On the stable limit cycle of a weight-driven pendulum clock

2010 Eur. J. Phys. 31 1249

(http://iopscience.iop.org/0143-0807/31/5/024)

IOP PUBLISHING EUROPEAN JOURNAL OF PHYSICS
Eur. J. Phys. **31** (2010) 1249–1254 doi:10.1088/0143-0807/31/5/024

On the stable limit cycle of a weight-driven pendulum clock

J Llibre[1] and M A Teixeira[2]

[1] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain
[2] Departamento de Matemática, Universidade Estadual de Campinas, 13083-859, Campinas, SP, Brazil

E-mail: jllibre@mat.uab.cat and teixeira@ime.unicamp.br

Received 6 July 2010, in final form 22 July 2010
Published 31 August 2010
Online at stacks.iop.org/EJP/31/1249

Abstract
In a recent paper (Denny 2002 *Eur. J. Phys.* **23** 449–58), entitled 'The pendulum clock: a venerable dynamical system', Denny showed that in a first approximation the steady-state motion of a weight-driven pendulum clock is shown to be a stable limit cycle. He placed the problem in a historical context and obtained an approximate solution using the Green function. In this paper we obtain the same result with an alternative proof via known issues of classical averaging theory. This theory provides a useful means to study a planar differential equation derived from the pendulum clock, accessible to Master and PhD students.

1. Introduction and statement of the results

For students, weight-driven pendulum clocks provide an interesting, practical and historically important dynamical system to be considered. In a nice paper, Denny [3] showed that in a first approximation the steady-state motion of a weight-driven pendulum clock is shown to be a stable limit cycle. He obtains an approximate solution using the Green function. Here we obtain the same result with an alternative proof via the averaging theory.

The linearized equation of the pendulum with friction and escapement is

$$\ddot{\theta} + b\dot{\theta} + \omega^2\theta \approx \frac{1}{\Delta t} p(t, \dot{\theta}), \tag{1}$$

where b is the friction coefficient and the right-hand side of this expression is the escapement. We expect small pendulum amplitudes for grandfather clocks, <5, and so the linear approximation is a very good one. The function $p(t, \dot{\theta})$ represents the (angular) momentum transferred to the pendulum by the escapement mechanism, during the short time interval Δt. It can be written as

$$p(t, \dot{\theta}) = \begin{cases} \bar{k}_+\delta(t) & \text{if } \dot{\theta} > 0, \\ \bar{k}_-\delta(t) & \text{if } \dot{\theta} < 0, \end{cases} \tag{2}$$

and

$$\delta(t) = \begin{cases} 1 & \text{if } |t - 2n\pi/\omega| < \Delta t/2, \\ 0 & \text{otherwise,} \end{cases}$$

where $\Delta t > 0$ is a small parameter, $\bar{k}_+ > 0$, $\bar{k}_- < 0$ and the dot denotes derivative with respect to the time t. For more details on this linearized equation of the pendulum, see [5].

Roughly speaking, our first concern is to bring the above system to the standard form $\mathbf{x}'(t) = F_0(t, \mathbf{x}) + \varepsilon F_1(t, \mathbf{x}) + \varepsilon^2 F_2(t, \mathbf{x}, \varepsilon)$, where ε indicates a small parameter, and F_0, F_1 and F_2 are 2π-periodic maps in the variable t. At this step, it is possible from the averaging method to show the existence of periodic solutions of the system under convenient conditions on the functions F_0, F_1 and F_2.

