The many faces of superradiance

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Inertial motion superradiance, the emission of radiation by an initially unexcited system moving inertially but superluminally through a medium, has long been known. Rotational superradiance, the amplification of radiation by a rotating rigid object, was recognized much later, principally in connection with black hole radiances. Here we review the principles of inertial motion superradiance and prove thermodynamically that the Ginzburg-Frank condition for superradiance coincides with the condition for superradiant amplification of already existing radiation. Examples we cite include a new type of black hole superradiance. We correct Zel'dovich's thermodynamic derivation of the Zel'dovich-Misner condition for rotational superradiance by including the radiant entropy in the bookkeeping. We work out in full detail the electrodynamics of a Zel'dovich rotating cylinder, including a general electrodynamic proof of the Zel'dovich-Misner condition, and explicit calculations of the superradiant gain for both types of polarization. Contrary to Zel'dovich's pessimistic conclusion we conclude that, if the cylinder is surrounded by a dielectric jacket and the whole assembly is placed inside a rotating cavity, the superradiance is measurable in the laboratory. [S0556-2821(98)08516-6]

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I. INTRODUCTION

A free structureless particle moving inertially in vacuum cannot absorb or emit a photon. This well known fact follows solely from Lorentz invariance and four-momentum conservation. But a free object endowed with internal structure can, of course, absorb photons, and can also emit them provided it is initially excited above its ground state (rest mass M larger than minimum possible value $M_{\rm gr}$). Somewhat surprisingly, when the object, which may be electrically neutral overall, moves uniformly through a medium, emission may be allowed even when the object starts off in its ground state. The early recognition of this possibility by Ginzburg and Frank [1] (Ginzburg [2] gives a modern review) marks the beginning of our subject, which we term superradiance.

The term superradiance, introduced by Dicke [3], originally referred to amplification of radiation due to coherence in the emitting medium. Many years later Zel'dovich [4] pointed out that a cylinder made of absorbing material and rotating about its axis with frequency Ω is capable of amplifying those modes of scalar or electromagnetic radiation impinging on it which satisfy the condition

$$\omega - m\Omega < 0 \tag{1}$$

where ω is the waves' frequency and m the azimuthal quantum number with respect to the axis of rotation. Zel'dovich realized that, when quantum physics is allowed for, the rotating object should be able to emit spontaneously in the regime (1), and anticipated that a rotating (Kerr) black hole should show both amplification and spontaneous emission when condition (1) is satisfied. Misner [5] independently made a suggestion that the Kerr black hole will amplify waves, and supported it with unpublished calculations. The corresponding spontaneous emission was first put into evidence field theoretically by Unruh [6].

Following Misner's observation one of us noted [7] that in the Kerr black hole case superradiant amplification is classically required when condition (1) holds because that is the only way to fulfill Hawking's classically rigorous horizon area theorem [8] (see also Ref. [9]). From the same logic it followed [7] that superradiance of electrically charged waves by a charged black hole is required whenever

$$\omega - q\Phi/\hbar < 0 \tag{2}$$

where q is the elementary charge of the field and Φ the electrostatic potential of the black hole measured at the horizon.

Following the emergence of black hole thermodynamics [10] it became clear that black horizon area plays the role of entropy for black holes. This correspondence and the cited argument for superradiance from black hole area immediately suggests that the necessity of superradiance in ordinary objects is solely a consequence of thermodynamics. In fact, Zel'dovich [11] used a thermodynamic argument in his discussion to show that superradiance of the rotating cylinder must take place. Following this idea we extend in this paper the superradiance condition to a broad range of circumstances. Indeed, we make the point that superradiance is a useful and broad guiding principle for radiating systems in electrodynamics and elsewhere.

This paper is organized as follows. In Sec. II we review and elaborate on the Ginzburg-Frank argument for spontaneous emission in certain modes by an object moving inertially and superluminally in a medium. We then give a thermodynamic argument for superradiant amplification of those classical waves which satisfy the Ginzburg-Frank condition. In Sec. III we discuss a number of phenomena involving am-

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plification of waves which can be reduced to the inertial superradiance motif; one is a new effect. Sec. IV discusses, in the footsteps of Zel'dovich [4,11], the thermodynamic basis of spontaneous emission and superradiant amplification by *rotating* objects. In Sec. V we give an electrodynamic proof of rotational superradiance, and calculate in detail the superradiant gain of a conducting (and/or otherwise dissipative) rotating cylinder for one type of polarization. This new treatment improves and corrects Zel'dovich's semiquantitative estimates [11], extends the results to situations where the material's permeabilities are not unity, and can easily be extended to relativistic rotation (as may be relevant for pulsars). It is followed by a description of a device which can make rotational superradiance measurable in the laboratory. The Appendix extends some of the above results to the second polarization.

Unless otherwise noted we use units in which the speed of light *in vacuum* is unity: $c=1$.

II. INERTIAL MOTION SUPERRADIANCE: PRINCIPLES

A. Spontaneous superradiance

Let E and $E'=E-\hbar\omega$ denote the object's total energy in the *laboratory* frame before and after the emission of a photon with energy $\hbar\omega$ and momentum $\hbar\mathbf{k}$ (both measured in the laboratory frame), while $\mathbf{P}$ and $\mathbf{P}'=\mathbf{P}-\hbar\mathbf{k}$ denote the corresponding momenta; $\mathbf{v}=\partial E/\partial\mathbf{P}$ is the initial velocity of the object. The object's rest mass M is nothing but the energy measured in the rest frame, $M=\gamma(E-\mathbf{v}\cdot\mathbf{P})$ with $\gamma\equiv(1-\mathbf{v}^2)^{-1/2}$, while after the emission, with obvious notation, $M'=\gamma'(E'-\mathbf{v}'\cdot\mathbf{P}')$. Then a straightforward calculation to $\mathcal{O}(\omega)$, $\mathcal{O}(\mathbf{k})$ and $\mathcal{O}(\mathbf{v}'-\mathbf{v})$ gives

$$M'-M=-\gamma\hbar(\omega-\mathbf{v}\cdot\mathbf{k})+\hbar\omega\cdot\mathcal{O}(\mathbf{v}'-\mathbf{v})\,. \tag{3}$$

As written, this formula is relevant for emission; for absorption the sign in front of $(\omega-\mathbf{v}\cdot\mathbf{k})$ should be reversed. The factor $\mathcal{O}(\mathbf{v}'-\mathbf{v})$ represents recoil effects; it is of order $\hbar\omega/M$ and becomes negligible for a sufficiently heavy object. In this recoilless limit

$$M'-M=-\gamma\hbar(\omega-\mathbf{v}\cdot\mathbf{k});\quad\text{(emission)} \tag{4a}$$

$$M'-M=\gamma\hbar(\omega-\mathbf{v}\cdot\mathbf{k});\quad\text{(absorption)}. \tag{4b}$$

We note that in vacuum $\omega=|\mathbf{k}|>\mathbf{v}\cdot\mathbf{k}$ so that emission is possible only with de-excitation $(M'-M<0)$, while absorption is coupled with excitation $(M'-M>0)$, as plain intuition would have.

Now suppose the object moves uniformly through an isotropic medium transparent to electromagnetic waves possessing an index of refraction $n(\omega)>1$. The $\hbar\omega$ and $\hbar\mathbf{k}$ are still the energy and momentum of the photon, but now $\omega=|\mathbf{k}|/n(\omega)$. Whenever $|\mathbf{v}|<1/n(\omega)$ (subluminal motion for the relevant frequency) we recover the connections ''de-excitation↔emission'' and ''excitation↔absorption.'' Ginzburg and Frank [1,2] refer to this kind of emission or absorption as the ordinary Doppler effect, because the relation between ω and $\mathbf{k}$ and the rest frame transition frequency $\omega_0\equiv|M-M'|/\hbar$,

$$\omega_0=\gamma(\omega-\mathbf{v}\cdot\mathbf{k}), \tag{5}$$

is the standard Doppler shift formula.

In the case $|\mathbf{v}|>1/n(\omega)$ the object moves faster than the *phase* velocity of electromagnetic waves of frequency ω. If ϑ denotes the angle between $\mathbf{k}$ and $\mathbf{v}$, a photon in a mode with $\cos\vartheta>[n(\omega)|\mathbf{v}|]^{-1}$ has negative $\omega-\mathbf{v}\cdot\mathbf{k}$, and can thus be *emitted* only in consonance with *excitation* of the object $(M'-M>0)$. Ginzburg and Frank refer to this eventuality as the *anomalous* Doppler effect. They note a variety of circumstances other than superluminal motion in a dielectric for which the conditions for the anomalous Doppler effect can be met: a particle moving in vacuum through a narrow channel drilled into a dielectric, a particle shot into a gap between dielectric slabs, emission from a collection of sources which are successively excited so that the active source moves along with superluminal *phase* velocity, etc. [1,2]

Thus an object in its ground state may become excited and emit a photon, provided it moves superluminally through a medium. The energy source must be the bulk motion. Emission is not just allowed by the conservation laws; it will occur spontaneously, as follows from thermodynamic reasoning. The object in its ground state with no photon around constitutes a low entropy state; the excitation of the object to one of a number of possible excited states with emission of a photon with momentum in a variety of possible directions evidently entails an increase in entropy. Thus the emission is favored by the second law.

Recall that according to Eq. (4b), when $\omega-\mathbf{v}\cdot\mathbf{k}<0$ *absorption* of a photon is possible only if accompanied by a *de-excitation* of the object $(M'-M<0)$. Thus a superluminally moving object in the ground state is forbidden from absorbing in certain modes.

