Divergent diagrams of folds and simultaneous conjugacy of involutions

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Abstract

In this work we show that the smooth classification of divergent diagrams of folds $(f_1, \ldots, f_s) : (\mathbb{R}^n, 0) \to (\mathbb{R}^n \times \cdots \times \mathbb{R}^n, 0)$ can be reduced to the classification of the s-tuples $(\varphi_1, \ldots, \varphi_s)$ of associated involutions. We apply the result to obtain normal forms when $s \leq n$ and $\{\varphi_1, \ldots, \varphi_s\}$ is a transversal set of linear involutions. A complete description is given when $s = 2$ and $n \geq 2$. We also present a brief discussion on applications of our results to the study of discontinuous vector fields and discrete reversible dynamical systems.

Key words: divergent diagram of folds, involution, singularities, normal form, discontinuous vector fields, reversible diffeomorphisms.

Mathematics Subject Classification: 58K40, 58K70, 37G05, 37G40

1 Introduction

The study presented here is contained in an area that reveals an important connection between singularities of smooth mappings and dynamical systems, with particular applications to discontinuous vector fields and to reversible diffeomorphisms. We are interested in the smooth classification of divergent diagrams

$$(f_1, \ldots, f_s) : (\mathbb{R}^n, 0) \to (\mathbb{R}^n \times \cdots \times \mathbb{R}^n, 0),$$

where each map-germ $f_i : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is a fold, $i = 1, \ldots, s$. We discuss the relationship between the classification of divergent diagrams of folds $(f_1, \ldots, f_s)$ and the classification of the s-tuples $(\varphi_1, \ldots, \varphi_s)$ of involutions associated with these diagrams, that is, s-tuples of germs of diffeomorphisms $\varphi_i : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $\varphi_i \circ \varphi_i = \text{Id}$, $\varphi_i \neq \text{Id}$ and $f_i \circ \varphi_i = f_i$, for $i = 1, \ldots, s$. The concepts of equivalence for s-tuples of involutions and for divergent diagrams in these classifications are given in Definition 2.2 and Definition 3.1.

The problem of simultaneous behavior of diffeomorphisms has been treated by many authors and several interesting results have been obtained in different contexts. Among such results, we mention the Bochner-Montgomery Theorem [8], which is a well known and useful result about linearization of a compact group of transformations around a fixed point. This theorem is preceded by the remarkable work by Cartan [1]. In this direction, it is also worth mentioning the work by Hermann [4]. On the other hand, there exists a vast literature on divergent diagrams; we mention here Dufour [2]. In the present work, we treat the problem of simultaneous conjugacy of involutions and its relationship with the classification of divergent diagrams of folds.

For any two divergent diagrams of pairs of folds (topologically) equivalent by (h, k_1, k_2), where h is a germ of diffeomorphism (homeomorphism) acting on the source and k_1, k_2 are germs of diffeomorphisms (homeomorphisms) acting on each target, h realizes the equivalence of the pairs of involutions associated with the corresponding diagrams. Also, the same h realizes a conjugacy of the compositions of the involutions. Problems related to the topological stability of pairs of involutions and applications to the study of divergent diagrams of folds, discontinuous vector fields and reversible systems have been treated by Teixeira in [10], [11], [12] and [13]. In [12], he studies pairs of involutions on the plane with the help of their compositions. In [15] Voronin presents a list of problems including the analytic classification of divergent diagrams of pairs of folds and pairs of associated involutions on $(\mathbb{C}, 0)$. This problem is concerned with the following question: Let $(f_1, \ldots, f_s)$ and $(g_1, \ldots, g_s)$ be divergent diagrams of folds on $(\mathbb{R}^n, 0)$ and let $(\varphi_1, \ldots, \varphi_s)$ and $(\psi_1, \ldots, \psi_s)$ be the s-tuples of involutions associated with each of the diagrams respectively.

Then, under what conditions the equivalence of $(\varphi_1, \ldots, \varphi_s)$ and $(\psi_1, \ldots, \psi_s)$ implies the equivalence of $(f_1, \ldots, f_s)$ and $(g_1, \ldots, g_s)$? Under C^0-equivalence, this question is answered by Teixeira in [11] for the case $s = n = 2$ when the fixed-point spaces of the involutions coincide.

The main results of this work answer the question above for the general smooth case and give the classification for special pairs of involutions and divergent diagrams of folds associated with them. In Section 3 we prove that two divergent diagrams of folds on $(\mathbb{R}^n, 0)$ are equivalent if, and only if, the associated s-tuples of involutions are equivalent (Theorem 3.3). Following this result, we have an invariant for the equivalence class of a divergent diagram of folds $(f_1, \ldots, f_s)$, namely the trace of the linearization at the origin $d(\varphi_1 \circ \cdots \circ \varphi_s)(0)$ of the composition $\varphi_1 \circ \cdots \circ \varphi_s$, where the s-tuple $(\varphi_1, \ldots, \varphi_s)$ is associated with $(f_1, \ldots, f_s)$. In Section 4 we obtain normal forms when $s \leq n$ and $\{\varphi_1, \ldots, \varphi_s\}$ is a transversal set of linear involutions. We also obtain in Theorem 4.4 and Theorem 4.5 the normal forms for the situation where the set of involutions is transversal and generates an Abelian group, the involutions not necessarily linear. In addition, we present in Section 5 a characterization of the orbits based on the parameters that appear in the normal forms of Section 4. In Sections 6 and 7 we treat the transversal linear cases for $s = n = 2$ and $s = 2$, $n \geq 3$, respectively. We describe the partition of the space of parameters whose elements correspond to the orbits in order to obtain the classifications of pairs of involutions and of divergent diagrams of folds associated with them (Theorem 6.2 and Theorem 6.4 for $s = n = 2$ and Theorem 7.3 and Theorem 7.4 for $s = 2$, $n \geq 3$). The normal forms show that, up to equivalence, for almost all diagrams (f_1, f_2), the knowledge of the invariant $\mathrm{tr}(\varphi_1 \circ \varphi_2)$ determines the class of (f_1, f_2). Finally, in Section 8 we summarize an interaction between discontinuous vector fields and divergent diagram of folds (Subsection 8.1) and apply our results to discrete reversible systems (Subsection 8.2).

