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Burgers turbulence and the continuous spontaneous localization model

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Abstract. – There is a striking convergence between Burgers turbulence and the continuous spontaneous localization (CSL) model of quantum mechanics. In this paper we exploit this analogy, identify the CSL counterparts of quantities in turbulence, such as the Reynolds number and the injected energy, show that the velocity of a particle subjected to the localization process satisfies Burgers equation and indicate a time region for experimental tests of the CSL models.

Introduction. – The study of nonlinear growth processes has recently become of central interest in physics, an example is provided by the KPZ equation [1, 2]. Originally devised to describe crystal growth, it has been since then applied to a wide range of systems, from bacterial growth [3, 4] to directed polymers [5]. One of the essential features of the KPZ equation is its nonlinear term introduced in order to account for lateral growth beyond the linear approximation such as described by the Edwards-Wilkinson model [6].

The KPZ equation is given by

$$\frac{\partial h(\boldsymbol{x},t)}{\partial t} = \nu\nabla^2 h(\boldsymbol{x},t) + \frac{1}{2}(\nabla h(\boldsymbol{x},t))^2 + \phi(\boldsymbol{x},t), \tag{1}$$

where $h(\boldsymbol{x},t)$ is the surface position, ν is the viscosity and $\phi(\boldsymbol{x},t)$ is a space- and time-dependent noise.

There is an interesting connection between the KPZ equation and the Burgers equation. The latter is a nonlinear diffusion equation for the velocity field of a fluid in N dimensions. The velocity is a gradient field, $\boldsymbol{v} = -\boldsymbol{\nabla} h$, and its equation is written as

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = \nu\nabla^2\boldsymbol{v} + \boldsymbol{f}(\boldsymbol{x},t), \tag{2}$$

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where the external stirring force is given by $\boldsymbol{f} = -\boldsymbol{\nabla}\phi$ and its correlation is [7]

$$\langle f^\mu(\boldsymbol{x},t) f^\nu(\boldsymbol{x},t')\rangle = \epsilon\delta(t-t')\left[\delta^{\mu\nu} - \frac{(\boldsymbol{x}-\boldsymbol{x}')^\mu(\boldsymbol{x}-\boldsymbol{x}')^\nu}{N\Delta^2}\right]\exp\left[-\frac{(\boldsymbol{x}-\boldsymbol{x}')^2}{2N\Delta^2}\right]. \tag{3}$$

ϵ is the energy injected into the fluid per unit time and unit mass and Δ is the length scale at which energy is injected.

In terms of the random potential the correlation function is

$$\langle \phi(\boldsymbol{x},t)\phi(\boldsymbol{x}',t')\rangle = \epsilon\Delta^2 N\delta(t-t')\exp\left[-\frac{(\boldsymbol{x}-\boldsymbol{x}')^2}{2N\Delta^2}\right]. \tag{4}$$

Through the Hopf-Cole transformation [5], $\boldsymbol{v} = -2\nu[\boldsymbol{\nabla} Z(\boldsymbol{x},t)]/Z(\boldsymbol{x},t)$, the Burgers equation can be put into a linear form with a multiplicative noise

$$\frac{\partial Z(\boldsymbol{x},t)}{\partial t} = \nu\nabla^2 Z(\boldsymbol{x},t) + \frac{\phi(\boldsymbol{x},t)}{2\nu} Z(\boldsymbol{x},t). \tag{5}$$

The above equation is a Schrödinger equation in the imaginary time. A similar equation for real time has been considered by several authors in a different context, such as the description of quantum open systems [8,9] or the continuous spontaneous localization of the wave function (CSL model) in an attempt to solve the quantum measurement problem [10, 11].