A further case is absorption or emission by a superluminal object of photons with the directions given precisely by $\cos\vartheta=[n(\omega)|\mathbf{v}|]^{-1}$. According to Eqs. (4a,b) both are possible and *do not* require a change in the object. In fact, both processes can occur consecutively, thus constituting scattering of a photon with no change in the object. Consequently, for superluminal motion, scattering with both initial and final directions specified by $\cos\vartheta=[n(\omega)|\mathbf{v}|]^{-1}$ can be coherent scattering. In particular, all these processes are possible for a structureless particle which, of course, has only one state (structureless is a relative concept; we mean the particle looks structureless at the relevant energy scale).

B. Superradiant amplification

Section II A deals with *spontaneous superradiance* by an elementary object; Ginzburg [2] has in mind a two level atom (a dipole). The condition for superradiance is that the emitted quantum's energy in the moving object's frame, $\hbar(\omega-\mathbf{v}\cdot\mathbf{k})$, be negative. Then its emission may be interpreted, in the medium's frame, as absorption of a positive energy quantum with consequent excitation of the object. When the object has complicated structure, it is simpler to derive the condition for amplification directly from the second law of thermodynamics. We thus assume here that the

object may dissipate energy internally. This allows us to develop the following classical argument.

Suppose that the incident radiation is exclusively in modes with frequency near ω and propagating within $\Delta\mathbf{n}$ of the direction $\mathbf{n}$. Let $I(\omega,\mathbf{n})$ denote the corresponding intensity (power per unit area, unit solid angle and unit bandwidth). Experience tells us that the body will absorb power $a(\omega,\mathbf{n})\Sigma(\mathbf{n})I(\omega,\mathbf{n})\Delta\omega\Delta\mathbf{n}$, where $\Sigma(\mathbf{n})$ is the geometric cross section normal to $\mathbf{n}$, and $a(\omega,\mathbf{n})<1$ is a characteristic absorptivity of the body. Simultaneously the object will scatter power $[1-a(\omega,\mathbf{n})]\Sigma(\mathbf{n})I(\omega,\mathbf{n})\Delta\omega\Delta\mathbf{n}$. By conservation of energy

$$\frac{dE}{dt}=a\Sigma I\Delta\omega\Delta\mathbf{n}-W \tag{6}$$

where W is the overall power spontaneously emitted by the body (including any thermal emission). We ignore energy going into scattered photons because it will not show up in Eq. (9) below.

Now the linear momentum conveyed by the radiation is $\mathbf{n}n(\omega)$ times the energy conveyed. The easiest way to see this is to think of the radiation as composed of quanta, each with energy $\hbar\omega$ and momentum $\hbar\mathbf{k}$ with $\omega n(\omega)=|\mathbf{k}|$. However, the result also holds classically, and can be derived, for instance, by comparing the temporal-spatial and spatial-spatial components of the energy-momentum tensor for the field. Thus absorption and spontaneous emission cause the linear momentum $\mathbf{P}$ of the body to change at a rate

$$\frac{d\mathbf{P}}{dt}=\mathbf{n}n(\omega)a\Sigma I\Delta\omega\Delta\mathbf{n}-\mathbf{U} \tag{7}$$

where $\mathbf{U}$ signifies the rate of spontaneous momentum emission. We have not included the transfer of momentum due to scattering because this has no influence on Eq. (9) below.

As already hinted, in calculating the rate of change of rest mass of the body, M, we may forget the effects of elastic scattering. For in the frame of the body waves are scattered with no Doppler shift (since there is no motion), which means that they contain the same energy before and after the scattering. Thus the scattering cannot contribute to dM/dt. Because M is just the body's energy in its own frame, the rest mass changes at a rate given by a Lorentz transformation:

$$dM/dt=\gamma(dE/dt-\mathbf{v}\cdot d\mathbf{P}/dt)\,. \tag{8}$$

Of course, a change in the proper mass means that the number of microstates accessible to the object has changed, i.e., that its entropy S has changed. Defining an effective temperature for the body, $T=\partial M/\partial S$, we see by Eqs. (6), (7) that

$$\frac{dS}{dt}=\frac{\gamma}{T}[\omega^{-1}(\omega-\mathbf{v}\cdot\mathbf{k})a\Sigma I\Delta\omega\Delta\mathbf{n}-W+\mathbf{v}\cdot\mathbf{U}] \tag{9}$$

where we have replaced $\mathbf{n}\omega n(\omega)\to\mathbf{k}$.

Let us now take into account the rate of change of radiation entropy, $d\mathcal{S}/dt$. We get an upper bound on it by ignoring any entropy carried *into* the object by the waves. Now the entropy in a single mode of a field containing on the mean N quanta is at most [13]

$$\mathcal{S}_{\max}=(N+1)\ln(N+1)-N\,\ln N\approx\ln N \tag{10}$$

where the approximation applies for $N\gg1$. The scattered waves carry a mean number of quanta proportional to $I(\omega,\mathbf{n})$. Hence, for large N, the outgoing waves' contribution to $d\mathcal{S}/dt$ is bounded from above by a quantity of $\mathcal{O}[\ln I(\omega,\mathbf{n})]$. There is an additional contribution to $d\mathcal{S}/dt$ of $\mathcal{O}(W)$ coming from the spontaneous emission. Hence

$$d\mathcal{S}/dt<\mathcal{O}[\ln I(\omega,\mathbf{n})]+\mathcal{O}(W). \tag{11}$$

If any dissipation takes place, the second law of thermodynamics demands $dS/dt+d\mathcal{S}/dt>0$. As $I(\omega,\mathbf{n})$ is made larger and larger, the total entropy rate of change becomes dominated by the term proportional to $I(\omega,\mathbf{n})$ in Eq. (9) because W and $\mathbf{U}$ are kept fixed. Positivity of $dS/dt+d\mathcal{S}/dt$ then requires

$$(\omega-\mathbf{v}\cdot\mathbf{k})a(\omega,\mathbf{n})>0. \tag{12}$$

Thus whenever the Ginzburg-Frank condition,

$$\omega-\mathbf{v}\cdot\mathbf{k}<0 \tag{13}$$

for the anomalous Doppler effect is fulfilled, we necessarily have $a(\omega,\mathbf{n})<0$. This result was obtained by assuming $a\Sigma I\Delta\omega\Delta\mathbf{n}\gg W$. But since—barring nonlinear effects—a must be independent of the incident intensity, the result must be true for any intensity which can still be regarded as classical. Now $a<0$ means that the scattered wave, with power proportional to $1-a$, is stronger than the incident one (which is represented by the ''1'' in the previous expression). Thus the moving object amplifies preexisting radiation in modes satisfying the Ginzburg-Frank condition. We say that the object superradiates. For modes with $\omega-\mathbf{v}\cdot\mathbf{k}>0$, $a>0$ and so the object absorbs on the whole.

As a rule of thumb amplification of waves may be regarded as the classical counterpart of stimulated emission at the quantum level. By Einstein's argument stimulated emission goes hand in hand with spontaneous emission in the same mode. The spontaneous emission corresponding to superradiance amplification is just the Ginzburg-Frank emission discussed in connection with Eqs. (4a,b).

Obviously a switches sign at the superradiance threshold $\omega=\mathbf{v}\cdot\mathbf{k}$. This switch cannot take place by a having a pole since $a<1$. *If* a is analytic in $\omega-\Omega m$, it must thus have the expansion

$$a=\alpha(\mathbf{v},\mathbf{n})(\omega-\mathbf{v}\cdot\mathbf{k})+\cdots \tag{14}$$

in the vicinity of the neutral frequency $\omega=\mathbf{v}\cdot\mathbf{k}$. However, we must emphasize that thermodynamics does not require the function a to be continuous at $\omega=\mathbf{v}\cdot\mathbf{k}$.

The superradiance discussed here and in Sec. II A will evidently occur also for fields other than the electromagnetic. All that is required is that the energy and momentum of a quantum be expressible in terms of frequency and wavevec-

tor in the usual way. Thus one can replace above ''photons'' and ''electromagnetic waves'' by phonons and sound waves, etc.

III. INERTIAL MOTION SUPERRADIANCE: EXAMPLES

We now give four examples of phenomena that can be understood as manifestations of inertial motion superradiance. One is novel.

A. Vavilov-Cherenkov effect

A point charge moving at speed v through a transparent isotropic dielectric medium faster than the phase speed of electromagnetic radiation for some range of frequencies will emit radiation at all those frequencies; for each frequency the radiation front is a cone with opening angle $2\Theta_C(\omega)$, where

$$\sin\Theta_C(\omega)=[vn(\omega)]^{-1}. \tag{15}$$

This Vavilov-Cherenkov effect, discovered experimentally in 1934, and explained theoretically by Tamm and Frank [12], was the first example of coherent radiation from an unaccelerating source. We now elaborate on Ginzburg's [2] discussion of the effect in terms of superradiance.

Since the charge has no internal degrees of freedom, its rest mass is fixed. We may thus set $M'-M=0$ in Eqs. (4a,b). Those conditions cannot thus be satisfied for $v<1/n(\omega)$ since their right hand side (RHS) would then be strictly positive: no absorption or emission is possible from a subluminal particle. However, for $v>1/n(\omega)$ the RHS vanishes when the photon's direction makes an angle ϑ to the particle's velocity, where $\cos\vartheta=[vn(\omega)]^{-1}$. But then the front of photons emitted as the charge goes by forms a cone with opening angle $2(\pi/2-\vartheta)$ which evidently coincides with $2\Theta_C$. As argued in Sec. II A, the growth of entropy associated with the multiplicity of possible azimuthal directions of the emitted photon favors emission; the emitted photons constitute the Vavilov-Cherenkov radiation.