We point out that the method of averaging is a classical and matured tool that provides a useful means to study the behaviour of nonlinear dynamical systems under periodic forcing. The method of averaging has a long history that starts with the classical works of Lagrange and Laplace who provided an intuitive justification of the process. The first formalization of this procedure was given by Fatou in 1928 [4]. Very important practical and theoretical contributions in the averaging theory were made by Krylov and Bogoliubov [6] in the 1930s and Bogoliubov [1] in 1945. We refer to the book of Sanders and Verhulst [9] for a general introduction of this subject. Of particular note, formulae for the average of time-periodic vector fields are presented. The principle of averaging has been extended in many directions for both finite- and infinite-dimensional differentiable systems. Here we are interested in its extensions for studying the periodic orbits of the differential systems which are near the integrable ones; see for instance the works of Malkin [7] and Roseau [8]. In section 2 we present a summary of the results on the averaging theory for studying periodic orbits that we need for studying the steady-state motion of a weight-driven pendulum clock.

Our main result is the following.

Theorem 1. *The differential equation*

$$\ddot{\theta} + \bar{b}\dot{\theta} + \omega^2\theta = \frac{1}{\Delta t} p(t, \dot{\theta}), \tag{3}$$

where the function $p(t, \dot{\theta})$ *is given in (2), for* $\Delta t > 0$ *sufficiently small has a stable limit cycle which tends to the periodic orbit*

$$\frac{\bar{k}_+}{b\pi}\sin(\omega t) \quad \text{if } \bar{b}\,\omega > 0, \qquad -\frac{\bar{k}_-}{b\pi}\sin(\omega t) \quad \text{if } \bar{b}\,\omega < 0,$$

when $\Delta t \to 0$.

Theorem 1 is proved in section 3. In section 2 we recall the basic results on the averaging theory that we need for proving it.

2. Basic results

We consider the problem of the bifurcation of T-periodic solutions from the differential system

$$\mathbf{x}'(t) = F_0(t, \mathbf{x}) + \varepsilon F_1(t, \mathbf{x}) + \varepsilon^2 F_2(t, \mathbf{x}, \varepsilon), \tag{4}$$

with $\varepsilon = 0$ to $\varepsilon \neq 0$ sufficiently small. Here, the functions $F_0, F_1 : \mathbb{R} \times \Omega \to \mathbb{R}^n$ and $F_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are $\mathcal{C}^2$ functions, T-periodic in the first variable and Ω is an open subset of $\mathbb{R}^n$. One of the main assumptions is that the unperturbed system

$$\mathbf{x}'(t) = F_0(t, \mathbf{x}) \tag{5}$$

has a submanifold of periodic solutions. A solution of this problem is given using the averaging theory. For a general introduction to the averaging theory see for instance the book of Sanders and Verhulst [9].

Let $\mathbf{x}(t, \mathbf{z})$ be the solution of the unperturbed system (5) such that $\mathbf{x}(0, \mathbf{z}) = \mathbf{z}$. We write the linearization of the unperturbed system along the periodic solution $\mathbf{x}(t, \mathbf{z})$ as

$$\mathbf{y}' = D_{\mathbf{x}} F_0(t, \mathbf{x}(t, \mathbf{z}))\mathbf{y}. \tag{6}$$

In what follows we denote by $M_{\mathbf{z}}(t)$ some fundamental matrix of the linear differential system (6).

We assume that there exists an open set V with $\mathrm{Cl}(V) \subset \Omega$ such that for each $\mathbf{z} \in \mathrm{Cl}(V)$, $\mathbf{x}(t, \mathbf{z}, 0)$ is T-periodic, where $\mathbf{x}(t, \mathbf{z}, 0)$ denotes the solution of the unperturbed system (5) with $\mathbf{x}(0, \mathbf{z}, 0) = \mathbf{z}$. The set $\mathrm{Cl}(V)$ is *isochronous* for the system (5), i.e. it is a set formed only by periodic orbits, all of them having the same period. Then, an answer to the problem of the bifurcation of T-periodic solutions from the periodic solutions $\mathbf{x}(t, \mathbf{z}, 0)$ contained in $\mathrm{Cl}(V)$ is given in the following result.