In this paper we extend an analogy we developed before [12] between the CSL model and enhanced diffusion, by exploring the similarities between the CSL modified Schrödinger equation and the KPZ, Burgers equations. The next section establishes an analogy between the KPZ and the CSL models, which provides a dictionary we use throughout the paper to move from one system to another and which allows us to introduce a Reynolds number for the CSL model. In the third section we develop the stochastic picture underlying our beable interpretation of the CSL model and connect it with the Burgers equation through the velocity field. This enables us to introduce the intermittency corrections and a Fokker-Planck equation (FPE) corresponding to the stochastic equation for the CSL model. An analysis of this system of equations allows us to introduce a time scale characterizing the dominance of enhanced diffusion over the standard (Wiener) diffusion. The fourth section presents our conclusions.

A dictionary for KPZ, Burgers and CSL. – The CSL model modifies the Schrödinger equation by introducing a multiplicative noise in the evolution of the wave function. In one dimension, the evolution equation in the Stratonovich form for a free particle of mass M is given by

$$\frac{\partial\psi(x,t)}{\partial t} = \left\{ i\nu\frac{\partial^2}{\partial x^2} - \lambda + \sqrt{\gamma}\int \mathrm{d}z w(z,t) G(x-z)\right\}\psi(x,t), \tag{6}$$

where $w(z,t)$ is a white noise, so that $\langle w(z,t)\rangle = 0$ and $\langle w(z,t)w(z',t')\rangle = \delta(z-z')\delta(t-t')$, and

$$G(x-z) = \sqrt{\frac{\alpha}{2\pi}}\exp\left[-\alpha\frac{(x-z)^2}{2}\right] \tag{7}$$

characterises the localization of the wave function. The diffusion constant ν is equal to $\hbar/2M$ and γ is related to the localization scale $1/\sqrt{\alpha}$ and the frequency of collapse λ as $\gamma = \lambda\sqrt{4\pi/\alpha}$.

These parameters are chosen in such a way that the new evolution equation does not give different results from the usual Schrödinger unitary evolution for microscopic systems with few degrees of freedom, but when a macroscopic system is described there is a fast decay of the macroscopic linear superpositions which are quickly transformed into statistical mixtures [10, 13].

The above-modified Schrödinger equation is similar to eq. (5) in the imaginary time. From this analogy the noise term may be written as $\phi(x,t) = 2\nu\sqrt{\gamma}\int \mathrm{d}z w(z,t) G(x-z)$, which gives the following correlation function:

$$\langle \phi(x,t)\phi(x',t')\rangle = 4\nu^2\lambda\delta(t-t')\exp\left[-\alpha\frac{(x-x')^2}{4}\right]. \tag{8}$$

Comparing this equation with eq. (4), we identify the injection length scale Δ with $\sqrt{2/\alpha}$ and the injected energy ϵ with $2\nu^2\alpha\lambda$ and can then obtain the Reynolds number for the CSL model by using the relation [5]: $Re = (\epsilon\Delta^4/\nu^3)^{1/3}$, which gives $Re = (8\lambda/\alpha\nu)^{1/3}$. For large Reynolds number we are in the domain of fully developed turbulence, which in the CSL case corresponds to a quantum system undergoing frequent collapses as λ is large.

The analogy between turbulence and the CSL model also provides a clarification regarding the issue of energy non-conservation in the original model for the collapse of the wave function proposed by Ghirardi, Rimini and Weber (GRW) [13]. The celebrated Kolmogorov analysis of 1941 (K41) [14] identifies Δ as the scale at which energy ϵ is injected into the system and a much smaller scale l_d at which energy is dissipated. For distances r such that $l_\mathrm{d} \ll r \ll \Delta$, we have what is called the inertial regime, where energy is transferred to ever smaller length scales. The analogous result for the collapse model has that the energy is injected at the localization scale $1/\sqrt{\alpha}$, which in turn is much larger than the atomic scale. In the framework of the collapse model, energy non-conservation coming from the collapse has always been discussed in terms of the magnitude of the effect itself and the constants were chosen in such a way as to make it a hardly observable effect at all. Here we stress, in analogy with turbulence, that the system does not gain energy indefinitely, for the energy injected in the collapse will be dissipated at the atomic scale. The disparity of scales is such that this fact was naturally unnoticed by the proponents of the collapse model.

