Dissipative chaotic scattering

Adilson E. Motter[1,*] and Ying-Cheng Lai[1,2]
1. Department of Mathematics, Center for Systems Science and Engineering Research, Arizona State University, Tempe, Arizona 85287
2. Departments of Electrical Engineering and Physics, Arizona State University, Tempe, Arizona 85287
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We show that weak dissipation, typical in realistic situations, can have a metamorphic consequence on nonhyperbolic chaotic scattering in the sense that the physically important particle-decay law is altered, no matter how small the amount of dissipation. As a result, the previous conclusion about the unity of the fractal dimension of the set of singularities in scattering functions, a major claim about nonhyperbolic chaotic scattering, may not be observable.

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Chaotic scattering [1–3] is a physical manifestation of transient chaos [4], which is due to the existence of nonattracting chaotic invariant sets, i.e., chaotic saddles, in the phase space. As a result, a Cantor set of singularities arises in physically measurable scattering functions relating an output variable after the scattering to an input variable before the scattering [2]. Generally, the dynamics of chaotic scattering may be characterized as either *hyperbolic* or *nonhyperbolic*. In hyperbolic chaotic scattering, all the periodic orbits are unstable and there are no Kol'mogorov-Arnol'd-Moser (KAM) tori in the phase space and, as such, the survival probability of a particle in the scattering region typically decays exponentially with time. In nonhyperbolic chaotic scattering [5,6], there are both KAM tori and chaotic sets in the phase space. Due to the stickiness effect of KAM tori, a particle initialized in the chaotic region can spend a long time in the vicinity of KAM tori, leading to an algebraic decay [7] of the survival probability of the particle in the scattering region. A surprising result in nonhyperbolic chaotic scattering is that, because of the algebraic decay, the fractal dimension of the set of singularities in a scattering function is unity [6].

A physically important issue in the study of nonlinear dynamics is to understand how robust a phenomenon is against perturbations or deviations between the underlying mathematical model and physical reality. In the case of chaotic scattering, most of the theoretical investigations so far have been restricted to Hamiltonian or conservative systems. In a realistic situation, a small amount of dissipation can be expected. Take, for example, chaotic scattering arising in the context of particle advection in hydrodynamical flows [8]. In most existing studies, the condition of incompressibility is assumed of the underlying flow [8], which allows the problem to be casted in the context of Hamiltonian dynamics as the particle velocities can be related to flow's stream function in a way that is completely analogous to the Hamilton's equations in classical mechanics. Real hydrodynamical flows cannot be perfectly incompressible, and the effects of inertia and finite mass of the particles advected by the flow are effectively those due to friction, or dissipation [9].

The aim of this paper is to study the effect of dissipation on chaotic scattering dynamics. We first consider hyperbolic chaotic scattering and argue that weak dissipations have a negligible effect on the physical observables of chaotic scattering, such as scattering functions. We then focus on nonhyperbolic chaotic scattering and find that, in contrast to the hyperbolic case, the scattering dynamics can be altered by weak dissipation in a fundamental way. The major consequence of dissipation is that it typically converts KAM tori into periodic attractors. As a result, the underlying chaotic saddle can undergo a metamorphic bifurcation to a structurally different chaotic set, playing the role of chaotic invariant set that generates fractal basin boundaries [10]. There is an immediate transformation of the decay law of scattering particle from being algebraic in the Hamiltonian case to being exponential in the dissipative case, no matter how small the amount of dissipation. As a result, the fractal dimension of the chaotic saddle decreases from the integer value in the Hamiltonian case. These findings have striking implications to the study of chaotic scattering: they suggest that the algebraic-decay law, regarded to hold universally in nonhyperbolic chaotic scattering, is apparently structurally unstable against weak dissipations. More importantly, the previously believed integer dimensions of the chaotic saddles [6] in nonhyperbolic chaotic scattering may not be observable in realistic physical situations where dissipation is present.