Theorem 2 (Perturbations of an isochronous set)**.** *We assume that there exists an open and bounded set V with $\mathrm{Cl}(V) \subset \Omega$ such that for each $\mathbf{z} \in \mathrm{Cl}(V)$, the solution $\mathbf{x}(t, \mathbf{z})$ is T-periodic; then we consider the function $\mathcal{F} : \mathrm{Cl}(V) \to \mathbb{R}^n$*

$$\mathcal{F}(\mathbf{z}) = \frac{1}{T}\int_0^T M_{\mathbf{z}}^{-1}(t, \mathbf{z}) F_1(t, \mathbf{x}(t, \mathbf{z}))\, \mathrm{d}t. \tag{7}$$

(a) If there exists $a \in V$ with $\mathcal{F}(a) = 0$ and $\det((\mathrm{d}\mathcal{F}/\mathrm{d}\mathbf{z})(a)) \neq 0$, then there exists a T-periodic solution $\varphi(t, \varepsilon)$ of the system (4) such that $\varphi(0, \varepsilon) \to a$ as $\varepsilon \to 0$.

(b) If all the eigenvalues of $(\mathrm{d}\mathcal{F}/\mathrm{d}\mathbf{z})(a)$ have a modulus different from 1, then for $|\varepsilon| > 0$ sufficiently small the corresponding periodic solution $\varphi(t, \varepsilon)$ of the system (4) is hyperbolic and of the same stability type of the singular point as the singular point a of the averaged differential system $\mathbf{x}'(t) = \mathcal{F}(x)$.

For a shorter proof of theorem 2 see the corollary 1 of [2]. In fact this result goes back to Malkin [7] and Roseau [8].

3. Proof of theorem 1

Without loss of generality we take

$$\bar{b} = \varepsilon b, \qquad \varepsilon = \Delta t, \qquad \bar{k}_+ = \varepsilon k_+, \qquad \bar{k}_- = \varepsilon k_-, \qquad \omega > 0, \tag{8}$$

in the differential equation (3).

Denoting by $x = \theta$ and $y = \dot{\theta}$ the differential equation (3) becomes the differential system

$$\dot{x} = y, \qquad \dot{y} = -\omega^2 x - \varepsilon\, by + q(t, y), \tag{9}$$

where

$$q(t, y) = \begin{cases} k_+\delta(t) & \text{if } y > 0,\\ k_-\delta(t) & \text{if } y < 0,\end{cases}$$

with

$$\delta(t) = \begin{cases} 1 & \text{if } |t - 2n\pi/\omega| < \varepsilon/2,\\ 0 & \text{otherwise.}\end{cases}$$

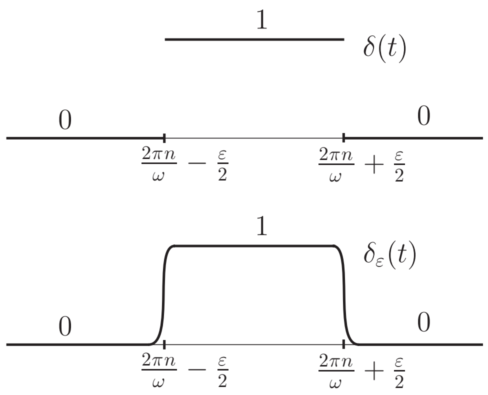

Figure 1. The functions $\delta(t)$ and $\delta_\varepsilon(t)$.

We work with the differential system (9) but instead with the discontinuous function $\delta(t)$ with the smooth function $\delta_\varepsilon(t)$ defined in figure 1, such that

$$\lim_{\varepsilon\to 0} \delta_\varepsilon(t) = \delta(t).$$

In this way the system (9) with the function $\delta_\varepsilon(t)$ satisfies the assumptions of theorem 2. We compute the function $\mathcal{F}(\mathbf{z})$ of the system (9) with the function $\delta(t)$ as the limit of the function $\mathcal{F}_\varepsilon(\mathbf{z})$ of the system (9) with the function $\delta_\varepsilon(t)$ when $\varepsilon \to 0$. We must note that the Poincaré map of both systems for $\delta(t)$ and $\delta_\varepsilon(t)$ are smooth, because in the first case it is the composition of smooth functions, and in the second by the general results on ordinary smooth differential equations.