In truth the above description is somewhat simplistic. It is well known that the Vavilov-Cherenkov radiation actually comes from those regions in the dielectric that feel strongly the electromagnetic field of the charge [12]. In effect the charges carries along with it a polarization cloud of dielectric material. As the charges advances, the atomic constituents of this cloud are replaced continuously by fresh atoms from upstream. Because of this renewal, the system ''charge+cloud'' is a dissipative one: part of the energy that goes into exciting an atom in the cloud is inexorably carried away into the wake of the charge.

The argument of Sec. II B then tells us that the moving charge (and its polarization cloud) must also amplify ambient radiation which satisfies condition (13). Writing $\mathbf{v}\cdot\mathbf{k}=v\omega n(\omega)\cos\vartheta$, it follows that amplification occurs for $\cos\vartheta>1/vn(\omega)$. Thus radiation modes inside the Cherenkov cone (those with wavevector more aligned with the charge's motion than the Vavilov-Cherenkov modes') must be amplified. This Vavilov-Cherenkov superradiant amplification has not yet been observed.

B. Gravitational generation of electromagnetic waves

We now discuss a new phenomenon. Suppose an electrically neutral black hole of mass M moves with constant velocity $\mathbf{v}$ through a uniform and isotropic dielectric with an index of refraction whose real part is $n(\omega)$. In order to avoid questions regarding the destructive effect of the hole on the dielectric, it is convenient to imagine that the dielectric is solid, and that the hole travels down a narrow straight channel drilled through the dielectric. Thus the hole does not accrete material, but its gravitational field certainly influences the dielectric.

Let a spectrum of electromagnetic waves pervade the dielectric. Those wave modes for which $\omega-\mathbf{v}\cdot\mathbf{k}=\omega[1-\mathbf{v}\cdot\mathbf{n}n(\omega)]<0$ can undergo superradiant amplification from the black hole. In the argument of Sec. II B the entropy of the object is now replaced by black entropy together with entropy of the surrounding dielectric. Now black hole entropy is proportional to the horizon area, and Hawking's area theorem [8] tells us that black hole area will increase in any classical process, such as absorption of electromagnetic waves by the hole. If the dielectric can dissipate, it will also contribute to the increase in entropy through changes it undergoes in the vicinity of the passing hole. Thus the argument of Sec. II B tells us that the black hole plus surrounding dielectric will amplify the radiation in the mentioned modes at the expense of the hole's kinetic energy. Likewise, even if there are no waves to start with, the argument of Sec. II A tells us that the black hole plus dielectric will spontaneously emit photons into modes that obey the Ginzburg-Frank superradiance condition (13).

In the conversion of kinetic energy to waves, gravitation must obviously play a role. For the black hole is assumed uncharged, so that the process is distinct from the Vavilov-Cherenkov effect. Since the waves cannot classically emerge from within the hole, we must look for their source in the polarization cloud accompanying the hole. This cloud forms because gravity pulls on the positively charged nuclei in the dielectric stronger than on the enveloping electrons. As a result the array of nuclei sags with respect to the electrons, and produces an electrical polarization of the dielectric accompanied by an electric field which ultimately balances the tendency of gravity to rip out nuclei from electrons. It is this electric structure which is to be viewed as the true source of the photons, some of which are absorbed by the black hole and cause the area increase, while the rest constitute emission.

In special circumstances the present problem may be mapped onto that of the Vavilov-Cherenkov effect by noting that the induced electric field $\mathbf{E}$ is related to the gravitational one, $\mathbf{g}$, by $e\mathbf{E}=-\delta\mu\mathbf{g}$ where $\delta\mu\approx Am_p$ is the nucleus-electron mass difference (A is the mass number of the atoms, m_p the proton's mass), and $e>0$ the unit of charge. From the gravitational Poisson equation it follows that $\nabla\cdot\mathbf{E}=4\pi GM(\delta\mu/e)\delta(\mathbf{r}-\mathbf{r}_0)$ where $\mathbf{r}_0$ denotes the momentary black hole position. The electric field accompanying the black hole is thus that of a pointlike charge $Q\equiv GAMm_p/e$. There is a big assumption here that the dielectric has time to relax to form the above compensating

field. Such relaxation does occur for sufficiently small $|\mathbf{v}|$, but since we need $|\mathbf{v}|$ to be sufficiently large for the Ginzburg-Frank condition to hold, stringent conditions are required of the dielectric (high n and fast relaxation). When these are satisfied the electromagnetic radiation will be of the Vavilov-Cherenkov form for the equivalent charge Q moving with velocity $\mathbf{v}$. Q/e is about 10^3A times the gravitational radius of the hole measured in units of the classical radius of the *electron*. Hence a fast 10^{15} g primordial black hole moving in a suitable dielectric would radiate just like an equally fast particle bearing $\sim 10^3A$ elementary charges. This is relevant for the experimental search for primordial black holes.

When things are looked at this way, the black hole character of the object is not critical. What matters is that it is endowed with a gravitational field. This tells us that an ordinary object with the same mass would have similar effect as a black hole, so long as both are smaller than the channel's width. It is also worthwhile noting that the effects here discussed will be significant only when the wavelengths involved are large compared to the width of the channel. Otherwise, the object acts as if in vacuum, and we expect no superradiance.

C. Critical speed for superfluidity

A superfluid can flow through thin channels with no friction. However, when the speed of flow is too large, the superfluidity is destroyed. Landau gave a criterion [14] for the critical speed v_c for removal of superfluidity. Although in practice superfluidity disappears already at much lower speeds as the superfluid develops turbulence through the formation of vortices, the Landau critical speed is the top speed at which superfluidity can survive no matter how carefully tailored the channel is to the flow. The Landau critical speed is

$$v_c \equiv \min\, \varepsilon(\mathbf{p})/|\mathbf{p}| \tag{16}$$

where $\varepsilon(\mathbf{p})$ is the dispersion relation of the quasiparticles (phonons and rotons) that can occur as excitations above the condensate constituting the superfluid. In superfluid He^4 $v_c \approx 6\times 10^3$ cm s^{-1}. Landau's argument is that at speeds of flow above v_c it becomes energetically permissible for bulk kinetic energy of the superfluid to transform into energy of one internal excitation—a quasiparticle. Once an abundance of quasiparticles has appeared, there is a normal component to the fluid, which, of course, is not a superfluid.

The Landau argument is usually framed in the rest frame of the fluid with respect to which the walls of the channel are in motion [14]. In the following argument we also employ that frame. Now the walls play the role of the object in our superradiance argument, and the waves of frequency $\omega = \varepsilon/\hbar$ and wavenumber $\mathbf{k}=\mathbf{p}/\hbar$ associated with the quasiparticles, are surrogates of the electromagnetic waves in the arguments of Sec. II. When the walls move with speed $> v_c \equiv \min\, \varepsilon(\mathbf{p})/|\mathbf{p}|$, the quantity $\omega - \mathbf{v}\cdot\mathbf{k} = (\varepsilon - \mathbf{v}\cdot\mathbf{p})/\hbar$ becomes negative for at least one quasiparticle mode. It then becomes entropically preferable for the wall material to become excited *and* simultaneously create a quasiparticle in that mode, as discussed in Sec. II A. Furthermore, the quasiparticles thus created can undergo superradiant amplification upon impinging on other parts of the walls (Sec. II B). As a consequence an avalanche of quasiparticle formation ensues, which acts to convert the superfluid into a normal fluid. It is clear that the transition away from superfluidity is a literal example of the superradiance phenomenon. In this phenomenon the sound speed, of order v_c, plays the role of the speed of light in our original arguments.

D. Superradiance in Mach shocks

It is well known that when a solid object travels through an originally quiescent fluid with a speed $v=|\mathbf{v}|$ exceeding that of sound c_s in the fluid, a shock (density discontinuity) in the form of a circular cone is formed in its wake [16]. The interior of this Mach cone is filled by perturbations originating in the object, while the fluid exterior to the cone is still unperturbed. The opening angle of the cone, $2\Theta_M$, is easily determined by considering the locus of sound signals emitted by the object and traveling in all directions at speed c_s with respect to the fluid which convects them downstream [16]:

$$\sin\,\Theta_M = c_s/v; \quad 0<\Theta_M<\pi/2. \tag{17}$$

The cone's opening angle is the same in both the object's and the fluid's rest frames.

Let us look at Mach shocks from the vantage point of superradiance. In the rest frame of the fluid, the object—considered structureless—can emit *phonons* spontaneously when these satisfy the Ginzburg-Frank condition in the form

$$\omega - \mathbf{v}\cdot\mathbf{k} = \omega - vk\,\cos\,\vartheta = 0. \tag{18}$$

Now for phonons $\omega = c_s k$; hence they are spontaneously emitted at an angle ϑ to the object's velocity $\mathbf{v}$ such that $\cos\,\vartheta = c_s/v$. These phonons thus have components of velocity $c_s\sqrt{1-c_s^2/v^2}$ and c_s^2/v normal and parallel to $\mathbf{v}$, respectively. A Galilean transformation (velocity $\mathbf{v}$) to the rest frame of the object gives for the angle ϑ' of superradiance emission in the new frame

$$\sin\,\vartheta' = \frac{c_s\sqrt{1-c_s^2/v^2}}{[(c_s\sqrt{1-c_s^2/v^2})^2+(c_s^2/v-v)^2]^{1/2}} = \frac{c_s}{v};$$

$$\pi/2<\vartheta'<\pi. \tag{19}$$

The range of ϑ' is so chosen because in the new frame the component of phonon velocity collinear with the object's velocity, $c_s^2/v - v$, is negative indicating that the emission occurs into the back hemisphere, that containing the fluid's velocity. Because $\sin\,\vartheta' = \sin\,\Theta_M$ we conclude that the superradiant phonons are emitted from the object *along* the shock discontinuity.