We begin by presenting a picture for the formation of the fractal sets in chaotic scattering. Consider the idealized model of the hierarchical construction of Cantor sets in the unit interval. For hyperbolic scattering, an open subinterval in the middle of the unit interval is removed first. From each one of the two remaining subintervals, the same fraction from their middle is removed, and so on. Each step of this construction can be thought of as an iteration of the hyperbolic tent map with slope larger then two. The total length that remains decays exponentially with the number of iterations and the resulting Cantor set has a fractal dimension (e.g., box-counting dimension) smaller than one. For nonhyperbolic chaotic scattering, the same construction applies but the fraction removed at each step decreases with time, say, is inversely proportional to time. This simple reduction of the fraction removed captures the essence of the effect of KAM tori: their "stickiness" to particle trajectories in the phase space [6]. The remaining length decays algebraically with time and, even though the measure of the remaining set asymptotes to zero, the resulting Cantor set has dimension one [6].

How does dissipation change the above construction? The skeleton of the underlying chaotic saddle is formed by periodic orbits. When the system is hyperbolic, its structural stability guarantees the survival of all periodic orbits under small changes of the system parameters. Accordingly, the structure of the Cantor set in the presence of a small amount of dissipation is expected to be the same as before. When the dynamics is nonhyperbolic, however, qualitatively different behavior can take place. Marginally stable periodic orbits in KAM islands can become stable, turning their nearby phase-space regions into the corresponding basins of attraction [11]. This means that, part of the previous chaotic saddle now becomes part of the basins of the attractors. Most importantly for the scattering dynamics, the converted subset supports orbits in the neighborhood of the KAM islands that otherwise would be scattered after a long, algebraic time. These orbits are solely responsible for the nonhyperbolic character of the scattering in the conservative case. Due to the existence of dense orbits in the original chaotic saddle, the noncaptured part of the invariant set remains in the boundaries of basins of the periodic attractors. Therefore, the invariant set is the asymptotic limit of the boundaries between scattered and *captured* orbits, rather than those between scattered and *scattered* orbits as in the conservative case. Chaos thus occurs on the nonattracting invariant set whose stable manifold becomes the boundary separating the basins of the attractors and of the scattering trajectories. Through this simple reasoning we can see that the structure and the meaning of the Cantor set is fundamentally altered: in successive steps we remove a *constant* instead of a decreasing fraction in the middle of each interval. As a result, the scattering dynamics becomes hyperbolic with exponential decay. The dimension of the Cantor set immediately decreases from unity as a dissipation parameter is turned on. There is now more than one possible outcome: some of the removed intervals correspond to scattered orbits and the others correspond to orbits captured by the attractors. We stress that the appearance of attractors accompanied by a metamorphosis of the chaotic saddle can occur for arbitrarily small dissipation.

We now present numerical support for the effect of weak dissipation on chaotic scattering, particularly the metamorphic transformation of particle decay and fractal dimension in the nonhyperbolic case. Our model is a dissipative version of the two-dimensional area-preserving map utilized in Ref. [6] to establish the unity of the fractal dimension, a particularly convenient model for studying nonhyperbolic chaotic scattering. The map reads

$$M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} \lambda[x-(x+y)^2/4-\nu(x+y)] \\ \lambda^{-1}[y+(x+y)^2/4] \end{cases}, \qquad (1)$$

where $\lambda > 1$ and $\nu \geq 0$ are parameters. The map is conservative for $\nu = 0$ and dissipative for $\nu > 0$. For $\nu = 0$, almost all orbits started from negative values of y are scattered to infinity. In this case, the dynamics is nonhyperbolic for $\lambda \lesssim 6.5$ and hyperbolic for $\lambda \gtrsim 6.5$. The computation of the particle decay and fractal dimension in nonhyperbolic scattering requires examining very small scales, which makes the numerical computation a highly nontrivial task [5]. The advantage of using map (1) instead of a continuous flow is that it makes high-precision computation possible. Our results are, however, expected to hold in typical nonhyperbolic systems with KAM tori.