For $\varepsilon = 0$ the differential system (9) becomes

$$\dot{x} = y, \qquad \dot{y} = -\omega^2 x. \tag{10}$$

Now we apply theorem 2 to the differential equation (9) taking $n = 2$ and

$$\begin{aligned} &\mathbf{x} = (x, y), \\ &F_0(\theta, \mathbf{x}) = (y, -\omega^2 x), \\ &\Omega = \mathbb{R}^2, \\ &\mathbf{z} = (x_0, y_0). \end{aligned} \tag{11}$$

Clearly the differential equation (10) is $T = 2\pi/\omega$-periodic in the variable t. Moreover this equation for $\varepsilon = 0$ has all its solutions $\mathbf{x}(t, \mathbf{z}) = (x(t, \mathbf{z}), y(t, \mathbf{z}))$ $2\pi/\omega$-periodic and given by

$$\begin{aligned} x(t, \mathbf{z}) &= x_0 \cos(t\omega) + \frac{y_0}{\omega} \sin(t\omega), \\ y(t, \mathbf{z}) &= y_0 \cos(t\omega) - x_0\omega \sin(t\omega). \end{aligned} \tag{12}$$

The V and $\mathbf{z}$ of theorem 2 are

$$V = \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 < \rho\},$$

for some real number $\rho > 0$, and $\mathbf{z} = (x_0, y_0) \in V$.

For the function F_0 given in (11) and the periodic solution $(x(t, \mathbf{z}), y(t, \mathbf{z}))$ given in (12), the 2×2 fundamental matrix $M(\theta)$ of the differential equation (6) such that $M(0)$ is the identity is given by

$$M(\theta) = \begin{pmatrix} \cos(t\omega) & \frac{1}{\omega}\sin(t\omega) \\ -\omega\sin(t\omega) & \cos(t\omega) \end{pmatrix}.$$

We remark that for the system (10) the fundamental matrix does not depend on the particular periodic solution $(x(t, \mathbf{z}), y(t, \mathbf{z}))$, i.e. it is independent of the initial condition z. Therefore,

$$M^{-1}(\theta) = \begin{pmatrix} \cos(t\omega) & -\frac{1}{\omega}\sin(t\omega) \\ \omega\sin(t\omega) & \cos(t\omega) \end{pmatrix}.$$

Since all the assumptions of theorem 2 are satisfied, we must study the zeros in V of the function $\mathcal{F}(\mathbf{z})$, where $\mathcal{F}$ is given by (7). More precisely, we have

$$\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon(\mathbf{z}) = \mathcal{F}(\mathbf{z})$$
$$= \int_{\frac{\varepsilon}{2}}^{\frac{2\pi}{\omega}-\frac{\varepsilon}{2}} M^{-1}(t)\begin{pmatrix} 0 \\ \varepsilon b y(t, \mathbf{z}) \end{pmatrix} \mathrm{d}t + \int_{\frac{2\pi}{\omega}-\frac{\varepsilon}{2}}^{\frac{2\pi}{\omega}+\frac{\varepsilon}{2}} M^{-1}(t)\begin{pmatrix} 0 \\ \varepsilon b y(t, \mathbf{z}) + k_\pm \end{pmatrix} \mathrm{d}t,$$

with k_+ if $y_0 > 0$ and k_- if $y_0 < 0$. Therefore,

$$\mathcal{F}(\mathbf{z}) = \begin{cases} \left(-\frac{1}{2}bx_0, -\frac{1}{2}by_0 + \frac{k_+}{\pi}\sin\left(\frac{\varepsilon\omega}{2}\right)\right) & \text{if } y_0 > 0, \\ \left(-\frac{1}{2}bx_0, -\frac{1}{2}by_0 + \frac{k_-}{\pi}\sin\left(\frac{\varepsilon\omega}{2}\right)\right) & \text{if } y_0 < 0. \end{cases}$$

