2} \ll 1. \tag{23}$$

In this case I_0 goes to unity[9] and solution (21) can be approximated by a constant. This solution corresponds to the case in which the displacement currents that appear in Eq. (20) are neglected (see the Appendix).

Following the same procedure used in Sec. II, we get the magnetic field in region 2 the solution

$$H_2(r) = C_1[\mathrm{ber}_0(K_2 r) + i\,\mathrm{bei}_0(K_2 r)] + D_1[\mathrm{ker}_0(K_2 r) + i\,\mathrm{kei}_0(K_2 r)], \tag{24}$$

where

$$K_2 = (\omega\mu_0\sigma)^{1/2}. \tag{25}$$

The constants A_1, C_1, and D_1 are calculated by imposing the following boundary conditions:

(i) The continuity of the tangential magnetic field at $r = b$:

$$H_0 = C_1[\mathrm{ber}_0(K_2 b) + i\,\mathrm{bei}_0(K_2 b)] + D_1[\mathrm{ker}_0(K_2 b) + i\,\mathrm{kei}_0(K_2 b)], \tag{26}$$

and at $r = a$:

$$A_1 = C_1[\mathrm{ber}_0(K_2 a) + i\,\mathrm{bei}_0(K_2 a)] + D_1[\mathrm{ker}_0(K_2 a) + i\,\mathrm{kei}_0(K_2 a)]. \tag{27}$$

(ii) The continuity of the tangential electric field at $r = a$:

$$\left.\frac{dH_2}{dr}\right|_{r=a} = \frac{i\omega\mu_0\sigma}{a}\int_0^a H_2(r) r\,dr \tag{28}$$

or

$$\alpha x A_1 = -(1+i)\sqrt{2}\{C_1[\mathrm{ber}_1(\alpha x) + i\,\mathrm{bei}_1(\alpha x)] + D_1[\mathrm{ker}_1(\alpha x) + i\,\mathrm{kei}_1(\alpha x)]\}, \tag{29}$$

where

$X \equiv K_2 b$ and $\alpha \equiv a/b$.

Using the definition for the relative magnetic permeability given by

$$\mu(\omega) = \frac{2.0}{b^2 H_0}\left(\int_0^a H_1(r) r\,dr + \int_a^b H_2(r) r\,dr\right), \tag{30}$$

we find for a hollow circular cylinder with conductivity σ

$$\mu_H(\omega) = \frac{[(2/x)\{\gamma_1(\alpha x)\lambda_1(x) - \lambda_1(\alpha x)\gamma_1(x)] + i\alpha[(1+i)/\sqrt{2}]\lambda_0(\alpha x)\gamma_1(x) - \gamma_0(\alpha x)\lambda_1(x)\}}{(1-1)/2[\gamma_0(x)\lambda_1(\alpha x) - \gamma_1(\alpha x)\lambda_0(x)] + [(1+i)/2\sqrt{2}]\alpha x[\gamma_0(\alpha x)\lambda_0(x) - \gamma_0(x)\lambda_0(\alpha x)]}, \tag{31}$$

where

$$\gamma_n(z) = \mathrm{ber}_n(z) + i\,\mathrm{bei}_n(z), \tag{32}$$

$$\lambda_n(z) = \mathrm{ker}_n(z) + i\,\mathrm{kei}_n(z). \tag{33}$$

The behavior of the real and imaginary parts of the susceptibility as a function of x is shown in Figs. 1(b), 1(c), and 1(d) for $\alpha = 0.5$, 0.7, and 0.9, respectively. One can see that for a metal with electrical conductivity σ, the crossover of the real and imaginary parts, denoted by x_M, increases considerably with α. This effect is caused by the magnetic screening boundary condition at the inner surface which tries to maintain the magnetic field in the hole so that the field must diffuse all the way to the other surface in order to escape.

In the limit of the massive cylinder, α goes to zero and as

$$\lim_{\alpha\to 0}\gamma_1(\alpha x) = -\infty.$$

Equation (31) can be simplified to the form

$$\lim_{\alpha\to 0}\mu_H(\omega) = -\frac{\sqrt{2}}{x}(1+i)\frac{\gamma_1(x)}{\gamma_0(x)},$$

which is exactly the same expression we got in (13), regardless of the strong boundary condition imposed on the inner surface of the hollow cylinder described in Eq. (30).

Another interesting limit is the case $\alpha \to 1$, which corresponds to the transition to a cylindrical metallic shell.

