Minimal Set of Local Measurements and Classical Communication for Two-Mode Gaussian State Entanglement Quantification

Luis F. Haruna, Marcos C. de Oliveira, and Gustavo Rigolin
Instituto de Física "Gleb Wataghin," Universidade Estadual de Campinas, 13083-970, Campinas, São Paulo, Brazil
(Received 8 February 2007; published 9 April 2007)

We develop the minimal requirements for the complete entanglement quantification of an arbitrary two-mode bipartite Gaussian state via local measurements and a classical communication channel. The minimal set of measurements is presented as a reconstruction protocol of local covariance matrices and no previous knowledge of the state is required but its Gaussian character. The protocol becomes very simple mostly when dealing with Gaussian states transformed to its standard form, since photocounting or intensity measurements define the whole set of entangled states. In addition, conditional on some prior information, the protocol is also useful for a complete global state reconstruction.

DOI: 10.1103/PhysRevLett.98.150501 PACS numbers: 03.67.Mn, 03.65.Ud, 42.50.Dv

Quantum communication protocols extend the information theoretical notion of channel [1] to the quantum domain by incorporating nonlocal entangled states. Those channels are generated by the preparation of a pair (or more) of quantum systems in an entangled state, which are then separated to establish nonlocal correlations [2], allowing several communication tasks otherwise unattainable via classical channels [3]. However, for most of the quantum protocols to work properly (deterministically) one has first to be able to prepare maximally pure entangled states and then to guarantee that those states stay pure or nearly pure during all the processing time. An important problem then arises in this whole process: One has to check the "quality" (the amount of entanglement and purity) of the quantum channel, while usually the only available tools for that are local measurements (operations) and one (or several) classical channel.

The quest for an optimal and general solution for this problem has generated a vast literature on the characterization of entangled states under local operations and classical communication (LOCC), either for qubits [4] or for continuous variable systems of the Gaussian type [5,6]. Gaussian states (completely described by up to second order moments) are particularly important since they can be easily generated with radiation field modes. Moreover, operations that keep the Gaussian character (so-called Gaussian operations) are given by the transformations induced by linear (active and passive) optical devices (beam splitters, phase shifters, and squeezers) [6]. A particular result for this kind of state is that it is impossible to distill entanglement out of a set of Gaussian states through Gaussian operations [7].

Assuming one is left with only Gaussian local operations and a classical channel (GLOCC), how is it possible to infer the quality of a quantum channel in use? For a two-mode Gaussian state one possibility is to access directly the entanglement properties of the system after a proper manipulation of the two modes [8,9]. This procedure requires, however, that the two parties (modes) be recombined in a beam splitter (nonlocal unitary operation) in which their entanglement content are transferred to local properties of one of the output modes. Another possible way is to completely reconstruct the bipartite quantum system, a resource demanding task [10] which also requires global operations here forbidden.

In this Letter we demonstrate a minimal set of GLOCC to completely quantify the entanglement of a two-mode Gaussian state. As a bonus of this procedure one can also assess the purity of the Gaussian state and, for some particular classes of states, reconstruct the bipartite covariance matrix. The protocol consists mainly in the attainment, via local measurements, of all the symplectic invariants that allows, for example, one to test the separability of the system, to know its P-representability properties, and to quantify its entanglement content. We also show that for a particular class of Gaussian states belonging to the set of symmetric Gaussian states [11], including the Einstein-Podolsky-Rosen (EPR) states and general mixed squeezed states, the protocol becomes straightforward due to the relative easiness with which one obtains the correlation matrix elements from local measurement outcomes. Moreover, since P-representability and separability for these kinds of states are equivalent, we show that for two-mode thermal squeezed states with internal noise [12] it is possible to decide whether or not they are separable via local photon number measurements.