2   Preliminaries

Definition 2.1 *An* involution *is a germ of diffeomorphism* $\varphi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ *satisfying* $\varphi \circ \varphi = Id$.

Let $\mathrm{Fix}(\varphi)$ denote the fixed-point set of φ,

$$\mathrm{Fix}(\varphi) = \{x \in (\mathbb{R}^n, 0) : \varphi(x) = x\}.$$

Definition 2.2 *Two s-tuples $(\varphi_1, \ldots, \varphi_s)$ and $(\psi_1, \ldots, \psi_s)$ of involutions on $(\mathbb{R}^n, 0)$ are said to be equivalent if there exists a germ of diffeomorphism h of $(\mathbb{R}^n, 0)$ such that $\psi_i = h \circ \varphi_i \circ h^{-1}$, for all $i = 1, \ldots, s$.*

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Note that in the situation of Definition 2.2 the germ of diffeomorphism h satisfies
$$h(\text{Fix}(\varphi_i)) = \text{Fix}(\psi_i), \quad i = 1, \ldots, s.$$

Consider a set $G_s = \{\varphi_1, \ldots, \varphi_s\}$ of involutions on $(\mathbb{R}^n, 0)$ and let $\Lambda_s = [\varphi_1, \ldots, \varphi_s]$ denote the group generated by the φ_i's. The next result follows from Bochner-Montgomery Theorem in [8]:

Theorem 2.3 *If Λ_s is an Abelian group, then the s-tuple $(\varphi_1, \ldots, \varphi_s)$ is equivalent to an s-tuple $(\psi_1, \ldots, \psi_s)$ of linear involutions.*

It is an immediate consequence of Theorem 2.3 that any involution $\varphi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is conjugate to a linear involution. This implies that $\text{Fix}(\varphi)$ is locally diffeomorphic to a linear subspace of $\mathbb{R}^n$; therefore, $\text{Fix}(\varphi)$ is a submanifold in $(\mathbb{R}^n, 0)$. This also implies that one of the following holds: (a) φ is the identity Id; (b) if codim $\text{Fix}(\varphi) = \ell \neq 0$, then φ is conjugate to the canonical form

$$(x_1, \ldots, x_n) \mapsto (-x_1, \ldots, -x_\ell, x_{\ell+1}, \ldots, x_n). \tag{2.1}$$

Another basic concept for this work is the following:

Definition 2.4 *A map-germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is a* fold *if it is $\mathcal{A}$-equivalent to the germ*

$$f^0 : (x_1, \ldots, x_n) \mapsto (x_1^2, x_2, \ldots, x_n), \tag{2.2}$$

that is, there exist germs of diffeomorphisms h and k of $(\mathbb{R}^n, 0)$ such that $f = k \circ f^0 \circ h^{-1}$.

Definition 2.5 *Given an involution φ on $(\mathbb{R}^n, 0)$ and a fold $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, we say that f is associated with φ, or φ is associated with f, if $\varphi \neq$ Id and $f \circ \varphi = f$.*

We now present some general results concerned with an involution and a fold associated with it.

Proposition 2.6 *Given a fold $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, there exists a unique involution associated with f.*

Proof. By Definition 2.4, there exist germs of diffeomorphisms h and k of $(\mathbb{R}^n, 0)$ such that $f = k \circ f^0 \circ h^{-1}$. Now, it is not difficult to show that there exists a unique involution φ^0 associated with f^0, namely $\varphi^0(x_1, \ldots, x_n) = (-x_1, x_2, \ldots, x_n)$. Therefore, $\varphi = h \circ \varphi^0 \circ h^{-1}$ is the unique involution associated with f.

From the proof of Proposition 2.6 above, for the involution φ associated with the fold f we have that codim $\mathrm{Fix}(\varphi) = 1$. In addition, it is easy to see that

$$\mathrm{Fix}(\varphi) = \Sigma(f),$$

where $\Sigma(f)$ denotes the singular set of f.

Proposition 2.7 *Given an involution φ on $(\mathbb{R}^n, 0)$ with codim $\mathrm{Fix}(\varphi) = 1$, there exists a fold $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ associated with φ.*

Proof. Being codim $\mathrm{Fix}(\varphi) = 1$, φ is conjugate to the involution φ^0, $\varphi^0(x_1, \ldots, x_n) = (-x_1, x_2, \ldots, x_n)$. Let h be a germ of diffeomorphism of $(\mathbb{R}^n, 0)$ such that $\varphi = h \circ \varphi^0 \circ h^{-1}$. The fold f^0 in (2.2) is associated with φ^0, so the fold $f = f^0 \circ h^{-1}$ is associated with φ.

Remark 2.8 *A fold associated with an involution φ is not uniquely determined. In fact, if f is a fold associated with φ, then any fold $g \in \mathcal{L} \cdot f$ is also associated with φ, where $\mathcal{L}$ is the group of left equivalences. Corollary 2.11 below states that the set of all folds associated with φ is precisely the orbit $\mathcal{L} \cdot f$.*

For our purposes we have only to consider involutions φ on $(\mathbb{R}^n, 0)$ for which codim $\mathrm{Fix}(\varphi) = 1$, so this condition is assumed from now on.