Expanding to the first order in $(D/a)[\equiv(b-a)/a \ll 1]$ for the Kelvin functions of argument αx, i.e.,

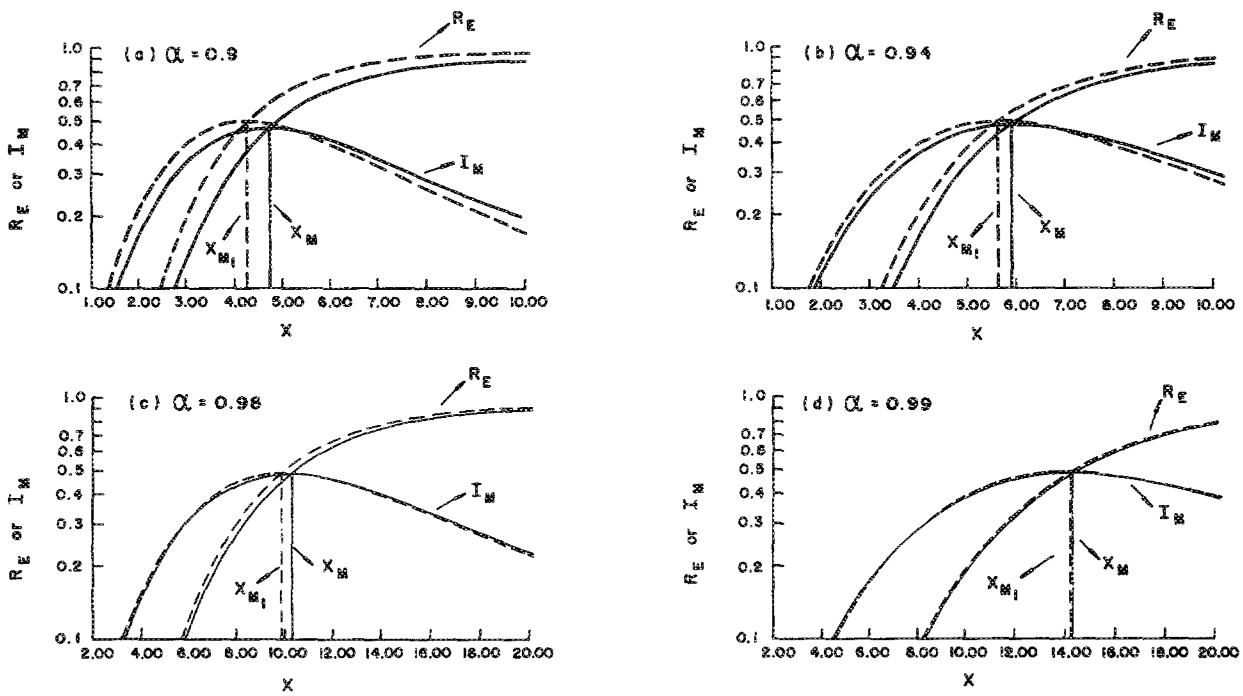

FIG. 3. Comparison between the results for magnetic susceptibilities for several α. Solid lines correspond to the exact calculation and the dashed lines to the first-order approximation.

$$\gamma_n(\alpha x) = \gamma_n(x) + K_2(b-a)\left[(1-i)/\sqrt{2}\right]\gamma_{n+1}(x) \quad (34)$$

and

$$\lambda_n(x) = \gamma_n(x) + K_2(b-a)\left[(1-i)/\sqrt{2}\right]\gamma_{n+1}(x), \quad (35)$$

one can find, after some algebra, for the magnetic permeability:

$$\lim_{\alpha \to 0} \mu_H(\omega) = \frac{1}{1 + i(x^2 D/2a)}. \quad (36)$$

The accuracy of the results in first order obtained for the magnetic susceptibility in shown in Fig. 3. One can see that the expansion gives good results as α goes to 1, even without considering two additional effects:

(a) The length-to-diameter ratio of the specimen is finite. Our treatment assumes infinite length for the sample. However, this can be taken into account when the measurements are done following the experimental procedure explained by Clark, Deason, and Powell.[10]

(b) Size effects: For a sample in which thickness is comparable to skin depth we must consider size effects.[11] A minimum sample thickness such that size effects could be neglected was considered in our treatment.

IV. CONCLUSION

A method was discussed to calculate magnetic properties of nonmagnetic metals and alloys with cylindrical geometries. The principal advantages of the Kelvin functions used in the determination of the real and imaginary parts of the relative magnetic permeability and susceptibility are that they allowed work with real arguments. The results that were obtained for the massive cylinder are exact and agree with the approximate calculation done by Chambers and Parker.[6] An equivalent calculation was performed for the hollow circular cylinder. In addition, an approximate expression was obtained for the case of the cylindrical shell.

APPENDIX

Consider Eq. (20):

$$\frac{1}{r}\frac{d}{dr}\left(\frac{r\,dH_1}{dr}\right) - \omega^2\epsilon_0\mu_0 H_1 = 0. \quad (A1)$$

Neglecting the displacement currents, one can find the general solution in the form

$$H_1(r) = A + B\ln(r). \quad (A2)$$

The logarithmic infinity at the origin requires $B = 0$, such that the final solution is

$$H_1(r) = A. \quad (A3)$$

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