Now as the shock follows the object with velocity $\mathbf{v}$, it advances normal to itself with speed $v\cdot\sin\,\Theta_M = c_s$. According to shock theory [16], a shock with speed c_s is a weak discontinuity, i.e., the fluid's density is nearly the same on both its sides. It thus seems possible that the shock itself is entirely made up of superradiant phonons.

Further, consider any sound waves, e.g., thermal phonons, present in the fluid before the arrival of the object. The object is—by assumption—structureless; however, it is accompanied in its motion by a boundary layer of fluid that partially ''sticks'' to it [16]. Because the layer is constantly being renewed as the ''old'' fluid in it is swept downstream, it is dissipative. Therefore, those waves which satisfy the Ginzburg–Frank condition (13) will be amplified as they are overtaken by the object. These waves propagate at angles ϑ to the object's direction which obey

$$\cos\ \vartheta > \omega(|\mathbf{k}||\mathbf{v}|)^{-1} = c_s/v \tag{20}$$

i.e., they are emitted inside the Mach cone. In addition, if we regard the object with its boundary layer as one with many possible energy states, then phonons can be emitted also by Ginzburg and Frank's anomalous Doppler emission (see Sec. II B). These also travel inside the Mach cone. Thus the entire acoustic ''noise'' originating from supersonic motion in a fluid has a superradiance interpretation.

IV. ROTATIONAL SUPERRADIANCE: PRINCIPLES

We focus on an axisymmetric *macroscopic* body rotating rigidly with constant angular velocity Ω about its symmetry axis which is supposed fixed. The assumption of axisymmetry is critical; otherwise precession of the axis would arise. We further assume the body contains many internal degrees of freedom, so that it can internally dissipate absorbed energy. We assume it has reached internal equilibrium and has well defined entropy S, rest mass M and temperature T.

The body is exposed to external radiation *in vacuum*. We classify the radiation modes by frequency ω and azimuthal number m. This last refers to the axis of rotation. Suppose that in the modes with azimuthal number m and frequencies in the range in $\{\omega, \omega+\Delta\omega\}$, power $I_m(\omega)\Delta\omega$ is incident on the body. Then, as is easy to verify from the energy-momentum tensor, or from the quantum picture of radiation, the radiative angular momentum is incident at rate $(m/\omega)I_m(\omega)\Delta\omega$. If $I_m(\omega)$ is large enough, we can think of the radiation as classical. Experience tells us that the body will absorb a fraction $a_m(\omega)$ of the incident power and angular momentum flow in the modes in question, where $a_m(\omega)<1$ is a characteristic coefficient of the body. A fraction $[1-a_m(\omega)]$ will be scattered into modes with the same ω and m. We may thus replace Eqs. (6), (7) by

$$\frac{dE}{dt} = a_m I_m \Delta\omega - W \tag{21}$$

and

$$\frac{dJ}{dt} = (m/\omega) a_m I_m \Delta\omega - U_J \tag{22}$$

where J is the body's angular momentum and U_J is the overall rate of spontaneous angular momentum emission in waves.

Now the energy ΔE_0 of a small system measured in a frame rotating with angular frequency $\mathbf{\Omega}$ is related to its energy ΔE and angular momentum $\Delta\mathbf{J}$ in the inertial frame by [15]

$$\Delta E_0 = \Delta E - \mathbf{\Omega}\cdot\Delta\mathbf{J}. \tag{23}$$

Thus, when as a result of interaction with the radiation, the energy of our rotating body changes by $dE/dt\times\Delta t$ and its angular momentum in the direction of the rotation axis by $dJ/dt\times\Delta t$, its rest mass-energy changes by $(dE/dt - \Omega dJ/dt)\times\Delta t$. From this we infer, in parallel to the derivation of Eq. (9), that the body's entropy changes at a rate

$$\frac{dS}{dt} = \frac{1}{T}\left[\frac{\omega - m\Omega}{\omega} a_m I_m \Delta\omega - W + \Omega U_J\right]. \tag{24}$$

As in the discussion involving Eqs. (10), (11) we would now argue that when $I_m(\omega)$ is large, the term proportional to $(\omega - m\Omega)a_m(\omega)$ in Eq. (24) dominates the overall entropy balance. The second law thus demands that

$$(\omega - m\Omega)a_m(\omega) > 0. \tag{25}$$

It follows that whenever the Zel'dovich-Misner condition (1) is met, $a_m(\omega)<0$ necessarily. As in Sec. II A, we can argue that the sign of $a_m(\omega)$ should not depend on the strength of the incident radiation if nonlinear radiative effects do not intervene. Hence, independent of the strength of $I_m(\omega)$, condition (1) is the generic condition for rotational superradiance.

Evidently $a_m(\omega)$ switches sign at $\omega=\Omega m$. This switch cannot take place by $a_m(\omega)$ having a pole there since $a_m(\omega)<1$. *If* $a_m(\omega)$ is analytic in $\omega-\Omega m$, it must thus have the expansion

$$a_m(\omega) = \alpha_m(\Omega)(\omega - \Omega m) + \cdots \tag{26}$$

in the vicinity of $\omega=\Omega m$. However, we must again stress that thermodynamics does not demand continuity of $a_m(\omega)$ at $\omega-\Omega m=0$. Specific examples like that of the rotating cylinder [Eq. (56) below] do show continuity.

V. SUPERRADIANCE OF A ROTATING CYLINDER

Devices for making rotational superradiance observable (see Sec. V E below) are modeled on Zel'dovich's rotating cylinder [11]. In this section we compute the classical superradiance of a Zel'dovich cylinder. Admittedly, by detailed balance there must also be (quantum) spontaneous emission by the cylinder, but we do not enter into this issue here.

We idealize the cylinder as infinitely long. Let its radius be R and let it be rotating rigidly in vacuum with constant angular frequency Ω. We suppose it to be made of material with *spatially uniform* permittivity $\epsilon(\omega)$ and permeability $\mu(\omega)$; these are not necessarily real because of the possibility of dissipative processes in the material. Alternatively, the material may be electrically conducting in which case we denote its conductivity by σ. Although it is possible to represent conductivity as an imaginary part of $\epsilon(\omega)$, we shall not

do so here. If σ is small, e.g. a semiconductor, one can allow nontrivial $\epsilon(\omega)$ and $\mu(\omega)$ alongside σ.

A. Constitutive relations and Maxwell equations

In the relativistic treatment we have in mind the electromagnetic field is described by the antisymmetric tensor $F^{\alpha\beta}$ composed in the usual way of the electric field **E** and magnetic induction **B**. The electric displacement **D** and magnetic field **H** form an analogous tensor $H^{\alpha\beta}$. The usual constitutive relations $\mathbf{D}=\epsilon\mathbf{E}$, $\mathbf{B}=\mu\mathbf{H}$ and $\mathbf{j}=\sigma\mathbf{E}$ can be expressed in covariant form as

$$H^{\alpha\beta}u_\beta=\epsilon F^{\alpha\beta}u_\beta \tag{27a}$$

$$*F^{\alpha\beta}u_\beta=\mu *H^{\alpha\beta}u_\beta \tag{27b}$$

$$j^\alpha=\sigma F^{\alpha\beta}u_\beta+\varrho u^\alpha. \tag{27c}$$

We have written the electric current as a sum of a conductive part (recall that electric and magnetic fields are observer dependent concepts and are here computed in the frame of the material whose 4-velocity is u^α) and a convective part with ϱ being the *proper* charge density. This last is included to give us the flexibility to treat, say, a dielectric bearing a net charge density (in which case we would set $\sigma=0$). We use the notation $*F^{\alpha\beta}\equiv\frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}$ with $\varepsilon^{\alpha\beta\gamma\delta}$ the Levi-Civita tensor. It should be observed that ϵ and μ are frequency dependent in general, so that equations involving them refer to time Fourier components of fields. And the arguments of ϵ or μ should be frequencies *in the frame of the rotating cylinder*.

In cylindrical coordinates $\{x^0,x^1,x^2,x^3\}=\{t,r,\phi,z\}$ with flat metric

$$ds^2=-dt^2+dr^2+r^2d\phi^2+dz^2 \tag{28}$$

we obviously have inside the cylinder

$$u_\beta=(-1,0,\Omega r^2,0)\gamma;\quad \gamma\equiv(1-\Omega^2r^2)^{-1/2}. \tag{29}$$

It is easy to generalize this to curved spacetime, but we shall not do so here.