A two-mode Gaussian state ρ_{12} is characterized by its Gaussian characteristic function $C(\boldsymbol{\alpha}) = e^{-(1/2)\boldsymbol{\alpha}^\dagger \mathbf{V} \boldsymbol{\alpha}}$, where $\boldsymbol{\alpha}^\dagger = (\alpha_1^*, \alpha_1, \alpha_2^*, \alpha_2)$ are complex numbers and a_1 ($a_1^\dagger$) and a_2 ($a_2^\dagger$) the annihilation (creation) operators for parties 1 and 2, respectively [13]. The covariance matrix $\mathbf{V}$ describing all the second order moments $V_{ij} = (-1)^{i+j} \times \langle v_i v_j^\dagger + v_j^\dagger v_i \rangle / 2$, where $(v_1, v_2, v_3, v_4) = (a_1, a_1^\dagger, a_2, a_2^\dagger)$, is given by

$$\mathbf{V} = \begin{pmatrix} \mathbf{V}_1 & \mathbf{C} \\ \mathbf{C}^\dagger & \mathbf{V}_2 \end{pmatrix} = \begin{pmatrix} n_1 + \frac{1}{2} & m_1 & m_s & m_c \\ m_1^* & n_1 + \frac{1}{2} & m_c^* & m_s^* \\ m_s^* & m_c & n_2 + \frac{1}{2} & m_2 \\ m_c^* & m_s & m_2^* & n_2 + \frac{1}{2} \end{pmatrix}.$$

$\mathbf{V}_1$ and $\mathbf{V}_2$ are local Hermitian matrices, while $\mathbf{C}$ is the correlation between the two parties. Any covariance matrix must be positive semidefinite $\mathbf{V} \geq \mathbf{0}$ and the generalized uncertainty principle, $\mathbf{V} + (1/2)\mathbf{E} \geq \mathbf{0}$, where $\mathbf{E} = \mathrm{diag}(\mathbf{Z}, \mathbf{Z})$ and $\mathbf{Z} = \mathrm{diag}(1, -1)$, must hold [14].

From local measurements on both modes of ρ_{12}, either through homodyne detection (see [15], and references therein) or alternatively by employing single-photon detectors [16], the local covariance matrices $\mathbf{V}_1$ and $\mathbf{V}_2$ can be reconstructed. Remark that for the reconstruction of the global matrix $\mathbf{V}$, and therefore the joint bipartite state, one has to obtain $\mathbf{C}$. Obviously, global joint measurements achieved through recombination of the two parties in a beam splitter followed by local homodyne detections are forbidden. Thus one has to deal only with local measurements whose results can be sent through classical communication channels to the other party. As we now show, there are minimal operations or measurements that can be performed locally on the system to attain $|\det\mathbf{C}|$ and $\det\mathbf{V}$. These quantities, together with $\det\mathbf{V}_1$ and $\det\mathbf{V}_2$, will be shown to be all that one needs to determine whether or not a two-mode Gaussian state is entangled as well as how much it is entangled. As it will become clear, the required set of operations is minimal in the sense that only two local measurement procedures are needed—one to characterize local covariance matrices and another to locally assess the parity of one of the modes.

First of all let us introduce an important result [17]. Given a two-mode Gaussian state with density operator ρ_{12} and covariance matrix $\mathbf{V}$ we can define the Gaussian operator $\sigma_1 = \mathrm{Tr}_2\{e^{i\pi a_2^\dagger a_2}\rho_{12}\}$, whose covariance matrix $\mathbf{\Gamma}_1$ is the Schur complement [18] of $\mathbf{V}$ relative to $\mathbf{V}_2$:

$$\mathbf{\Gamma}_1 = \mathbf{V}_1 - \mathbf{C}\mathbf{V}_2^{-1}\mathbf{C}^\dagger. \tag{1}$$

The meaning of σ_1 is best appreciated through a partial trace in the Fock basis: $\sigma_1 = \sum_{n_{\mathrm{even}}} {}_2\langle n|\rho_{12}|n\rangle_2 - \sum_{n_{\mathrm{odd}}} {}_2\langle n|\rho_{12}|n\rangle_2 = \rho_{1_e} - \rho_{1_o}$, being equal to the difference between Alice's mode states conditioned, respectively, to even and odd parity measurement results by Bob [17]. While ρ_{1_e} and ρ_{1_o} are not generally Gaussian, σ_1 is a Gaussian operator, and $\mathbf{\Gamma}_1$ can be built with only second order moments of these conditioned states.