Developing the analogy. – We have recently analysed the CSL model from a microscopic point of view [12]. In order to do so we used Vink's treatment [15], which shows that two alternative interpretations of quantum mechanics that treat position as a classical concept, the causal interpretation due to Bohm [16] and the stochastic interpretation due to Nelson [17], are actually particular cases of Bell's "beable" interpretation. The beable interpretation [18] is an attempt by Bell to treat physical quantities that exist independently of observation and therefore can be assigned well-defined values. His approach used fermion number, a discrete quantity, and Vink extended it to any observable that takes discrete values on small scales. One starts from the equation for the probability density $P_m(t)$ on a given basis

$$\partial_t P_m = \sum_n J_{mn}, \tag{9}$$

where the source matrix J_{mn} is given by

$$J_{mn}(t) = 2Im\{\langle\psi(t)|O_m\rangle\langle O_m|H|O_n\rangle\langle O_n|\psi(t)\rangle\}. \tag{10}$$

From a stochastic point of view, the probability distribution of O_m values satisfies the master equation

$$\partial_t P_m = \sum_n (T_{mn}P_n - T_{nm}P_m), \tag{11}$$

where $T_{mn}\mathrm{d}t$ is the transition probability for jumps from state n to state m.

To reconcile the quantum and stochastic views we equate (9) and (11):

$$J_{mn} = (T_{mn}P_n - T_{nm}P_m), \tag{12}$$

with $T_{mn} \geq 0$ and $J_{mn} = -J_{nm}$.

There is great freedom to find solutions of eq. (12). Bell chooses a particular one for $n \neq m$,

$$T_{mn} = \begin{cases} J_{mn}/P_n\,, & J_{mn} > 0\,, \\ 0\,, & J_{mn} \leq 0\,. \end{cases}$$

By discretising Schrödinger's equation, Vink shows that this choice leads to

$$T_{mn} = \frac{[S(an)]'}{Ma}\delta_{n,m-1}, \tag{13}$$

where use was made of the polar form of the wave function $\psi = Re^{iS/\hbar}$. This term considers only transitions between neighboring sites. Adding to T_{mn} the solution of the homogeneous equation derived from eq. (12):

$$T^o_{mn} \propto \exp\left[-\left[m-n-\frac{2\sigma ln(P_m/P_n)}{4(m-n)}\right]^2/2\sigma\right], \tag{14}$$

one introduces transitions between more distant sites.

The combination of eqs. (13) and (14) leads to the following equation for the particle position

$$\dot{x} = v(x,t) + (\beta\sigma a^2)^{\frac{1}{2}}\eta(t), \tag{15}$$

where β is a free parameter, $\eta(t)$ is a white noise, such that $\langle\eta(t)\rangle = 0$, $\langle\eta(t)\eta(t')\rangle = \delta(t-t')$ and

$$v(x,t) = \left[(\beta\sigma a^2)\frac{1}{R(x,t)}\frac{\partial R(x,t)}{\partial x} + \frac{1}{M}\frac{\partial S(x,t)}{\partial x}\right]. \tag{16}$$

Equation (15) coincides with Nelson's stochastic equation with $\beta\sigma a^2 = 2\nu$. A similar equation was obtained in our analysis of the microscopic dynamics of the CSL model. This is so because the new terms in the source matrix J_{mn} coming from the modified Schrödinger equation (6) do not contribute to the displacement $\mathrm{d}x$ [12].