We study map (1) in the nonhyperbolic regime, without and with dissipation. We set $\lambda = 4.0$. When there is no dissipation ($\nu = 0$), there is a major KAM island in the phase space, as shown in Fig. 1(a). The fractal boundaries of the basins of scattering trajectories to infinity are also shown, which correspond to the stable manifold of the chaotic saddle in the scattering region. When dissipation is present ($\nu > 0$), the fixed point in the center of the island becomes an attractor. Dynamically, it happens because the magnitudes of the eigenvalues of periodic orbits associated with islands are one, which are reduced by dissipation in general. The basin of attraction of this attractor "captures" the island itself and orbits close to the stable manifold of the previously existing invariant set, as shown in Fig. 1(b). The intricate character of the basin of attraction with apparent fractal boundaries comes from points of the invariant set that are arbitrarily close to the island for $\nu = 0$. The newly created basin of attraction contains these points and hence, all their preimages as well. These preimages extend in the phase space along the original stable manifold of the chaotic saddle, which is the reason that the boundaries mimic those of the original basins of scattering trajectories [Figs. 1(a) versus 1(b)]. Because of this similarity, the scattering functions and time-delay functions, which are physically measurable, *resemble* each other in both the conservative and weakly dissipative case, as shown, respectively, in Figs. 1(c) and 1(d), where the time delay of particles launched from the horizontal line $y = -2$ toward the scattering region is plotted against their x-coordinates on the line.

To examine the decay laws of the scattering particles, we approximate the survival probability of a particle in the scattering region by $R(n)$, the fraction of a large number of particles still remaining in the scattering region (defined by $\sqrt{x^2+y^2} < r$) at time n, which are initiated in subregions close to the boundaries of the scattering basins. For convenience, we choose $r = 100$ and choose initial conditions from the horizontal line at $y_0 = -2$.

When the dynamics is nonhyperbolic and conservative, the decay of $R(n)$ with time is exponential for small n and algebraic for large n, as shown in Fig. 2(a) for $\lambda = 4.0$: $R \sim e^{-\alpha n}$ for $n \lesssim 250$ and $R \sim n^{-\beta}$ for $n \gtrsim 250$, where $\alpha \cong 0.08$ and $\beta \cong 1.0$. In the presence of a small dissipation, the time decay becomes strictly exponential and with the same decay rate of the exponential regime of the conservative case, $\alpha \cong 0.08$, as shown in Fig. 2(b) for $\nu = 0.001$. The original algebraic decay in the conservative case is destroyed by the dissipation because orbits with points close to the island, and that otherwise would be stuck, are captured by the periodic attractor. The decay rate in general changes under further increases of the dissipation. For $\nu = 0.01$, for instance, we obtain $\alpha \cong 0.06$. In the hyperbolic region the time decay is always exponential. For $\lambda = 8.0$, for example, the decay rate α remains essentially constant and equal to 0.9 in the range $0 \leq \nu \leq 0.01$.

The uncertainty algorithm [10] can be used to compute the fractal dimension D of the set of intersection points between the stable manifold of the chaotic saddle and a line from which scattering particles are initiated. As above, we choose the line $y_0 = -2$. In the absence of attractors, D is the dimension of the set of singularities in scattering functions. In Ref. [6], it is argued that $D = 1$ when map (1) is nonhyperbolic and conservative. A technical point about the numerical evaluation of the dimension in this case is that the result converges *slowly* to unity, and the convergence rate is determined by the reduction of length scales in the computation [6]. When a small amount of dissipation is present, $D + 1$ becomes the dimension of the boundaries between scattered and captured basins. The numerical convergence of D is in this case *faster* and essentially independent of the size of the interval under consideration. For $\lambda = 4.0$ and $\nu = 0.01$, we obtain $D \cong 0.8$, a well-convergent value as the length scale is reduced over six orders of magnitude. The dimension is much less sensitive to the presence of dissipation if the dynamics is hyperbolic. For $\lambda = 8.0$, for instance, we obtain $D \cong 0.44$ for both $\nu = 0$ and 0.01.

Dissipation can also lead to several coexisting attractors for some values of λ in the originally nonhyperbolic region. Periodic attractors are created through saddle-node bifurcations as λ is varied. These attractors then undergo period doubling cascades until the accumulation point where chaos appears. For small dissipation, however, the chaotic interval in the parameter space can be so small that it is difficult to detect chaotic attractors numerically. In fact, for ν on the order of 0.01 or smaller, the dynamics of map (1) is dominated by low-period periodic attractors. Periodic attractors of high periods either have small basins of attraction or exist in small intervals in the parameter space. Since periodic attractors result from the stabilization of periodic orbits in KAM islands, the parameter regions in which these attractors exist are approximately the same as those of the corresponding islands. In addition, the sizes of the basins are of the same order of the sizes of the original islands in the phase space [12].