Now suppose that Alice and Bob share many copies of a two-mode Gaussian state. The protocol works as follows: (i) First, in a subensemble of the copies, each party performs a set of local measurements in such a manner to obtain the covariance matrices $\mathbf{V}_1$ and $\mathbf{V}_2$, corresponding to the reduced operators $\rho_1 = \mathrm{Tr}_2\{\rho_{12}\}$ and $\rho_2 = \mathrm{Tr}_1\{\rho_{12}\}$; (ii) then Bob informs Alice, via a classical communication channel, the matrix elements of $\mathbf{V}_2$; (iii) after that, for the remaining copies, Bob performs parity measurements on his mode, letting Alice know to which copies does that operation correspond and the respective outcomes, i.e., even parity (eigenvalue 1) or odd parity (eigenvalue -1); (iv) Alice then separates her copies in two groups, the even (e) and the odd (o) ones. The first group (e) contains all the copies conditioned on an even parity measurement on Bob's copies. The other one (o) contains all the remaining copies, namely, those conditioned on an odd parity measurement at Bob's; (v) for each group, Alice measures the respective correlation matrices $\mathbf{V}_{1e}$ and $\mathbf{V}_{1o}$; (vi) finally, she obtains $\mathbf{\Gamma}_1$ [Eq. (1)] subtracting the odd correlation matrix from the even one [17]: $\mathbf{\Gamma}_1 = \mathbf{V}_{1e} - \mathbf{V}_{1o}$. Remarkably, with $\mathbf{V}_1$, $\mathbf{V}_2$, and $\mathbf{\Gamma}_1$ in hand Alice is able to completely characterize the Gaussian state's entanglement content as well as its purity without any global or nonlocal measurements.

Remembering that a two-mode Gaussian state's purity $\mathcal{P}$ is equal to $1/(4\sqrt{\det\mathbf{V}})$ [19] and using the identity [18]

$$\det\mathbf{V} = \det\mathbf{V}_2 \det\mathbf{\Gamma}_1, \tag{2}$$

Alice readily obtains the purity of the channel: $\mathcal{P} = 1/(4\sqrt{\det\mathbf{V}_2 \det\mathbf{\Gamma}_1})$. Her next task is to decide whether or not she deals with an entangled two-mode Gaussian state. Using the Simon separability [20] test she knows that it is not entangled if, and only if,

$$I_1 I_2 + (1/4 - |I_3|)^2 - I_4 \geq (I_1 + I_2)/4, \tag{3}$$

where $I_1 = \det\mathbf{V_1}$, $I_2 = \det\mathbf{V_2}$, $I_3 = \det\mathbf{C}$, and $I_4 = \mathrm{Tr}(\mathbf{V_1 Z C Z V_2 Z C^\dagger Z})$. These four quantities are the local symplectic invariants, belonging to the $Sp(2,R) \otimes Sp(2,R)$ group [20], that characterizes all the entanglement properties of a two-mode Gaussian state. Alice already has I_1 and I_2. We must show, however, how she can obtain $|I_3|$ and I_4. Since one can prove that [9]

$$I_4 = 2|I_3|\sqrt{I_1 I_2}, \tag{4}$$

we just need to show how $|I_3|$ is obtained from I_1, I_2, and $I_V = \det\mathbf{V}$, the three pieces of information locally available to Alice. To achieve this goal we first note that a direct calculation gives $I_V = I_1 I_2 - I_4 + I_3^2$. Using Eq. (4) we see that $|I_3|$ follows from $|I_3|^2 - 2|I_3|\sqrt{I_1 I_2} + I_1 I_2 - I_V = 0$. One of its roots is not acceptable since it implies $\mathbf{V} < 0$. Therefore, we are left with

$$|I_3| = \sqrt{I_1 I_2} - \sqrt{I_V}. \tag{5}$$

Hence, substituting Eqs. (4) and (5) in Eq. (3), Alice is able to unequivocally tell whether or not she shares an entangled two-mode Gaussian state with Bob. Finally, if her state is entangled then $I_3 < 0$ [20] and, for a symmetric state ($I_1 = I_2$), Alice can quantify its entanglement via the entanglement of formation (E_f) [21,22]:

$$E_f(\rho_{12}) = f\left(2\sqrt{I_1 + |I_3| - \sqrt{I_4 + 2I_1|I_3|}}\right), \tag{6}$$

where $f(x) = c_+(x)\log_2[c_+(x)] - c_-(x)\log_2[c_-(x)]$ and

$c_{\pm}(x) = (x^{-1/2} \pm x^{1/2})^2/4$. For arbitrary two-mode Gaussian states ($I_1 \neq I_2$) Alice can work with lower bounds for E_f [21] or calculate its negativity or logarithmic negativity [23]. These last two quantities are the best entanglement quantifiers for nonsymmetric two-mode Gaussian states and are given as analytical functions [19,24] of the four invariants here obtained from local measurements: I_1, I_2, $|I_3| = \sqrt{I_1 I_2} - \sqrt{I_V}$, and $I_4 = 2|I_3|\sqrt{I_1 I_2}$, with $I_V = \det\mathbf{V}$ given by Eq. (2). It is worth mentioning that I_1 (I_2) can easily be determined by the measurement of the purity (Wigner function at the origin of the phase space) of Alice's (Bob's) mode alone [16,25]. This measurement is less demanding than the ones required to reconstruct $\mathbf{V}_1$ and $\mathbf{V}_2$ [9].