The connection between the CSL (beable) equation and the Burgers equation is now straightforward. In imaginary time $S(x,t) = 0$ and $v = 2\nu(\nabla R)/R$, which, replacing R by Z together with the transformation $x \to -x$, corresponds to the Hopf-Cole transformation. The velocity then satisfies the Burgers equation if we assume the relation between the random force and the random potential without the minus sign $f = \nabla\phi$ (a point that was already noticed by Garbaczewski *et al.* [19]).

To simplify the following calculations, the Hamiltonian is set equal to zero as done in [20, 21]. The normalised solution of the modified Schrödinger equation is easily obtained:

$$\phi(x,t) = \frac{\psi(x,t)}{||\psi||} = \frac{1}{||\psi||}\exp[-\lambda t]\exp\left[\sqrt{\gamma}\int_0^t \mathrm{d}t'\int \mathrm{d}z G(x-z)w(z,t')\right]\psi(x,0). \tag{17}$$

By choosing the initial wave function as a single Gaussian

$$\psi(x,0) = \frac{1}{(2\pi\Delta x)^{\frac{1}{4}}}\exp\left[-\frac{(x-\langle x\rangle)^2}{4\Delta x}\right]\times \\ \times\exp\left[\frac{i}{\hbar}\left[\frac{(\frac{x^2}{2}-x\langle x\rangle)\sqrt{\Delta x\Delta p-\hbar^2/4}}{\Delta x}+\langle p\rangle x\right]\right], \tag{18}$$

where $\Delta x = \langle x^2\rangle - \langle x\rangle^2$ and $\langle x\rangle$ is the mean value of x (the same is valid for p), the stochastic differential equation for position is

$$\dot{x} = D_{\mathrm{S}} + 2\nu\sqrt{\gamma}\left\{\int_0^t \mathrm{d}t'\int \mathrm{d}z w(z,t')[-\alpha(x-z)]G(x-z)\right\} + \sqrt{2\nu}\eta(t). \tag{19}$$

This equation describes the evolution of the position of a tracer in a turbulent medium. The first term on the right-hand side describes a single free particle deterministic evolution and we choose D_{S} as a short notation for the term derived from the initial wave function $D_{\mathrm{S}} = [2\nu\nabla R_{\mathrm{S}}(x,0)/R_{\mathrm{S}}(x,0) + \nabla S_{\mathrm{S}}(x,0)/M] = [\langle x\rangle - x][\nu/\Delta x - \sqrt{\Delta x\Delta p - \hbar^2/4}/M\Delta x] + \langle p\rangle/M$. The two other terms describe the stochastic processes. The last term corresponds to a Brownian diffusion and the second one is responsible for the t^3 behavior of the mean-square displacement, which is the same time dependence obtained by Richardson in his pioneering studies of turbulence [22]. Notice that the coefficient of $\langle x^2\rangle$ corresponds to the injected energy $2\alpha\lambda\nu^2$ as expected from hydrodynamical turbulence [23] and coincides with the result in the previous section.

Moreover, the time dependence and nonlocal character of the second term on the r.h.s. of the equation above are in accordance with the concept of Lévy walk as introduced by Shlesinger *et al.* [24] when studying the phenomenon of enhanced diffusion. The basic difference between the more familiar Lévy flight [25] and Lévy walk is that for the latter, although the walker visits all sites visited by the flight, the jumps do not occur instantaneously, but there may be a time delay before the next jump. By introducing time, Shlesinger *et al.* obtained an integral transport equation involving a scaled memory which is nonlocal in space and time. Contrary to the infinite mean-square displacement obtained in a Lévy flight, the solution of such transport equation leads to a finite mean-square displacement such as the one obtained by Richardson.

From eq. (19), it follows that momentum satisfies

$$\dot{p} = 2M\nu\sqrt{\gamma}\int \mathrm{d}z w(z,t)[-\alpha(x-z)]G(x-z). \tag{20}$$

The stochastic process for momentum derives from the introduction of the random potential in the modified Schrödinger equation, which in turn is responsible for the localization of the wave function. This process vanishes when the GRW parameters go to zero.