There is a unique solution of $\mathcal{F}(\mathbf{z}) = 0$, namely

$$(x_0, y_0) = \begin{cases} \left(0, \frac{2k_+}{b\pi}\sin\left(\frac{\varepsilon\omega}{2}\right)\right) \approx \left(0, \frac{\varepsilon\omega k_+}{b\pi}\right) & \text{if } b\omega > 0, \\ \left(0, -\frac{2k_-}{b\pi}\sin\left(\frac{\varepsilon\omega}{2}\right)\right) \approx \left(0, -\frac{\varepsilon\omega k_-}{b\pi}\right) & \text{if } b\omega < 0. \end{cases} \tag{13}$$

Since the determinant of $(\mathrm{d}\mathcal{F}/\mathrm{d}\mathbf{z})(x_0, y_0)$ at solution (13) is $b^2\pi^2/\omega^2 \neq 0$, from theorem 2(a) it follows that the differential system (9) has a periodic orbit $(x(t), y(t))$ which tends to the periodic solution (12) with (x_0, y_0) given by (13) when $\varepsilon \to 0$, i.e which tends to

$$\frac{\bar{k}_+}{b\pi}\sin(\omega t) \quad \text{if } \bar{b}\omega > 0, \qquad -\frac{\bar{k}_-}{b\pi}\sin(\omega t) \quad \text{if } \bar{b}\omega < 0, \tag{14}$$

when $\varepsilon \to 0$.

The eigenvalue of $(\mathrm{d}\mathcal{F}/\mathrm{d}\mathbf{z})(x_0, y_0)$ at solution (13) is $-b\pi/\omega$ with multiplicity 2. Therefore, by theorem 2(b) we obtain that the periodic solution which tends to (14) when $\varepsilon \to 0$ is a stable limit cycle. Hence theorem 1 is proved.

Acknowledgments

The first author is partially supported by a MCYT/FEDER grant MTM2008-03437, by a CIRIT grant number 2009SGR-0410 and by ICREA Academia. The second author is partially supported by a FAPESP–BRAZIL grant 2007/06896-5. All the authors are also supported by the joint project CAPES-MECD grant PHB-2009-0025-PC.

References

[1] Bogoliubov N N 1945 On some statistical methods in mathematical physics *Izv. vo Akad. Nauk Ukr. SSR, Kiev*

[2] Buică A, Françoise J P and Llibre J 2007 Periodic solutions of nonlinear periodic differential systems with a small parameter *Commun. Pure Appl. Anal.* **6** 103–11

[3] Denny M 2002 The pendulum clock: a venerable dynamical system *Eur. J. Phys.* **23** 449–58

[4] Fatou P 1928 Sur le mouvement d'un systeme soumis à des forces a courte periode *Bull. Soc. Math.* **56** 98–139

[5] Jordan D W and Smith P 1999 *Nonlinear Ordinary Differential Equations* (Oxford: Oxford University Press)

[6] Krylov N and Bogoliubov N N 1934 The application of methods of nonlinear mechanics in the theory of stationary oscillations *Publ. 8 of the Ukrainian Acad. Sci. Kiev*

[7] Malkin I G 1956 Some problems of the theory of nonlinear oscillations *Gosudarstv. Izdat. Tehn.-Teor. Lit. (Moscow, Russian)*

[8] Roseau M 1966 Vibrations non linéaires et théorie de la stabilité *Springer Tracts in Natural Philosophy (French)* vol 8 (Berlin: Springer)

[9] Sanders J A and Verhulst F 1985 *Averaging Methods in Nonlinear Dynamical Systems* (*Applied Mathematical Sciences* vol 59) (Berlin: Springer)