Let i be a fixed integer, $1 \leq i \leq n$. Consider the involution φ_i^0 given by $\varphi_i^0(x_1, \ldots, x_n) = (x_1, \ldots, -x_i, \ldots, x_n)$ and the fold f_i^0, $f_i^0(x_1, \ldots, x_n) = (x_1, \ldots, x_i^2, \ldots, x_n)$, associated with φ_i^0. Then:

Lemma 2.9 *A fold $g : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is associated with φ_i^0 if, and only if, g is $\mathcal{L}$-equivalent to f_i^0.*

Proof. Suppose that g is a fold associated with φ_i^0. The equality $g \circ \varphi_i^0 = g$ implies that the components of g are even in x_i. So we can write

$$g(x_1, \ldots, x_n) = \left(k_1(x_1, \ldots, x_i^2, \ldots, x_n), \ldots, k_n(x_1, \ldots, x_i^2, \ldots, x_n) \right) =$$
$$= (k \circ f_i^0)(x_1, \ldots, x_n),$$

where $k = (k_1, \ldots, k_n)$. Now, since g is a fold, k is a germ of diffeomorphism. Therefore, g is $\mathcal{L}$-equivalent to f_i^0. The converse is immediate.

The next proposition generalizes Lemma 2.9 above.

Proposition 2.10 *Let φ be an involution on $(\mathbb{R}^n, 0)$, and let h be a germ of diffeomorphism of $(\mathbb{R}^n, 0)$ such that $\varphi = h \circ \varphi_i^0 \circ h^{-1}$. Consider the fold $f_i^0 \circ h^{-1}$ associated with φ. Then a fold $g : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is also associated with φ if, and only if, g is $\mathcal{L}$-equivalent to $f_i^0 \circ h^{-1}$.*

Proof. If g is a fold associated with φ, then $g \circ h$ is a fold associated with φ_i^0. From Lemma 2.9 it follows that $g \circ h$ is $\mathcal{L}$-equivalent to f_i^0, that is, g is $\mathcal{L}$-equivalent to $f_i^0 \circ h^{-1}$. The converse is immediate.

More generally, we can rewrite the statement of Proposition 2.10 replacing the fold $f_i^0 \circ h^{-1}$ by any fold f associated with φ:

Corollary 2.11 *Let φ be an involution on $(\mathbb{R}^n, 0)$, and let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a fold associated with φ. Then a fold $g : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is also associated with φ if, and only if, g is $\mathcal{L}$-equivalent to f.*

3 Divergent diagram of folds

A diagram of map-germs of the type

$$
(f_1, \ldots, f_s) : (\mathbb{R}^n, 0)
\begin{array}{l}
\xrightarrow{f_1} (\mathbb{R}^n, 0) \\
\xrightarrow{f_2} (\mathbb{R}^n, 0) \\
\vdots \\
\xrightarrow{f_s} (\mathbb{R}^n, 0)
\end{array}
$$

is called divergent diagram. In the space of these diagrams, the concept of equivalence is given by the following definition:

Definition 3.1 *Two divergent diagrams*

$$
(f_1, \ldots, f_s) : (\mathbb{R}^n, 0)
\begin{array}{l}
\xrightarrow{f_1} (\mathbb{R}^n, 0) \\
\xrightarrow{f_2} (\mathbb{R}^n, 0) \\
\vdots \\
\xrightarrow{f_s} (\mathbb{R}^n, 0)
\end{array}
\qquad , \qquad
(g_1, \ldots, g_s) : (\mathbb{R}^n, 0)
\begin{array}{l}
\xrightarrow{g_1} (\mathbb{R}^n, 0) \\
\xrightarrow{g_2} (\mathbb{R}^n, 0) \\
\vdots \\
\xrightarrow{g_s} (\mathbb{R}^n, 0)
\end{array}
$$

are equivalent if there exist germs of diffeomorphisms $h, k_1, \ldots, k_s$ of $(\mathbb{R}^n, 0)$ such that $g_i = k_i \circ f_i \circ h^{-1}$, for all $i = 1, \ldots, s$.

We shall identify a divergent diagram

$$
(f_1, \ldots, f_s) : (\mathbb{R}^n, 0)
\begin{array}{c}
\xrightarrow{\ f_1\ } (\mathbb{R}^n, 0) \\
\xrightarrow{\ f_2\ } (\mathbb{R}^n, 0) \\
\vdots \\
\xrightarrow{\ f_s\ } (\mathbb{R}^n, 0)
\end{array}
$$

with the map-germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^n \times \ldots \times \mathbb{R}^n, 0)$, $f(x) = (f_1(x), \ldots, f_s(x))$, and maintain the notation $(f_1, \ldots, f_s)$ for f. Under this identification, the equivalence given in Definition 3.1 corresponds to the action of a subgroup of the group $\mathcal{A}$ of right-left equivalences, consisting of elements such that the germ of diffeomorphism in the target is of product type, i. e., preserves the product structure of $\mathbb{R}^n \times \ldots \times \mathbb{R}^n$.

Our attention is addressed to divergent diagrams $(f_1, \ldots, f_s) : (\mathbb{R}^n, 0) \to (\mathbb{R}^n \times \ldots \times \mathbb{R}^n, 0)$ where each $f_i : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is a fold, $i = 1, \ldots, s$.