The FPE corresponding to eq. (19), which is a stochastic differential equation with a colored noise introduced by $v(t)$, is obtained as done in [26] and is equal to

$$\frac{\partial P(x,p,t)}{\partial t} = -\frac{\langle p\rangle}{M}\frac{\partial P(x,p,t)}{\partial x} + \\ + \left[\frac{\hbar}{2M}\frac{\partial^2}{\partial x^2} + \sqrt{\frac{\hbar^3\alpha\lambda}{2M}}\frac{\partial^2}{\partial x\partial p} + \frac{\hbar^2\alpha\lambda}{4}\frac{\partial^2}{\partial p^2}\right]P(x,p,t). \tag{21}$$

A nice feature of our model is to obtain the above phase-space equation of evolution. Differently from Richardson [24], we started from purely theoretical arguments and took into account the discontinuous nature of the particle velocity.

A) Intermittency corrections. Having eqs. (19) and (20), we can now proceed to obtain the Mandelbrot intermittency corrections [27] to Richardson's law. In order to do so, we replace a white noise in time by an affine one [28] called fractional Brownian noise:

$$\langle w(z,t)\,w(z',t')\rangle = t^{A-1}\delta(t-t')\,\delta(z-z'), \tag{22}$$

which gives for the anomalous diffusion term [12]

$$\langle x^2(t)\rangle \sim t^{A-1+3}. \tag{23}$$

This corresponds to one of the intermittency corrections obtained by Shlesinger *et al.* [24] provided we identify $A-1$ with $3\mu/(4-\mu)$, where $\mu = E - \mathrm{d}f$, E being the Euclidean dimension and $\mathrm{d}f$ the fractal dimension.

For the momentum variable the nonwhite noise gives

$$\langle p^2\rangle \sim t^A, \tag{24}$$

which leads to the scaling relation obtained by Shlesinger *et al.* [24] for the root-mean-square velocity.

B) Time scales. Neglecting the deterministic term in eq. (19), we find the mean-square displacement $\langle x^2\rangle$:

$$\langle x^2\rangle = 2\nu t + \frac{2}{3}\alpha\lambda\nu^2 t^3, \tag{25}$$

which has two contributions: the usual Brownian one, coming from the $\eta(t)$ term, and the enhanced diffusion [12], arising from the multiplicative noise in the modified Schrödinger equation. Comparison of these two terms allows us to determine the time scale beyond which the enhanced diffusion dominates over the Brownian one:

$$t_{\mathrm{enh}} > \sqrt{\frac{3}{\alpha\lambda\nu}}. \tag{26}$$

Using the GRW parameters [13]: $\alpha = 10^{10}$ cm^{-2}, $\lambda(\text{micro}) = 10^{-16}$ s^{-1}, $\lambda(\text{macro}) = 10^7$ s^{-1}, $M(\text{micro}) = 10^{-23}$ g, $M(\text{macro}) = 1$ g, we estimate the time scale, which is approximately 2.4×10^5 s, independently of the macro or microscopic nature of the system.

A second time scale is given by the characteristic collapse time, which is of the order of λ^{-1}. For a microscopic system, enhanced diffusion ($t \sim 2.4 \times 10^5$ s) manifests itself even before collapse occurs ($t \sim 10^{16}$ s), opening an interesting window for experimental tests of this scenario, which would help to put stricter bounds on the parameters of the GRW/CSL models [29].

Conclusions. – In this article we found further points of contact between turbulence and the CSL model, identifying the CSL counterparts of important quantities in turbulence, such as the Reynolds number and the injected energy.

A microscopic stochastic picture for the CSL model allowed us to show that the velocity field of the tracer satisfies the Burgers equation.

Finally, a study of the different time scales governing enhanced and Brownian diffusion indicates that there is a region of parameter space of the CSL model amenable to experimental test, an investigation we plan to follow in a forthcoming publication.

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