In summary, our qualitative and quantitative examinations indicate that weak dissipation, no matter how small, can fundamentally alter the nature and dynamics of nonhyperbolic chaotic scattering. The algebraic-decay law, commonly believed to hold in such a case, is typically converted into an exponential-decay law in the metamorphic sense that the conversion can be induced by arbitrarily small amount of dissipation. A consequence of such a metamorphosis is that the previously claimed [6] unity of the fractal dimension of the set of singularities in scattering functions may not be physically meaningful. To our knowledge, there has been no previous attempt to address the effect of dissipation in open Hamiltonian systems, but this is a physically important issue of nonlinear dynamics that deserves further attention.

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* Permanent address: Departamento de Matemática Aplicada, Universidade Estadual de Campinas, 13083-970 Campinas, SP, Brazil.

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[12] Small additive noise, another perturbation likely to appear in physical processes, does not destroy the attractors, at least for time scales achievable in numerical simulations. In the presence of uniform noise, periodic attractors become extended stochastic attractors and the fine structures of the basin boundary become smeared. Increases in the noise amplitude eventually lead to the escape of the attracted orbits after a transient period of time near the destroyed attractors.

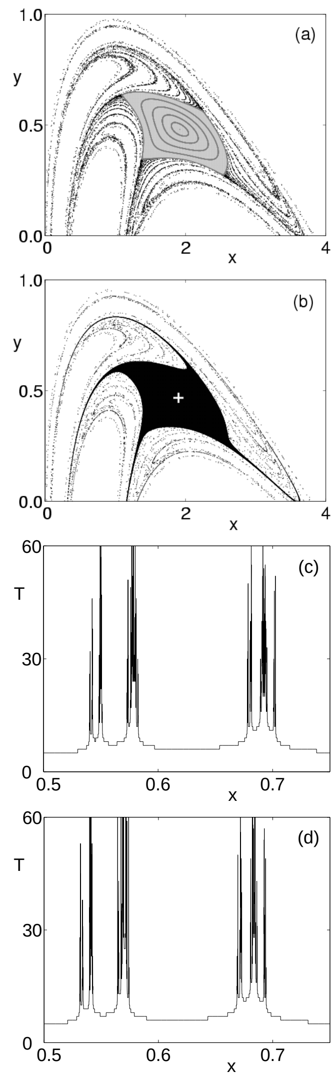

FIG. 1. Phase-space structure and time-delay function for map (1) with $\lambda = 4.0$. (a) $\nu = 0$: KAM island (in grey), scattered orbits (in blank), and the fractal boundaries of the scattered orbits (in black). (b) $\nu = 0.01$: captured orbits (in black) and scattered orbits (in blank). The plus sign is the fixed point attractor. (c),(d) Time delay in the conservative and dissipative cases of (a) and (b), respectively. T is the time taken by particles to reach $\sqrt{x^2 + y^2} \geq 100$.

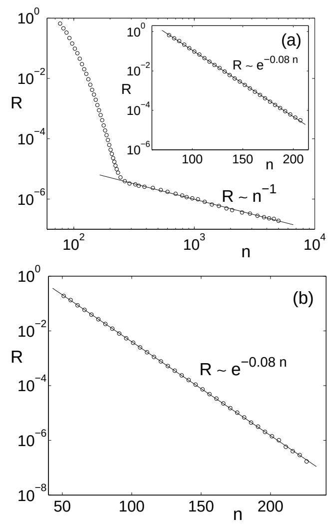

FIG. 2. Time decay for map (1) with $\lambda = 4.0$, in the interval $[x_0, x_0 + 10^{-7}]$. (a) $\nu = 0$, $x_0 = 0.5770050$. The inlet corresponds to the initially exponential decay. (b) $\nu = 0.001$, $x_0 = 0.5760006$.