Definition 3.2 *Let* $\varphi_1, \ldots, \varphi_s$ *be involutions on* $(\mathbb{R}^n, 0)$ *and let* $(f_1, \ldots, f_s) : (\mathbb{R}^n, 0) \to (\mathbb{R}^n \times \cdots \times \mathbb{R}^n, 0)$ *be a divergent diagram of folds. We say that* $(f_1, \ldots, f_s)$ is associated with the s-tuple $(\varphi_1, \ldots, \varphi_s)$, or $(\varphi_1, \ldots, \varphi_s)$ is associated with $(f_1, \ldots, f_s)$, if f_i is a fold associated with φ_i for all $i = 1, \ldots, s$.

In view of Proposition 2.6, for a given divergent diagram of folds $(f_1, \ldots, f_s)$, there exists a unique s-tuple of involutions $(\varphi_1, \ldots, \varphi_s)$ associated with it.

We now present the key result which establishes that the classification of divergent diagrams of folds can be reduced to the classification of the associated s-tuples of involutions.

Theorem 3.3 *Let* $(f_1, \ldots, f_s)$ *be a divergent diagram of folds associated with* $(\varphi_1, \ldots, \varphi_s)$ *and let* $(g_1, \ldots, g_s)$ *be a divergent diagram of folds associated with* $(\psi_1, \ldots, \psi_s)$. *Then,* $(f_1, \ldots, f_s)$ *and* $(g_1, \ldots, g_s)$ *are equivalent if, and only if,* $(\varphi_1, \ldots, \varphi_s)$ *and* $(\psi_1, \ldots, \psi_s)$ *are equivalent.*

Proof. Suppose that $(f_1, \ldots, f_s)$ and $(g_1, \ldots, g_s)$ are equivalent. By definition, there exist germs of diffeomorphisms h, k_1, $\ldots$, k_s of $(\mathbb{R}^n, 0)$ such that $g_i = k_i \circ f_i \circ h^{-1}$, for all $i = 1, \ldots, s$. So, the s-tuples of involutions $(h \circ \varphi_1 \circ h^{-1}, \ldots, h \circ \varphi_s \circ h^{-1})$ is associated with $(g_1, \ldots, g_s)$. By the uniqueness just mentioned above, it follows that $\psi_i = h \circ \varphi_i \circ h^{-1}$, for all $i = 1, \ldots, s$. Therefore, $(\varphi_1, \ldots, \varphi_s)$ and $(\psi_1, \ldots, \psi_s)$ are equivalent.

Conversely, suppose that there exists a germ of diffeomorphism h of $(\mathbb{R}^n, 0)$ such that $\psi_i = h \circ \varphi_i \circ h^{-1}$, $i = 1, \ldots, s$. Then, the divergent diagram of folds $(f_1 \circ h^{-1}, \ldots, f_s \circ h^{-1})$ is associated with $(\psi_1, \ldots, \psi_s)$. Hence, by Corollary 2.11,

$$(g_1, \ldots, g_s) \in (\mathcal{L} \times \ldots \times \mathcal{L}) \cdot (f_1 \circ h^{-1}, \ldots, f_s \circ h^{-1}).$$

Therefore, $(f_1, \ldots, f_s)$ and $(g_1, \ldots, g_s)$ are equivalent.

As a consequence of this result, for the case of divergent diagrams of folds $(f_1, \ldots, f_s)$, we have that $\mathrm{tr}(d(\varphi_1 \circ \cdots \circ \varphi_s)(0))$ is an invariant up to equivalence, where $(\varphi_1, \ldots, \varphi_s)$ is the s-tuple associated with $(f_1, \ldots, f_s)$.

4 Transversal sets of involutions

In this section we obtain normal forms for special classes of divergent diagrams of folds. These are given in Proposition 4.3 and Theorem 4.5. We start with a definition.

Definition 4.1 *A set $G_s = \{\varphi_1, \ldots, \varphi_s\}$ of involutions on $(\mathbb{R}^n, 0)$, $s \leq n$, is said to be* transversal *if $\mathrm{Fix}(\varphi_i)$ is transversal to $\mathrm{Fix}(\varphi_j)$ at 0 for $i \neq j$ and codim $\cap_{i=1}^{s} \mathrm{T}_0 \mathrm{Fix}(\varphi_i) = \sum_{i=1}^{s}$ codim $\mathrm{Fix}(\varphi_i)$, where $\mathrm{T}_0 \mathrm{Fix}(\varphi_i)$ denotes the tangent space to $\mathrm{Fix}(\varphi_i)$ at 0.*

The next result is essentially a result from Linear Algebra.

Proposition 4.2 *Let $G_s = \{\varphi_1, \ldots, \varphi_s\}$ be a transversal set of linear involutions on $(\mathbb{R}^n, 0)$. Then $(\varphi_1, \ldots, \varphi_s)$ is linearly equivalent to $(\psi_1, \ldots, \psi_s)$ such that, for each ψ_i, $\mathrm{Fix}(\psi_i)$ is given by the equation $x_i = 0$ and, therefore, ψ_i has the form*

$$\psi_i(x_1, \ldots, x_n) = (x_1 + a_{i1}x_i, \ldots, -x_i, \ldots, x_n + a_{in}x_i), \qquad (4.3)$$

for some constants a_{ij}, $j \neq i$, $1 \leq j \leq n$.