Besides furnishing all the entanglement properties of an arbitrary two-mode Gaussian state, the previous local protocol can also be employed to reconstruct the covariance matrix for some particular types of Gaussian states. To see this, let $\mathbf{\Gamma}_1$ be explicitly written as

$$\mathbf{\Gamma}_1 = \begin{pmatrix} \eta_1 + \frac{1}{2} & \mu_1 \\ \mu_1^* & \eta_1 + \frac{1}{2} \end{pmatrix}, \tag{7}$$

where

$$\eta_1 = \langle a_1^\dagger a_1 \rangle_e - \langle a_1^\dagger a_1 \rangle_o, \tag{8}$$

$$\mu_1 = \langle a_1^2 \rangle_e - \langle a_1^2 \rangle_o, \qquad \mu_1^* = \langle (a_1^\dagger)^2 \rangle_e - \langle (a_1^\dagger)^2 \rangle_o, \tag{9}$$

being $\langle \cdot \rangle_e$ and $\langle \cdot \rangle_o$ the mean values for Alice's even and odd subensembles, respectively. From this identity it is clear that $\mathbf{\Gamma}_1$ does not necessarily represent a physical state since η_1 can take negative values [17]. From Eq. (1) we obtain the following two relations:

$$n_1 - \eta_1 = \frac{1}{(n_2 + \frac{1}{2})^2 - |m_2|^2}\{(|m_c|^2 + |m_s|^2)(n_2 + \tfrac{1}{2}) - 2\Re e(m_2 m_s m_c^*)\}, \tag{10}$$

$$m_1 - \mu_1 = \frac{1}{(n_2 + \frac{1}{2})^2 - |m_2|^2}\{2 m_s m_c (n_2 + \tfrac{1}{2}) - m_2^* m_c^2 - m_2 m_s^2\}. \tag{11}$$

Equations (10) and (11) give the matrix elements of $\mathbf{\Gamma}_1$ as a function of the matrix elements of $\mathbf{V}$. If m_c and m_s are real (if either m_c or m_s is zero) Eqs. (10) and (11) can be inverted to give m_c and m_s (either m_s or m_c).

Let us explicitly solve the previous equations for an important case, namely, the ones in which $\mathbf{C}\mathbf{C}^\dagger = |m_i|^2\mathbf{I}$, where $i = c$ or s and $\mathbf{I}$ is the identity matrix. The states comprehending this class are the ones where $\mathbf{C}$ has only diagonal or non diagonal elements, i.e., $m_s = 0$ and $m_c \neq 0$ or $m_c = 0$ and $m_s \neq 0$, reducing the unknown quantities to two, namely, the absolute value and the phase of m_s or m_c. Recall that if $i = s$ the system is separable, since $\det\mathbf{C} = |m_s|^2 \geq 0$; i.e., the correlation between the two modes is strictly classical [20]. Otherwise, if $i = c$ the state is not necessarily separable, possibly being entangled, for in this case $\det\mathbf{C} = -|m_c|^2 \leq 0$. This last case is more interesting since it represents a class of states that might show nonlocal features [20].

From Eqs. (10) and (11) the diagonal (off-diagonal) elements of $\mathbf{C}$, $m_i = |m_i|e^{i\phi_i}$, for $i = s$ ($i = c$), are

$$|m_i|^2 = \frac{(n_1 - \eta_1)}{n_2 + 1/2}[(n_2 + 1/2)^2 - |m_2|^2], \tag{12}$$

$$e^{2i\phi_i} = \left(\frac{\mu_1 - m_1}{n_1 - \eta_1}\right)\frac{n_2 + 1/2}{m_{2i}}, \tag{13}$$

where $m_{2c} = m_2^*$ and $m_{2s} = m_2$. Note that whenever $m_2 = 0$, ϕ_i becomes undetermined. This problem can be solved by locally (unitary) transforming the two-mode squeezed state to a matrix V_2' with $m_2' \neq 0$, where ϕ_i' can be determined. Then, transforming back, we get ϕ_i. Fortunately, there are various experimentally available bipartite Gaussian states in which all the parameters are real, $m_s(m_c) = m_1 = m_2 = 0$, and $m_c(m_s) \neq 0$. For these states, Eq. (12) is sufficient to determine $\mathbf{C}$.