By successively taking $\alpha=0,1,3$ in Eqs. (27a,b) and converting components of duals to components of the original fields we get

$$\epsilon^{-1}H^{02}=F^{02}\equiv r^{-1}E_\phi \tag{30a}$$

$$\mu H^{31}=F^{31}\equiv B_\phi \tag{30b}$$

$$\epsilon^{-1}(H^{01}+\Omega r^2H^{12})=F^{01}+\Omega r^2F^{12}\equiv\gamma^{-1}E_r \tag{30c}$$

$$\mu(H^{23}-\Omega H^{03})=F^{23}-\Omega F^{03}\equiv(r\gamma)^{-1}B_r \tag{30d}$$

$$\epsilon^{-1}(H^{03}-\Omega r^2H^{23})=F^{03}-\Omega r^2F^{23}\equiv\gamma^{-1}E_z \tag{30e}$$

$$\mu(H^{12}+\Omega H^{01})=F^{12}+\Omega F^{01}\equiv(r\gamma)^{-1}B_z. \tag{30f}$$

Here E_r, E_ϕ, E_z, B_r, B_ϕ, and B_z denote the physical components in the indicated directions of the electric field and magnetic induction *in the rotating frame*. Outside the cylinder one should set $\Omega=0$ and $\epsilon=\mu=1$ in these equations.

Let us now pass to the Maxwell equations:

$$F_{[\alpha\beta,\gamma]}=0 \tag{31a}$$

$$H^{\alpha\beta}{}_{,\beta}=4\pi j^\alpha. \tag{31b}$$

In view of the symmetries of the problem we shall look for solutions where the fields vary as $f(r)e^{\imath(m\phi+kz-\omega t)}$ with m an integer, and ω and k real constants. Here ω is the frequency in the laboratory frame; in the cylinder's (rotating) frame, the azimuthal coordinate is $\phi'=\phi-\Omega t$, and hence the frequency is $\omega'=\omega-m\Omega$. Our choice of modes means that in writing the equations one can simply replace $\partial/\partial\phi\to\imath m$, etc. From Eq. (31a) we get, after raising indices,

$$\partial(F^{02}r^2)/\partial r-\imath\omega r^2F^{12}-\imath mF^{01}=0 \tag{32a}$$

$$\partial(F^{23}r^2)/\partial r+\imath kr^2F^{12}+\imath mF^{31}=0 \tag{32b}$$

$$\partial F^{03}/\partial r+\imath\omega F^{31}-\imath kF^{01}=0 \tag{32c}$$

$$\imath kF^{02}+\imath\omega F^{23}-\imath mr^{-2}F^{03}=0. \tag{32d}$$

Finally we take in Eq. (31b) successively $\alpha=0$, 1, 2 and 3:

$$\partial(H^{01}r)/\partial r+\imath mrH^{02}+\imath krH^{03}=4\pi\sigma\gamma\Omega r^2E_\phi+r\gamma\varrho \tag{33a}$$

$$\imath\omega H^{01}+\imath mH^{12}-\imath kH^{13}=4\pi\sigma E_r \tag{33b}$$

$$\partial(H^{12}r)/\partial r-\imath\omega rH^{02}-\imath krH^{23}=-4\pi\sigma\gamma E_\phi-r\gamma\varrho\Omega \tag{33c}$$

$$\partial(H^{31}r)/\partial r-\imath mrH^{23}+\imath\omega rH^{03}=4\pi\sigma E_z. \tag{33d}$$

Outside the cylinder one should put $\sigma=0$ and $\varrho=0$ in Eqs. (33a–d).

B. Axial electric and magnetic modes ($k=0$)

As in any electromagnetic problem of this type, there are here two distinct modes for each set $\{\omega,m,k\}$. Here we characterize them for the case $k=0$.

First assume, in harmony with Eq. (30a) that everywhere inside and outside the cylinder $F^{02}=H^{02}=E_\phi=0$. It will transpire that this is a consistent choice, and therefore characterizes the first mode. Equation (32a) then gives $\omega r^2F^{12}+mF^{01}=0$ everywhere, while outside the cylinder ($H^{\alpha\beta}=F^{\alpha\beta}$; $\sigma=\varrho=0$) Eq. (33b) gives $\omega F^{01}+mF^{12}=0$. These simultaneous equations require $F^{12}=F^{01}=H^{12}=H^{01}=0$ outside the cylinder. To connect these with the interior fields we go to Eqs. (33a,c). There may be a charge layer at $r=R$ of surface density $\Delta q=\int_{R^-}^{R^+}\varrho\,dr$. Integrating the two equations across the layer gives for the jumps in the fields $\Delta H^{01}=q\gamma(R)$ and $\Delta H^{12}=-q\Omega\gamma(R)$ so that $\Omega H^{01}+H^{12}$ must be continuous across the surface. If we now add Ω times Eq. (33a) to Eq. (33c) we find that $r(\Omega H^{01}+H^{12})$ is independent of r everywhere, including at $r=R$. Since it vanishes for $r>0$, it must vanish everywhere. Then by

Eq. (30f) $\Omega F^{01}+F^{12}=0$ everywhere. But as we mentioned, $\omega r^2 F^{12}+mF^{01}=0$ everywhere; these two simultaneous equations force F^{01} and F^{12} to vanish everywhere. It is now evident by solving Eqs. (30c,f) simultaneously that H^{01} and H^{12} must also vanish everywhere.

As we shall show in Sec. V C, one can construct F^{31} and F^{23} out of F^{03} which obeys an autonomous equation. Thus the ansatz $F^{01}=F^{02}=F^{12}=H^{01}=H^{02}=H^{12}=0$ defines a mode of the system. We call it the axial electric (AE) mode because its electric field (only component F^{03}) points along the cylinder's axis. It corresponds to Zel'dovich's [11] first mode.

Now we look for a mode which has [see Eq. (30b)] $F^{31}=H^{31}=B_\phi=0$. Again, it will transpire that this is a consistent choice. From Eqs. (32b,c) it follows that $r^2F^{23}=C_1$ and $F^{03}=C_2$ with C_1 and C_2 independent of r. Equation (33d) implies that outside the cylinder $mF^{23}-\omega F^{03}=0$. This last is inconsistent with the previous expressions unless we put $C_1=C_2=F^{03}=F^{23}=H^{03}=H^{23}=0$ in the exterior. Now since F^{23} is the magnetic field component *normal* to the cylinder's surface, it must be continuous there. Thus C_1 along with F^{23} must also vanish inside the cylinder. The *tangential* electric field F^{03} must likewise be continuous at the surface; thus C_2 and F^{03} have to vanish inside as well. By solving Eqs. (30d,e) simultaneously we find that also $H^{03}=H^{23}=0$ inside.

As we show in the Appendix, one can construct F^{02}, F^{12} and F^{01} out of a single function obeying an autonomous equation. Thus the ansatz $F^{03}=F^{31}=F^{23}=H^{03}=H^{31}=H^{23}=0$ defines a second mode. We call it the axial magnetic (AM) mode because its magnetic field (only component H^{12}) points along the cylinder's axis. It corresponds to Zel'dovich's [11] second mode.

C. Electrodynamic proof of superradiance for AE modes ($k=0$)

Here we give a new basically electrodynamic proof that for $\omega-m\Omega<0$ the cylinder superradiates. We shall first obtain the radial equation governing the shape of the AE mode with $k=0$. First we note that according to Eqs. (32c,d),

$$F^{31}=\imath\omega^{-1}\partial F^{03}/\partial r \tag{34a}$$

$$F^{23}=m\omega^{-1}r^{-2}F^{03}. \tag{34b}$$

Next we solve for H^{03}, H^{23} and H^{31} from Eqs. (30b,d,e) and substitute these and Eq. (34a) in Eq. (33d) to get

$$r^{-1}\partial(r\partial F^{03}/\partial r)/\partial r+m\omega\gamma(B_r+\epsilon\mu\Omega E_z) -\omega^2\gamma(\epsilon\mu E_z+\Omega r^2 B_r)=4\pi\imath\sigma\mu E_z. \tag{35}$$

But by combining the definitions of E_z and B_r in Eqs. (30a–f) with Eq. (34b) we have that

$$E_z=\gamma\omega^{-1}(\omega-m\Omega)F^{03} \tag{36a}$$

$$B_r=\gamma(\omega r^2)^{-1}(m-\omega\Omega r^2)F^{03}. \tag{36b}$$

If we now substitute these in Eq. (35) and cancel out the common phase $e^{\imath(m\phi-\omega t)}$ we get

$$r^2f''+rf'-\gamma^2[(m-\omega\Omega r^2)^2-\epsilon\mu(\omega-m\Omega)^2r^2 -4\pi\imath\gamma^{-1}\mu\sigma(\omega-m\Omega)r^2]f=0 \tag{37}$$

where $f(r)\equiv F^{03}e^{-\imath(m\phi-\omega t)}$ and $'$ denotes an ordinary radial derivative. All this is for $r<R$. In the cylinder's exterior we just set $\epsilon\mu\to1$ and $\sigma\to0$. This is the promised exact radial equation for the AE mode; the fields F^{31} and F^{23} can be recovered from Eqs. (34a,b).

Now we are ready to discuss the energy flux. Both inside and outside the cylinder the *radial* energy flux is [12]

$$S_r=(\mathbf{E}\times\mathbf{H})_r/4\pi=(F^{02}H^{12}-F^{03}H^{31})/4\pi. \tag{38}$$

But F^{02} and H^{12} both vanish, so this reduces to $-F^{03}H^{31}/4\pi$. This is the instantaneous flux; of more interest is the time-averaged flux which can be obtained by substituting [12]

$$F^{03}\to\frac{1}{2}[fe^{\imath(m\phi-\omega t)}+f^*e^{-\imath(m\phi-\omega t)}] \tag{39a}$$

$$H^{31}=F^{31}/\mu\to\frac{1}{2\omega}[\imath(f'/\mu)e^{\imath(m\phi-\omega t)} -\imath(f^{*\prime}/\mu^*)e^{-\imath(m\phi-\omega t)}] \tag{39b}$$

and then averaging. Here we have used Eq. (34a) to simplify. Note that the complex conjugate of the primary field f contributes with weight $1/\mu^*$. We thus have for the time-averaged radial flux

$$\overline{S_r}=\frac{1}{16\pi\imath\omega}(f^*f'/\mu-ff^{*\prime}/\mu^*). \tag{40}$$

In the process two terms involving exponents $e^{\pm2\imath(m\phi-\omega t)}$ have averaged out.