Proposition 4.3 *Let $G_s = \{\varphi_1, \ldots, \varphi_s\}$ be as in Proposition 4.2. Then any divergent diagram of folds $(f_1, \ldots, f_s)$ associated with $(\varphi_1, \ldots, \varphi_s)$ is equivalent to the diagram of folds $(g_1, \ldots, g_s)$ associated with $(\psi_1, \ldots, \psi_s)$, where ψ_i is given by (4.3) and*

$$g_i(x_1, \ldots, x_n) = (x_1 + \frac{a_{i1}}{2}x_i, \ldots, x_i^2, \ldots, x_n + \frac{a_{in}}{2}x_i). \qquad (4.4)$$

Proof. For each i, $i = 1, \ldots, s$, consider the germ at zero of the isomorphism h_i of $\mathbb{R}^n$ given by

$$h_i(x_1, \ldots, x_n) = (x_1 - \frac{a_{i1}}{2} x_i, \ldots, x_i, \ldots, x_n - \frac{a_{in}}{2} x_i).$$

We have $\psi_i = h_i \circ \varphi_i^0 \circ h_i^{-1}$. Moreover, the formula (4.4) defines the fold $g_i = f_i^0 \circ h_i^{-1}$, which is associated with ψ_i. Hence, by Theorem 3.3, $(f_1, \ldots, f_s)$ is equivalent to $(g_1, \ldots, g_s)$.

The following two results give the normal forms for the case when a transversal set of involutions generates an Abelian group, the involutions not being necessarily linear.

Theorem 4.4 *If $G_s = \{\varphi_1, \ldots, \varphi_s\}$ is transversal and $\Lambda_s = [\varphi_1, \ldots, \varphi_s]$ is Abelian, then $(\varphi_1, \ldots, \varphi_s)$ is equivalent to $(\varphi_1^0, \ldots, \varphi_s^0)$, where*

$$\varphi_i^0(x_1, \ldots, x_n) = (x_1, \ldots, -x_i, \ldots, x_n), \quad i = 1, \ldots, s. \tag{4.5}$$

Proof. Since Λ_s is Abelian, by Theorem 2.3, $(\varphi_1, \ldots, \varphi_s)$ is equivalent to $(\bar{\varphi}_1, \ldots, \bar{\varphi}_s)$, with each $\bar{\varphi}_i$ linear. Since G_s is transversal, then $\bar{G}_s = \{\bar{\varphi}_1, \ldots, \bar{\varphi}_s\}$ is transversal, and using again the property of Λ_s being Abelian we also have $\bar{\Lambda}_s = [\bar{\varphi}_1, \ldots, \bar{\varphi}_s]$ Abelian. Now we can argue in two ways: One would be to use the fact that $\bar{\Lambda}_s$ is Abelian and deal with the normal forms (4.3). The other, more direct, is to use the simultaneous diagonalization of commuting operators. This result implies that we can assume that the linear involutions $\bar{\varphi}_i$'s have each diagonal matricial form. Finally, by using a permutation matrix if necessary, $(\bar{\varphi}_1, \ldots, \bar{\varphi}_s)$ is equivalent to $(\varphi_1^0, \ldots, \varphi_s^0)$, where φ_i^0 is given by (4.5).

From Theorem 3.3 and Theorem 4.4 we get the following:

Theorem 4.5 *If G_s is transversal and Λ_s is Abelian, then any divergent diagram of folds $(f_1, \ldots, f_s)$ associated with $(\varphi_1, \ldots, \varphi_s)$ is equivalent to the divergent diagram of folds $(f_1^0, \ldots, f_s^0)$, where*

$$f_i^0(x_1, \ldots, x_n) = (x_1, \ldots, x_i^2, \ldots, x_n), \quad i = 1, \ldots, s.$$

5   Characterization of orbits

In this section we characterize the orbits of divergent diagrams of folds $(f_1, \ldots, f_s) : (\mathbb{R}^n, 0) \to (\mathbb{R}^n \times \cdots \times \mathbb{R}^n, 0)$ associated with s-tuples $(\varphi_1, \ldots, \varphi_s)$ of linear involutions on $(\mathbb{R}^n, 0)$ when the set $G_s = \{\varphi_1, \ldots, \varphi_s\}$ is transversal.

According to Theorem 3.3 and Proposition 4.3, it suffices to characterize the orbits of the s-tuples $(\psi_1, \ldots, \psi_s)$, where ψ_i is as (4.3). This is given in the following result:

Proposition 5.1 *Consider the s-tuples of transversal linear involutions* $(\psi_{1_a}, \ldots, \psi_{s_a})$ *and* $(\psi_{1_b}, \ldots, \psi_{s_b})$, *where*

$$\psi_{i_a}(x_1, \ldots, x_n) = (x_1 + a_{i1}x_i, \ldots, -x_i, \ldots, x_n + a_{in}x_i),$$

$$\psi_{i_b}(x_1, \ldots, x_n) = (x_1 + b_{i1}x_i, \ldots, -x_i, \ldots, x_n + b_{in}x_i),$$

for all $i = 1, \ldots, s$. *Then* $(\psi_{1_a}, \ldots, \psi_{s_a})$ *and* $(\psi_{1_b}, \ldots, \psi_{s_b})$ *are equivalent if, and only if, there exists an invertible matrix*

$$H = \begin{pmatrix} \alpha_1 & & & & & & \\ & \alpha_2 & & 0 & & 0 & \\ & 0 & \ddots & & & & \\ & & & \alpha_s & & & \\ \hline \delta_{s+1} & \gamma_{s+1,2} & \cdots & \gamma_{s+1,s} & \beta_{s+1,s+1} & \cdots & \beta_{s+1,n} \\ \vdots & & \ddots & & & \ddots & \\ \delta_n & \gamma_{n2} & & \gamma_{ns} & \beta_{n,s+1} & & \beta_{nn} \end{pmatrix} \tag{5.6}$$

such that $\alpha_1 = 1$ *and if* $1 \le i, j \le s$, $i \ne j$, *then*

$$b_{ij} = \frac{\alpha_j}{\alpha_i} a_{ij}. \tag{5.7}$$

If $s + 1 \le j \le n$, *then*

$$b_{1j} = -2\delta_j + \sum_{k=2}^{s} \gamma_{jk} a_{1k} + \sum_{k=s+1}^{n} \beta_{jk} a_{1k}, \tag{5.8}$$

and for $s + 1 \le j \le n$ *and* $2 \le i \le s$,

$$b_{ij} = \frac{1}{\alpha_i}\left(\delta_j a_{i1} - 2\gamma_{ji} + \sum_{k=2, k \ne i}^{s} \gamma_{jk} a_{ik} + \sum_{k=s+1}^{n} \beta_{jk} a_{ik}\right). \tag{5.9}$$