A natural and important example belonging to this class is the two-mode thermal squeezed state [12], which is generated in a nonlinear crystal with internal noise. Its covariance matrix is

$$\mathbf{V} = \begin{pmatrix} n + \frac{1}{2} & 0 & 0 & m_c \\ 0 & n + \frac{1}{2} & m_c & 0 \\ 0 & m_c & n + \frac{1}{2} & 0 \\ m_c & 0 & 0 & n + \frac{1}{2} \end{pmatrix}, \tag{14}$$

where n and m_c are time dependent functions having as parameters the relaxation constant of the bath as well as the nonlinearity of the crystal [12]. In this case the protocol involves only simple local measurements, i.e., those to get n, $\langle a_1^\dagger a_1 \rangle_e$, and $\langle a_1^\dagger a_1 \rangle_o$ (or equivalently η_1) by Alice, and the parity measurements by Bob. The classical communication corresponds to Bob informing Alice the instances he performs the parity measurement in his mode and the respective outcomes. Hence, Eq. (12) reduces to

$$m_c^2 = (n - \eta_1)(n + 1/2). \tag{15}$$

Experimentally, n and η_1 [Eq. (8)] are readily obtained by photodetection, while the parity measurement is related to the determination of Bob's mode Wigner function at the origin of the phase-space [26], or alternatively to his mode's purity, both of which can be measured by photocounting experiments [16,25].

We can also study the P representability [27] for the state (14), which in this case is equivalent to the Simon separability test [11,20]. A two-mode Gaussian state is P representable iff $\mathbf{V} - \frac{1}{2}\mathbf{I} \geq 0$, where $\mathbf{I}$ is the unity matrix of dimension 4. Explicitly, this separability condition in terms of the elements of (14) is equivalent to $n \geq |m_c|$. From this inequality and Eq. (15) we see that for a given n there exists a bound for η_1 below which the states are entangled (upper solid curve in Fig. 1):

$$-\frac{n/2}{n + 1/2} \leq \eta_1 \leq \frac{n/2}{n + 1/2}. \tag{16}$$

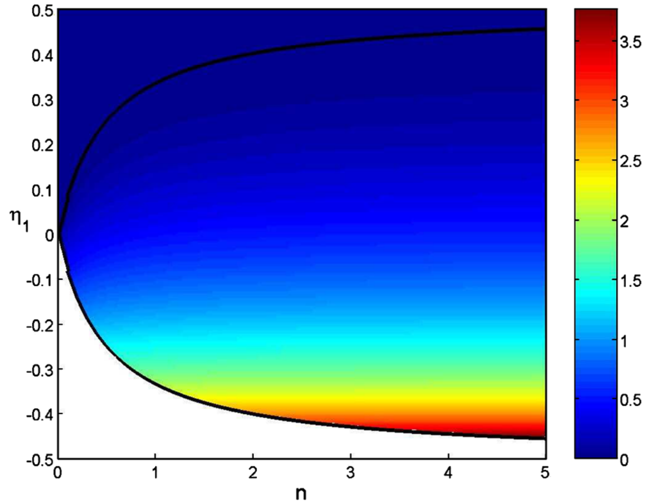

FIG. 1 (color online). Above the upper solid curve lie the separable states. Below it, entanglement is quantified via E_f [Eq. (6)] up to the lower curve, where the pure entangled states are located. Below this curve there exist no physical states.