We can get a useful equation for the Wronskian-like expression in the last equation by first dividing Eq. (37) by $r\mu$, multiplying it by f^*, and then subtracting from the result its complex conjugate:

$$\frac{d}{dr}[r(f^*f'/\mu-ff^{*\prime}/\mu^*)]=-2\imath r[\mathcal{A}|f'|^2+(\mathcal{B}+\mathcal{C})|f|^2] \tag{41}$$

with ($\Im$ means imaginary part)

$$\mathcal{A}\equiv\Im\mu/|\mu|^2 \tag{42a}$$

$$\mathcal{B}\equiv[\Im\epsilon(\omega-m\Omega)^2+\mathcal{A}(m-\omega\Omega r^2)^2r^{-2}]\gamma^2 \tag{42b}$$

$$\mathcal{C}\equiv4\pi\sigma(\omega-m\Omega)\gamma. \tag{42c}$$

In the vacuum outside the cylinder $\Im\epsilon=\Im\mu=\sigma=0$ so that according to Eqs. (40), (41) $S_r\propto1/r$. This just means that energy is conserved outside the cylinder, the overall outflow (inflow) at large distances equaling that at $r=R$. Thus to find

out which way energy flows at large distances, it is sufficient to determine the sign of $\overline{S_r}$ at $r=R$.

Now because f represents a physical electric field, it must be bounded at $r=0$. And then f' cannot diverge as fast as $1/r$. It follows that, barring the exceptional circumstance that $\mu=0$, $r(f^*f'/\mu - ff^{*\prime}/\mu^*)\to 0$ as $r\to 0$. Hence by integrating Eq. (41) from $r=0$ to $r=R$ we find

$$\overline{S_r}(r=R)=\frac{-1}{32\pi\omega R}\int_0^R r[\mathcal{A}|f'|^2+(\mathcal{B}+\mathcal{C})|f|^2]dr. \tag{43}$$

To determine the sign of this expression we note that it follows from the second law of thermodynamics [12] that $\sigma \geqslant 0$, and that $\Im\epsilon$ and $\Im\mu$ are both odd in the frequency and both positive for positive frequency. Of course, frequency here means frequency in the frame of the material, namely $\omega - m\Omega$. Hence $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ all bear the same sign as $\omega - m\Omega$. Thus regardless of the source of dissipation, there is an energy outflow to infinity (superradiance) if only if the Misner-Zel'dovich condition $\omega - m\Omega < 0$ is satisfied, as we might have guessed from the method of Sec. IV.

D. Gain in superradiance for nonrelativistic rotation: AE modes

In his pioneering study of superradiance of a rotating cylinder, Zel'dovich [11] concluded that for AE modes with $\omega - m\Omega < 0$, $m>0$ and $k=0$, the gain coefficient [defined precisely after Eq. (49)] is very small for nonrelativistic rotation. The gist of his argument is as follows. Outside the cylinder the radial equation (37) reduces exactly to

$$r^2f''+rf'-[m^2-\omega^2r^2]f=0; \quad r>R \tag{44}$$

whose solutions are the Hankel functions $H_m^{(1)}(\omega r)$ and $H_m^{(2)}(\omega r)$, the first (second) representing outgoing (ingoing) waves at infinity. Inside the cylinder Zel'dovich takes $\epsilon=\mu=1$, and neglects the effect of σ to argue that one may, to sufficient accuracy, approximate f by $J_m(\omega r)$ which is that combination of $H_m^{(1)}(\omega r)$ and $H_m^{(2)}(\omega r)$ regular at $r=0$. We may justify this form by realizing that

$$[(m-\omega\Omega r^2)^2-(\omega-m\Omega)^2r^2]\gamma^2=(m^2-\omega^2r^2) \tag{45}$$

so that in the stated limit Eq. (37) reduces to Eq. (44) also inside the medium. This is true even for relativistic rotation, a point not remarked on by Zel'dovich.

Working nonrelativistically Zel'dovich then calculates via Ohm's law the current **j** induced in the cylinder by the electric and magnetic fields **E** and **B** deriving from this f. Because the medium rotates, he finds that $j_z\propto(\omega-m\Omega)$. Thus the Joule work j_zE_z is negative: the cylinder does work on the field and superradiance ensues. Zel'dovich obtains a gain coefficient $\propto\sigma\cdot(m\Omega-m)(\omega R)^{2m}$. The factors ωR come from the small argument approximation $J_m(x)\sim x^m$ for $x \ll m$; recall that because of the Zel'dovich-Misner condition and the assumed nonrelativistic rotation, $\omega r < m\Omega R \ll m$. As Zel'dovich remarks, the physical reason for the smallness is that R lies deep within the near zone, which circumstance suppresses the matter-wave coupling.

Is Zel'dovich's pessimistic conclusion valid also when $\epsilon,\mu\neq 1$? One may be skeptic because when $\epsilon\mu$ differs significantly from unity, Eq. (37) does not reduce to Eq. (44), but rather to the Bessel equation (46a) below whose solution regular at $r=0$ is different from $J_m(\omega r)$. One also wonders what happens when the conductivity is large, so that the backreaction of the cylinder on the wave cannot be neglected, and when γ is significantly greater than unity? To answer these question we shall work with the full Eq. (37), and match its interior and exterior solutions. We can then be more specific about the prefactor in Zel'dovich's expression and the corrections it is subject to for large σ.

Let us assume that the ingoing wave generated by some external agency, $H_m^{(2)}(\omega r)$, has unit coefficient. Then the total radial wave amplitude outside the cylinder will be $f_{\rm out}=H_m^{(2)}(\omega r)+\rho H_m^{(1)}(\omega r)$ where ρ is the (possibly complex) amplitude for reflection off the cylinder. For superradiance we expect $|\rho|^2>1$.

Inside the cylinder the exact $f(r)$ is determined by Eq. (37) which in light of Eq. (45) can be rewritten in the more convenient form

$$r^2f''+rf'-[m^2-\kappa^2r^2]f=0; \quad r<R \tag{46a}$$

$$\kappa^2\equiv\omega^2+(1-\epsilon\mu)(\omega-m\Omega)^2\gamma^2+\imath 4\pi\gamma\mu\sigma(\omega-m\Omega). \tag{46b}$$

This is again a Bessel equation whose solution regular at $r=0$ is $J_m(\kappa r)$. The radial wave amplitude inside will thus be $f_{\rm in}=\tau J_m(\kappa r)$ where τ is the (possibly complex) amplitude for transmission into the cylinder.

Now we match interior with exterior solutions by the usual continuity conditions on electric and magnetic fields. By integrating Eq. (34a) from $r=R-\varepsilon$ to $r=R+\varepsilon$ and relying on the boundedness of F^{31} we conclude that $F^{03}|_{R_+}=F^{03}|_{R_-}$. But since $F^{03}=f(r)e^{\imath(m\phi-\omega t)}$, it is obvious that f must be continuous at $r=R$. By similarly integrating Eq. (33d) and invoking the boundedness of H^{23}, H^{03} and E_z we find $H^{31}|_{R_+}=H^{31}|_{R_-}$. Then from Eqs. (30b) and (34a) it follows that $f'|_{R_+}=(f'/\mu)|_{R_-}$. One checks that with these matching conditions $\overline{S_r}$ in Eq. (40) is continuous at $r=R$.

With the expressions for $f_{\rm in}$ and $f_{\rm in}$ written out earlier, the matching conditions are

$$\tau J_m(\kappa R)=\rho H_m^{(1)}(\omega R)+H_m^{(2)}(\omega R) \tag{47a}$$

$$(\tau\kappa/\mu)J_m{}'(\kappa R)=\rho\omega H_m^{(1)\prime}(\omega R)+\omega H_m^{(2)\prime}(\omega R) \tag{47b}$$

where $'$ here means derivative with respective to the argument. Solving these simultaneously for ρ and rearranging the result with help of the identity $H^{(1)}{}_m=H_m^{(2)*}$ gives

$$\rho=-\frac{1-\mu\chi_m(x)\,\eta_m(y)}{1-\mu\chi_m(x)^*\,\eta_m(y)}\cdot\frac{H_m^{(2)}(x)}{H_m^{(2)}(x)^*} \tag{48a}$$

$$\chi_m(x)\equiv xH^{(2)\prime}_m(x)/H^{(2)}_m(x);\quad \eta_m(y)\equiv J_m(y)/[yJ'_m(y)] \tag{48b}$$

with $x\equiv\omega R$ and $y\equiv\kappa R$.

When there is no dissipation, ϵ and μ are real while $\sigma=0$, and so y is real. It follows that numerator and denominator of Eq. (48a) are complex conjugates so that $|\rho|=1$. This is in harmony with the arguments of Sec. IV that superradiance goes hand in hand with dissipation.