Proof. The s-tuples $(\psi_{1_a}, \ldots, \psi_{s_a})$ and $(\psi_{1_b}, \ldots, \psi_{s_b})$ are equivalent if, and only if, there exists a germ of diffeomorphism h of $(\mathbb{R}^n, 0)$ such that

$$\psi_{i_b} = h \circ \psi_{i_a} \circ h^{-1}. \tag{5.10}$$

By taking the derivatives at zero we can assume that h is linear. It is now straightforward that the linear diffeomorphism has matrix of the form (5.6).

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The fact that α_1 can be taken to be 1 follows from the property that if h satisfies (5.10), then for any nonzero constant α, αh also satisfies (5.10).

In the next two sections we use Proposition 5.1 above to obtain normal forms for the pairs of transversal linear involutions on $(\mathbb{R}^n, 0)$, so $s = 2$ and $n \geq 2$. Let us observe that when $n > s$, the relations (5.7) to (5.9) show that the partition of the parameter space $\mathbb{R}^{s(n-1)}$ that characterizes the orbits of the s-tuples $(\psi_1, \cdots, \psi_s)$ projects onto the partition of the parameter space $\mathbb{R}^{s(s-1)}$ determined by the orbits of the s-tuples $(\phi_1, \cdots, \phi_s)$ of involutions on $(\mathbb{R}^s, 0)$, with

$$\phi_i = \pi_s \circ \psi_i \circ \iota_s, \tag{5.11}$$

where $\iota_s(x_1, \ldots, x_s) = (x_1, \ldots, x_s, 0, \ldots, 0)$ and $\pi_s(y_1, \ldots, y_n) = (y_1, \ldots, y_s)$. For this reason, we first study the case $s = n = 2$ (Section 6), which is the key for the analysis of the cases $s = 2$, $n \geq 3$ (Section 7).

We have already obtained normal forms for the Abelian case even in the nonlinear context (Theorem 4.4). However, in the following two sections we state the results including pairs of transversal linear involutions generating an Abelian group for completeness. In fact, the Abelian case appears naturally in our procedure to derive the normal forms.

6 The case $s = n = 2$

In this section we apply the results of Sections 4 and 5 to the pairs (φ_1, φ_2) of transversal linear involutions on $(\mathbb{R}^2, 0)$. In Subsection 6.1 we present the normal forms for these pairs of involutions and in Subsection 6.2 the normal forms for the divergent diagrams of folds associated with them. As we shall see, almost all normal forms depend on one parameter, namely the trace of the composition $\varphi_1 \circ \varphi_2$. This fact is applied in Section 8 in the discussion of reversible diffeomorphisms.

6.1 Normal forms of pairs of involutions

We start by considering only pairs of involutions (ψ_1, ψ_2) as in (4.3) and describing the orbits on the plane (a_{12}, a_{21}) of parameters that appear in these pairs. For this particular case of $s = n = 2$ we can easily explicit the partition of this plane determined by the orbits. More precisely, following the notation of Section 5, we consider the pairs (ψ_{1_a}, ψ_{2_a}) and (ψ_{1_b}, ψ_{2_b}), where

$$\psi_{1_a}(x, y) = (-x, y + a_{12}x),$$

$$\psi_{2_a}(x, y) = (x + a_{21}y, -y)$$

and

$$\psi_{1_b}(x, y) = (-x, y + b_{12}x),$$

$$\psi_{2_b}(x, y) = (x + b_{21}y, -y).$$

From (5.7) we have that (ψ_{1_a}, ψ_{2_a}) and (ψ_{1_b}, ψ_{2_b}) are equivalent if, and only if, there exists a nonzero constant α such that

$$b_{12} = \alpha a_{12}$$

$$b_{21} = \frac{1}{\alpha} a_{21}.$$

Therefore, the required partition of the plane (a_{12}, a_{21}) is given in Fig.1. Each orbit determines either a hyperbole, or an axis minus the origin or the origin itself. The origin corresponds to the group $\Lambda_2 = [\psi_1, \psi_2]$ being Abelian.
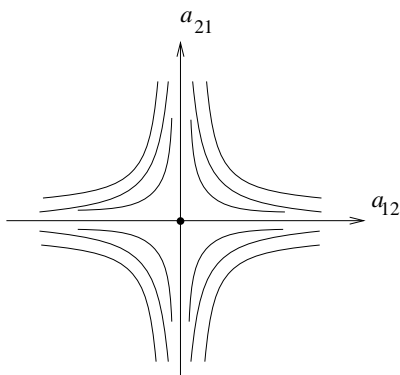

Figure 1: Partition of the plane (a_{12}, a_{21}) determined by the orbits of the pairs of involutions (ψ_1, ψ_2).