The left bound in Eq. (16) (lower solid curve in Fig. 1) is a consequence of the uncertainty principle, delimiting the set of all physical symmetric Gaussian states (SGS). This bound is marked by all the pure states and the upper curve bounds (from below) the subset of all separable (P-representable) states [11]. Thus, for the SGS class, photon number measurements, before and after Bob's parity measurements, are all Alice needs to discover whether or not her mode is entangled with Bob's. The exquisite symmetry of those two antagonistic bounds is quite surprising, and possibly valid only for the SGS class. There is another interesting feature for the SGS set that should be emphasized. Note that $\eta_1 = 0$ contains all the states where Bob has equal chances of getting even or odd outcomes for his parity measurements, delimiting two subsets (even and odd). The even subset contains all the states where Bob has greater probabilities of getting even outcomes while the odd subset contains all the states where he has greater probabilities of getting odd outcomes. The entanglement for states belonging to the SGS can be quantified through E_f [Eq. (6)] as depicted by the color scale in Fig. 1. It is remarkable that the most entangled states (including the pure ones) are concentrated in the $\eta_1 < 0$ odd subset.

In conclusion, we have presented the minimal set of local operations and classical communication that allows one to quantify the entanglement of an arbitrary two-mode Gaussian state. One important step towards the derivation of this protocol was the mathematical identity relating the two-mode covariance matrix determinant to the product of two local quantities, namely, the determinants of the one-mode correlation matrix and its Schur complement. In addition, we have also shown that the Schur complement of one of the modes' covariance matrix is obtained via a set of parity measurements on the other one. We have also explicitly discussed how the protocol works for a particular class of Gaussian states belonging to the SGS set. Within this class, for states written in its standard form, we have shown that only photon number measurements (made before and after a parity measurement on the other mode) are needed to completely characterize the state's entanglement.

This work is supported by FAPESP and CNPq.

[1] T. M. Cover and J. A. Thomas, *Elements of Information Theory* (Wiley-Interscience, New York, 1991).
[2] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[3] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge Univ. Press, Cambridge, U.K., 2000).
[4] M. A. Nielsen, Phys. Rev. Lett. **83**, 436 (1999).
[5] J. Eisert and M. B. Plenio, Phys. Rev. Lett. **89**, 097901 (2002); G. Giedke and J. I. Cirac, Phys. Rev. A **66**, 032316 (2002).
[6] J. Eisert and M. B. Plenio, Int. J. Quantum. Inform. **1**, 479 (2003).
[7] J. Eisert, S. Scheel, and M. B. Plenio, Phys. Rev. Lett. **89**, 137903 (2002); J. Fiurášek, Phys. Rev. Lett. **89**, 137904 (2002).
[8] M. C. de Oliveira, Phys. Rev. A **72**, 012317 (2005).
[9] G. Rigolin and M. C. de Oliveira, quant-ph/0608184.
[10] J. Laurat *et al.*, J. Opt. B **7**, S577 (2005); V. D'Auria *et al.*, J. Opt. B **7**, S750 (2005).
[11] M. C. de Oliveira, Phys. Rev. A **70**, 034303 (2004).
[12] S. Daffer, K. Wódkiewicz, and J. K. McIver, Phys. Rev. A **68**, 012104 (2003).
[13] We neglect first order moments since they can be set to zero locally without changing the entanglement content.
[14] B. G. Englert and K. Wódkiewicz, Int. J. Quantum. Inform. **1**, 153 (2003).
[15] A. Ourjoumtsev, R. Tualle-Brouri, and P. Grangier, Phys. Rev. Lett. **96**, 213601 (2006).
[16] J. Fiurášek and N. J. Cerf, Phys. Rev. Lett. **93**, 063601 (2004).
[17] L. F. Haruna and M. C. de Oliveira, quant-ph/0701196.
[18] R. A. Horn and C. R. Johnson, *Matrix Analysis* (Cambridge University Press, Cambridge, 1987).
[19] G. Adesso, A. Serafini, and F. Illuminati, Phys. Rev. Lett. **92**, 087901 (2004).
[20] R. Simon, Phys. Rev. Lett. **84**, 2726 (2000).
[21] G. Rigolin and C. O. Escobar, Phys. Rev. A **69**, 012307 (2004).
[22] G. Giedke, M. M. Wolf, O. Krüger, R. F. Werner, and J. I. Cirac, Phys. Rev. Lett. **91**, 107901 (2003).
[23] G. Vidal and R. F. Werner, Phys. Rev. A **65**, 032314 (2002).
[24] G. Adesso, A. Serafini, and F. Illuminati, Phys. Rev. A **70**, 022318 (2004).
[25] K. Banaszek and K. Wódkiewicz, Phys. Rev. Lett. **76**, 4344 (1996).
[26] A. Royer, Phys. Rev. A **15**, 449 (1977).
[27] A *P*-representable state has a non-negative probability distribution, being thus classical.