Let us now define the dimensionless parameters $v\equiv\Omega R$ (peripheral velocity of the cylinder in units of c) and $\xi\equiv 4\pi\mu\sigma R$ in terms of which

$$y^2=x^2+(1-\epsilon\mu)(x-mv)^2\gamma^2+\imath(x-mv)\xi\gamma. \tag{49}$$

The y shall be the square root which is positive in the limit $\sigma\to 0$. A useful approximation for the gain coefficient $-a_m\equiv|\rho|^2-1$ [this is the same as the coefficient $a_m(\omega)$ appearing in Sec. IV] can be obtained from Eqs. (48a,b) by passing to the nonrelativistic limit $v\ll 1$, $\gamma\approx 1$ which, for m not too large, implies $x\ll 1$.

First the recursion relation [17] $xH^{(2)\prime}_m=xH^{(2)}_{m-1}-mH_m{}^{(2)}$ allows us to write

$$\chi_m(x)=-m+\frac{xH^{(2)}_{m-1}(x)}{H^{(2)}_m(x)}. \tag{50}$$

For $x\ll 1$ the leading terms of the real and imaginary parts of the Hankel function are [17]

$$H^{(2)}_0\approx 1-\frac{2\imath}{\pi}\left(\ln\frac{x}{2}+\gamma_E\right) \tag{51a}$$

$$H^{(2)}_m(x)\approx\frac{x^m}{m!2^m}+\imath\frac{2^m}{\pi x^m};\quad m\geq 1 \tag{51b}$$

where $\gamma_E\approx 0.577216$ is the Euler-Mascheroni constant. Substituting in Eq. (50) we have to leading real and imaginary orders in x

$$\chi_m(x)\approx -m-\delta^1_m\left(\frac{1}{2}+\ln\frac{x}{2}+\gamma_E\right)x^2+\frac{x^2}{2}-\frac{\imath\pi x^{2m}}{(m-1)!2^{2m-1}}+\cdots. \tag{52}$$

We now substitute from Eq. (52) into Eq. (48a) and recall that the ratio of $H^{(2)}_m$ to its complex conjugate has unit modulus. Factoring out $1+\mu m\eta_m(y)$ from numerator and denominator, we find in each the function $h_m(y)\equiv\mu\eta_m(y)[1+\mu m\eta_m(y)]^{-1}$ multiplied in one by a small complex expression and in the other by the conjugate of this expression. As a result to leading $[\mathcal{O}(x^2)]$ order, h_m appears in ρ multiplied only by an imaginary factor, so that only the imaginary part of h_m remains in $|\rho|^2$. Retaining only dominant terms leads to

$$a_m\approx\frac{8\pi(x/2)^{2m}}{(m-1)!}\,\Im h_m(y)$$

$$=\frac{8\pi(x/2)^{2m}}{(m-1)!}\,\Im\frac{\mu J_m(y)}{(\mu-1)mJ_m(y)+yJ_{m-1}(y)} \tag{53}$$

where the last form follows from the recursion relation [17] $yJ'_m=yJ_{m-1}-mJ_m$. Since $-a_m$ is proportional to the small factor x^{2m}, superradiance is mostly confined to the $m=1$ mode (unless the ingoing wave only has $m>1$).

We went through the derivation of Eq. (53) with possibly complex ϵ and μ as a matter of principle, and because it will be required for the discussion in the Appendix. But in practice little need can arise to consider complex ϵ or μ. For low frequencies both these quantities are real with ϵ becoming complex in real materials only at frequencies $>10^{11}$ Hz (in ferromagnets μ can become dispersive at somewhat lower frequencies) [12]. Recall that the appropriate argument of ϵ or μ in our discussion is $\omega-m\Omega$ which must be negative. But a macroscopic cylinder rotating nonrelativistically will do so below $\Omega=10^{10}$ Hz. And as mentioned, m cannot be large without superradiance being suppressed. Thus in the laboratory we cannot arrange for $\omega-m\Omega$ to be negative and sufficiently large in magnitude to access the complex range of ϵ or μ. Henceforth we consider only real ϵ and μ.

As mentioned, for nonrelativistic rotation $v\ll 1$ and $x\ll 1$ and thus $|x-mv|\ll 1$. The low conductivity regime may be defined by the additional condition

$$|x-mv|\xi\ll 1. \tag{54}$$

When all these are valid, the argument y of the Bessel functions is a small *complex* number, and we can expand

$$J_m(y)=\frac{y^m}{2^m m!}\left[1-\frac{y^2}{4(m+1)}+\cdots\right]. \tag{55}$$

Substituting this, Eq. (49) and the definitions of ξ, x and v in Eq. (53) and reinstating c gives to leading order

$$a_m\approx\frac{16\pi^2\mu^2(\omega R/2c)^{2m}(\omega-m\Omega)\sigma R^2/c^2}{m(m+1)!(\mu+1)^2} \tag{56}$$

which shows clearly that for $(\omega-m\Omega)<0$ there is superradiance ($a_m<0$). The formula supports Zel'dovich's assertion that for low conductivity the gain coefficient is proportional to $\sigma R^2(m\Omega-\omega)(\omega R)^{2m}$. Our result gives the proportionality constant and shows that $-a_m$ is independent of ϵ. Numerical work shows that Eq. (56) remains accurate to within 1% up to $|x-mv|\xi\approx 1$.

For $|x-mv|\xi>1$ we return to Eq. (53). Because the gain coefficient falls off with growing m, we discuss here only the results for $m=1$. Clearly the terms x^2 and $(x-mv)^2$ in y^2 are negligible because $x<v$ and $v\ll 1$. (We presume that $\epsilon\mu$ is not too large, which is reasonable because for a good conductor ϵ and μ are formally unity.) Hence the argument y in Eq. (53) reduces to $[(x-v)\xi]^{1/2}$. The imaginary part is best evaluated numerically. As a function of $(x-v)\xi$ it sports single maximum of height 0.1887 located at $(x-v)\xi\approx -6.325$. From these last numbers and Eq. (53) we infer the maximal gain coefficient for given ω:

$$-(a_1)_{\max}=1.185(\omega R/c)^2 \quad \text{at} \quad \Omega=\omega+0.503c^2/\sigma R^2. \tag{57}$$

For a copper cylinder with $R=10$ cm, the minimum Ω required for the peak to be present is $0.06\ \mathrm{s}^{-1}$; this is also the offset between ω and the Ω giving maximum superradiance.

E. Rotational superradiance devices

From Eqs. (56), (57) it is clear that for superradiance of a nonrelativistically rotating cylinder the gain coefficient $-a_m$ is extremely small (basically $\omega R/c$ is very small). This would seem to imply that superradiance cannot be observed in the laboratory. But in fact this is not the case for two reasons. First by surrounding the cylinder with a jacket made of material where the speed of light is rather small, one achieves a more favorable ratio of cylinder radius to wavelength with a consequent improvement in $-a_m$. Second, a suitable device can cycle the amplified radiation any number of times to compound the gain coefficient. This raises the possibility of practical devices for amplification of signals at the expense of mechanical energy.

To explain the reason for the first improvement in the simplest terms we consider the jacket material to have $\mu=1$ but very large and *real* permittivity ϵ_{j}. Equations (30a)–(33d) can obviously be used outside the cylinder if we put everywhere $\sigma=\varrho=\Omega=0$. The arguments of Sec. V B characterizing the AE and AM modes can be repeated with like conclusions. For AE modes we need to replace the radial equation (44) outside the cylinder by [cf. Eq. (37)]

$$r^2f''+rf'-[m^2-\epsilon_{\mathrm{j}}\omega^2r^2]f=0; \quad r>R. \tag{58}$$

Therefore, the argument of the Hankel functions in Sec. V D is now $\sqrt{\epsilon_{\mathrm{j}}}\omega r$ rather than ωr. And the Hankel and χ_m functions in Eqs. (47a,b), (48a,b) now take argument $\sqrt{\epsilon_{\mathrm{j}}}x$. There is no change in the matching conditions $f|_{\mathrm{R}_+}=f|_{\mathrm{R}_-}$ and $f'|_{\mathrm{R}_+}=(f'/\mu)|_{\mathrm{R}_-}$ since μ has not been changed. In Eq. (47b) a factor $\sqrt{\epsilon_{\mathrm{j}}}$ appears alongside μ; it comes from the arguments of the differentiated Hankel functions. No change occurs in y, the argument of the Bessel functions, which is composed exclusively of quantities describing the cylinder.

Let us assume that even though ϵ_{j} is large, $\sqrt{\epsilon_{\mathrm{j}}}x\ll 1$ (remember we are in the superradiant regime so $\Omega R\ll 1$). The assumption means that the rotational velocity is still well below the speed of light in the jacket. Then we can expand the Hankel functions for small argument as before and arrive back at formulas (56) and (57) with the replacements $\mu\to\mu\sqrt{\epsilon_{\mathrm{j}}}$ and $x\to\sqrt{\epsilon_{\mathrm{j}}}x$. Since $\sqrt{\epsilon_{\mathrm{j}}}$ is assumed large, the μ dependent factor in Eq. (56) is here replaced by unity so that

$$a_m\approx\frac{16\pi^2(\epsilon_{\mathrm{j}})^m(\omega R/2c)^{2m}(\omega-m\Omega)\sigma R^2/c^2}{m(m+1)!} \tag{59a}$$

$$-(a_1)_{\max}=1.185\epsilon_{\mathrm{j}}(\omega R/c)^2 \quad \text{at} \quad \Omega=\omega+0.503c^2/\sigma R^2. \tag{59b}$$

In Eq. (59a) the factor $\omega-m\Omega$ is unchanged because it stems from y. Thus a jacket of high ϵ_{j} material provides, for $m=1$, a gain larger by a factor ϵ_{j} over the vacuum value.