Let us observe that each pair (ψ_1, ψ_2), with Λ_2 non-Abelian, is equivalent to $(\bar{\psi}_1, \bar{\psi}_2)$, where $\bar{\psi}_1(x, y) = (-x, y + \bar{a}_{12}x)$ and $\bar{\psi}_2(x, y) = (x + \bar{a}_{21}y, -y)$, with a unique representative point $(\bar{a}_{12}, \bar{a}_{21})$ either on an arbitrary fixed point of the a_{12}-axis (distinct from the origin) or on an arbitrary fixed line parallel to this axis in the plane (a_{12}, a_{21}). We choose

$$(\bar{a}_{12}, \bar{a}_{21}) = \begin{cases} (1, 0) & \text{if } a_{21} = 0 \\ (2 + \text{tr}(\psi_1 \circ \psi_2), 1) & \text{if } a_{21} \neq 0. \end{cases} \qquad (6.12)$$

For the second choice we use the equality

$$a_{12}a_{21} = 2 + \text{tr}(\psi_1 \circ \psi_2).$$
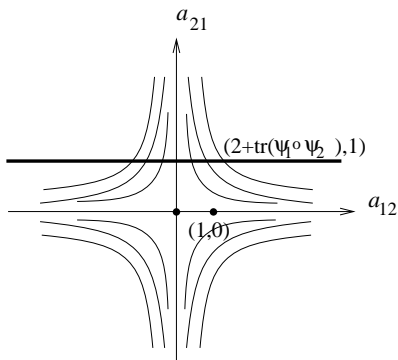
These ideas are schematically represented in Fig.2.

Figure 2: The point (1,0) and the points of the horizontal thick line represent all the orbits of pairs (ψ_1, ψ_2) of involutions that generate a non-Abelian group.

Remark 6.1 *Two pairs (ψ_1, ψ_2) and $(\tilde{\psi}_1, \tilde{\psi}_2)$ with orbit representatives out of the a_{12}-axis on the plane of Fig.1 are equivalent if, and only if, the compositions $\psi_1 \circ \psi_2$ and $\tilde{\psi}_1 \circ \tilde{\psi}_2$ are conjugate. In fact, the necessity follows from the definition of the equivalence of pairs of involutions; the sufficiency follows directly from the discussion above.*

This characterization of orbits has important applications to reversible systems as discussed in Section 8.

We now have that

$$a_{21} = 0 \Leftrightarrow \mathrm{Im}(\psi_2 - Id) = \mathrm{Fix}(\psi_1).$$

Furthermore, the equality $\mathrm{Im}(\psi_2 - Id) = \mathrm{Fix}(\psi_1)$ and the number $2 + \mathrm{tr}(\psi_1 \circ \psi_2)$ are invariant under linear simultaneous conjugacy. Therefore, Proposition 4.2 together with (6.12) gives the following theorem:

Theorem 6.2 *Let (φ_1, φ_2) be a pair of transversal linear involutions on $(\mathbb{R}^2, 0)$. Consider the group $\Lambda_2 = [\varphi_1, \varphi_2]$.*
(a) If Λ_2 is Abelian, then (φ_1, φ_2) is equivalent to the canonical pair $(\varphi_1^0, \varphi_2^0)$, where

$$\varphi_1^0(x, y) = (-x, y), \quad \varphi_2^0(x, y) = (x, -y). \qquad (6.13)$$

(b) Suppose now that Λ_2 is non-Abelian. If $\mathrm{Im}(\varphi_2 - Id) = \mathrm{Fix}(\varphi_1)$, then (φ_1, φ_2) is equivalent to $(\bar{\psi}_1, \bar{\psi}_2)$, where

$$\bar{\psi}_1(x, y) = (-x, y + x), \quad \bar{\psi}_2(x, y) = (x, -y). \qquad (6.14)$$

If $\mathrm{Im}(\varphi_2 - Id) \neq \mathrm{Fix}(\varphi_1)$, then (φ_1, φ_2) is equivalent to $(\bar{\psi}_1, \bar{\psi}_2)$, where

$$\bar{\psi}_1(x, y) = (-x, y + (2 + \mathrm{tr}(\varphi_1 \circ \varphi_2))x), \quad \bar{\psi}_2(x, y) = (x + y, -y). \qquad (6.15)$$

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The corollary below is the corresponding to the result presented in Remark 6.1 for general pairs of transversal linear involutions.

Corollary 6.3 *Two pairs (φ_1, φ_2) and $(\tilde{\varphi}_1, \tilde{\varphi}_2)$ of transversal linear involutions on $(\mathbb{R}^2, 0)$, with $Im(\varphi_2 - Id) \neq Fix(\varphi_1)$ and $Im(\tilde{\varphi}_2 - Id) \neq Fix(\tilde{\varphi}_1)$, are equivalent if, and only if, the compositions $\varphi_1 \circ \varphi_2$ and $\tilde{\varphi}_1 \circ \tilde{\varphi}_2$ are conjugate.*

6.2 Normal forms of divergent diagrams of folds

In this subsection we present the classification of divergent diagrams of folds

$$(f_1, f_2) : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2 \times \mathbb{R}^2, 0)$$

associated with pairs (φ_1, φ_2) of transversal linear involutions on $(\mathbb{R}^2, 0)$. This classification is obtained via the classification of pairs of involutions presented in the previous subsection and is given by the following theorem:

Theorem 6.4 *Let $(f_1, f_2) : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2 \times \mathbb{R}^2, 0)$ be a divergent diagram of folds associated with a pair (φ_1, φ_2) of transversal linear involutions on $(\mathbb{R}^2, 0)$. Consider the group $\Lambda_2 = [\varphi_1, \varphi_2]$.*
(a) If Λ_2 is Abelian, then (f_1, f_2) is equivalent to the canonical diagram (f_1^0, f_2^0), where

$$f_1^0(x, y) = (x^2, y), \quad f_2^0(x, y) = (x, y^2). \tag{6.16}$$

(b) Suppose Λ_2 non-Abelian. If $Im(\varphi_2 - Id) = Fix(\varphi_1)$, then (f_1, f_2) is equivalent to (g_1, g_2) with

$$g_1(x, y) = (x^2, y + \frac{1}{2}x), \quad g_2(x, y) = (x, y^2). \tag{6.17}$$

If $Im(\varphi_2 - Id) \neq Fix(\varphi_1)$, then (f_1, f_2) is equivalent to (g_1, g_2) with

$$g_1(x, y) = (x^2, y + (1 + \frac{1}{2}tr(\varphi_1 \circ \varphi_2))x), \quad g_2(x, y) = (x + \frac{1}{2}y, y^2). \tag{6.18}$$

Proof. It is a consequence of Proposition 4.3 and Theorem 6.2.