The second ingredient of the superradiant device is cycling through reflection. Suppose the rotating cylinder and its high-ϵ jacket are placed inside a concentric cylindrical reflecting cavity of radius $R_c>R$ (this is similar to Press and Teukolsky's idea for the ''black hole bomb'' [18]). Introduce in the intervening material an electromagnetic wave with low m components. One simple way to do this is to apply across the ends of the cylinder along one edge a voltage varying sinusoidally with frequency ω; this will produce preferentially low m waves with their electric field parallel to the cylinder's axis (hence AE modes). Each such wave which satisfies the Zel'dovich-Misner condition gains in power as per Eqs. (59a,b) as it interacts with the cylinder. Propagating out, the amplified wave is reflected back by the cavity for a second round of amplification, and so on. If the cavity is a perfect reflector, and the material between cylinder and cavity is perfectly transparent, there will be a net gain in power which increases linearly with the number of bounces. But if the cavity absorbs (or leaks radiation outward), the consequent loss in power may quench the process. However, absorption in the cavity may be turned to our advantage by making the cavity rotate in the same sense as the cylinder with sufficiently large angular frequency so as to cause it also to superradiate for the modes in question. If the cavity walls are thick enough to prevent leakage, then each of the waves mentioned will always gain power in each round trip, and the overall gain is limited only by the time one allows the process go on.

When estimating the efficiency of such devices, the principal question is how big can $-a_m$ be. For an isolated cylinder, and $m=1$ AE modes, Eq. (59b) gives for optimal parameters that $-(a_1)_{\max}\approx 1.2\epsilon_{\mathrm{j}}(\omega R/c)^2$. For the cylinder-cavity device, this optimum gain is acquired over the back-and-forth light travel time $2\sqrt{\epsilon_{\mathrm{j}}}(R_c-R)/c$; one must still add to it the gain due to the cavity. As mentioned, for a cylinder made of good conductor, the peak superradiance occurs at $\Omega\approx\omega$. Hence the e-folding time of the cylinder-cavity device is $T_e<1.67c(R_c-R)/(\sqrt{\epsilon_{\mathrm{j}}}\Omega^2R^2)$. With $R_c=2R=20$ cm and $\Omega=2\pi\times 10^2\ \mathrm{s}^{-1}$, $T_e\approx(4/\sqrt{\epsilon_{\mathrm{j}}})$ hour, so that the effect can become dramatic for large ϵ_{j}. Many materials made of polar molecules have big ϵ at low frequencies, e.g. $\epsilon(0)\approx 80$ for water ice while $\epsilon(0)\approx 300$ for lead telluride [19]. And ferroelectrics just above the Curie point have virtually unbounded $\epsilon(0)$ [12].

A variation on the above is to have a coaxial cable (with no filling) rotating about its axis. Wave modes which not only have angular variation, but also vary along the axis (the $k\neq 0$ case studied in Sec. V A) will travel along the cable while bouncing between inner and outer boundaries. So long as the Zel'dovich-Misner condition is satisfied for such a mode, it will be amplified—rather than damped—as it travels along the cable. This might prove useful in protecting signals from degradation. We should stress that similar amplification will take place whatever the nature of the wave, sound waves being another useful candidate.

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APPENDIX: SUPERRADIANCE IN AXIAL MAGNETIC MODES

For completeness we now work out the gain coefficient for the AM modes with $k=0$. We set $\epsilon=\mu=1$ inside the cylinder to simplify the equations. Thus $H^{\alpha\beta}=F^{\alpha\beta}$ everywhere. By the definition of the modes we have $F^{03}=F^{31}=F^{23}=0$.

We combine Eqs. (33a,c) judiciously to cause the charge density terms (wherever nonvanishing) to cancel:

$$\partial(F^{12}+\Omega F^{01})r/\partial r+\zeta^{-1}rF^{02}=0 \tag{A1a}$$

$$\zeta^{-1}\equiv 4\pi\sigma\gamma^{-1}\Theta(R-r)-\imath(\omega-m\Omega). \tag{A1b}$$

Here Θ denotes the Heaviside step function. The function $g(r)\equiv(F^{12}+\Omega F^{01})re^{-\imath(m\phi-\omega t)}$ shall here play a role analogous to $f(r)$ in Sec. V D. In terms of it Eq. (A1a) gives

$$F^{02}=-\zeta r^{-1}g'e^{\imath(m\phi-\omega t)}. \tag{A2}$$

Now for $r<R$ we eliminate E_r between Eqs. (30c) and (33b) to obtain F^{01}/F^{12} so that we may express g in terms of F^{12} alone. It follows that

$$F^{12}=(4\pi\sigma\gamma-\imath\omega)\zeta r^{-1}ge^{\imath(m\phi-\omega t)} \tag{A3a}$$

$$F^{01}=-(4\pi\sigma\gamma\Omega r^2-\imath m)\zeta r^{-1}ge^{\imath(m\phi-\omega t)}. \tag{A3b}$$

For $r>R$ we use solely Eq. (33b) to determine F^{01}/F^{12}; the result is again Eqs. (A3a,b) with $\sigma\to 0$. Hence all nonvanishing field components can be recovered from g.

Substituting all these results in Eq. (32a) we get the radial equation for the AM modes:

$$(\zeta rg')'-\zeta[m^2-\omega^2r^2-\imath 4\pi\gamma\sigma(\omega-m\Omega)r^2\Theta(R-r)]\times r^{-1}g=0. \tag{A4}$$

Because $\epsilon=\mu=1$ here, this equation is quite similar to that for f, Eqs. (46a,b); in fact the only difference between them is a term involving $d\zeta/dr$. This last will vanish in the nonrelativistic limit where ζ becomes constant (except at $r=R$), and in that limit the equations are identical both inside and outside the cylinder. Indeed

$$r^2g''+rg'-[m^2-\omega^2r^2]g=0;\quad r>R \tag{A5a}$$

$$r^2g''+rg'-[m^2-\tilde{\kappa}^2r^2]g=0;\quad r<R \tag{A5b}$$

$$\tilde{\kappa}^2\equiv\omega^2+\imath 4\pi\sigma(\omega-m\Omega). \tag{A5c}$$

By analogy with Sec. V D the solution outside the cylinder is $g|_{R_+}=H_m^{(2)}(\omega r)+\rho H_m^{(1)}(\omega r)$ while that inside is $g|_{R_-}=\tau J_m(\tilde{\kappa}r)$. To find the matching conditions at $r=R$ we note that the condition of continuity of tangential electric fields requires that $F^{02}|_{R_+}=F^{02}|_{R_-}$. By Eq. (A2) this means $(\zeta g')|_{R_+}=(\zeta g')|_{R_-}$. Further, by integrating Eqs. (A1a,b) over a small radial interval spanning $r=R$ and realizing that all quantities are bounded, we see that $g|_{R_+}=g|_{R_-}$. These matching conditions parallel those for f when one replaces $\mu|_{R_-}\to\zeta^{-1}|_{R_-}$ and $\mu|_{R_+}\to\zeta^{-1}|_{R_+}$. Recalling Eq. (A1b) we see that Eqs. (47a,b), (48a,b) and (53) are applicable here with the replacements $\kappa\to\tilde{\kappa}$, $y\to\tilde{y}$ and $\mu\to\tilde{\mu}$, where

$$\tilde{y}^2\equiv x^2+\imath\xi(x-mv) \tag{A6a}$$

$$\tilde{\mu}=\imath(\omega-m\Omega)^{-1}\zeta^{-1}|_{R_-}=1+\imath\xi(x-mv)^{-1}. \tag{A6b}$$

We now obtain a formula analogous to (56) valid for $x\ll 1$ and when the small conductivity condition (54) holds. We substitute the expansion (55) into Eq. (53) and retain terms to $\mathbf{O}(\tilde{y}^2)$. The isolation of the imaginary part is easier if the denominator is put in real form. Neglecting terms in the numerator of higher order in x and $x-mv$, and reverting to dimensional quantities we get

$$a_m\approx\frac{8\pi^2(\omega R/2c)^{2m}}{m!}\frac{(\omega-m\Omega)\sigma}{(\omega-m\Omega)^2+4\pi^2\sigma^2}. \tag{A7}$$

This formula again shows that superradiance occurs only for $\omega-m\Omega<0$, and is in harmony with the expansion (25). It supports the insight mentioned in Sec. V D that superradiance is significant only for $m=1$. It corrects Zel'dovich's approximate formula for the AM modes, $a_m\propto(\omega-m\Omega)\sigma[(\omega-m\Omega)^2+16\pi^2\sigma^2)]^{-1}$ and supplies the normalization. We note that for fixed ωR, a_1 has the peak

$$-(a_1)_{\max}=1.571(\omega R/c)^2 \quad \text{at} \quad \Omega=\omega+2\pi\sigma. \tag{A8}$$

This peak gain is similar to that for AE modes. But unless the cylinder's conductivity is small, the Ω required to reach the peak gain will not be a practical one. For example, for copper $\sigma\approx 10^{17}\ \mathrm{s}^{-1}$. Put another way, for given v the peak is accessible only if $\xi<2v$. For larger ξ we must resort to numerical evaluation of the imaginary part in Eq. (53) with the substitutions (A6a,b); it certifies that the peak gain (A8) is not even approached. In closing we should note that for small ξ faster rotation is necessary to reach the peak gain for AE modes than for AM modes.

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