As we have already remarked, the number $tr(\varphi_1 \circ \varphi_2)$ is an invariant for the equivalence class of (f_1, f_2). It is now a consequence of Theorem 6.2 that this invariant determines, up to equivalence, the class of almost all diagrams of folds. This is the result below:

Corollary 6.5 *Let (f_1, f_2) and $(\tilde{f}_1, \tilde{f}_2)$ be two divergent diagrams of folds on $(\mathbb{R}^2, 0)$ associated with pairs of transversal linear involutions (φ_1, φ_2) and $(\tilde{\varphi}_1, \tilde{\varphi}_2)$, respectively, such that $Im(\varphi_2 - Id) \neq Fix(\varphi_1)$ and $Im(\tilde{\varphi}_2 - Id) \neq Fix(\tilde{\varphi}_1)$. Then (f_1, f_2) and $(\tilde{f}_1, \tilde{f}_2)$ are equivalent if, and only if, $tr(\varphi_1 \circ \varphi_2) = tr(\tilde{\varphi}_1 \circ \tilde{\varphi}_2)$.*

7 The cases $s = 2$, $n \geq 3$

In this section we obtain a generalization of the results of Section 6, that is, we give normal forms for the pairs of transversal linear involutions on $(\mathbb{R}^n, 0)$, $n \geq 3$, and the normal forms for the divergent diagrams of folds associated with these pairs. Let us recall that for $n = 2$ almost all normal forms depend on one parameter, namely the trace of the composition of the two involutions. One can notice that the same goes for $n \geq 3$. However, we have now an interesting bifurcation phenomenon with respect to the $n - 2$ new coordinates.

7.1 Normal forms of pairs of involutions

As before, to obtain the normal forms for the non-Abelian case we first consider pairs of linear involutions of the type (4.3). This is given in the next Proposition, whose proof relies on the description of the orbits on the parameter space.

Proposition 7.1 *Consider the pair (ψ_1, ψ_2) of transversal linear involutions on $(\mathbb{R}^n, 0)$, $n \geq 3$, given by*

$$\psi_1(x_1, \ldots, x_n) = (-x_1, x_2 + a_{12}x_1, x_3 + a_{13}x_1, \ldots, x_n + a_{1n}x_1),$$

$$\psi_2(x_1, \ldots, x_n) = (x_1 + a_{21}x_2, -x_2, x_3 + a_{23}x_2, \ldots, x_n + a_{2n}x_2),$$

with $\Lambda_2 = [\psi_1, \psi_2]$ non-Abelian, i.e., $a_{12}^2 + a_{21}^2 \neq 0$.
(a) Suppose $a_{12}a_{21} \neq 4$. If $a_{21} = 0$, then (ψ_1, ψ_2) is equivalent to $(\bar{\psi}_1, \bar{\psi}_2)$, where

$$\begin{aligned}
\bar{\psi}_1(x_1, \ldots, x_n) &= (-x_1, x_2 + x_1, x_3, \ldots, x_n), \\
\bar{\psi}_2(x_1, \ldots, x_n) &= (x_1, -x_2, x_3, \ldots, x_n).
\end{aligned} \tag{7.19}$$

If $a_{21} \neq 0$, then (ψ_1, ψ_2) is equivalent to $(\bar{\psi}_1, \bar{\psi}_2)$, where

$$\begin{aligned}
\bar{\psi}_1(x_1, \ldots, x_n) &= (-x_1, x_2 + (4 - n + tr(\psi_1 \circ \psi_2))x_1, x_3, \ldots, x_n), \\
\bar{\psi}_2(x_1, \ldots, x_n) &= (x_1 + x_2, -x_2, x_3, \ldots, x_n).
\end{aligned} \tag{7.20}$$

(b) Suppose $a_{12}a_{21} = 4$. If $(a_{23}, \ldots, a_{2n}) = \frac{-a_{21}}{2}(a_{13}, \ldots, a_{1n})$, then (ψ_1, ψ_2) is equivalent to $(\bar{\psi}_1, \bar{\psi}_2)$, where

$$\begin{aligned}
\bar{\psi}_1(x_1, \ldots, x_n) &= (-x_1, x_2 + 4x_1, x_3, \ldots, x_n), \\
\bar{\psi}_2(x_1, \ldots, x_n) &= (x_1 + x_2, -x_2, x_3, \ldots, x_n).
\end{aligned} \tag{7.21}$$

If $(a_{23}, \ldots, a_{2n}) \neq \frac{-a_{21}}{2}(a_{13}, \ldots, a_{1n})$, then (ψ_1, ψ_2) is equivalent to $(\bar{\psi}_1, \bar{\psi}_2)$, where

$$\begin{aligned}
\bar{\psi}_1(x_1, \ldots, x_n) &= (-x_1, x_2 + 4x_1, x_3, \ldots, x_n), \\
\bar{\psi}_2(x_1, \ldots, x_n) &= (x_1 + x_2, -x_2, x_3 + x_2, x_4, \ldots, x_n).
\end{aligned} \tag{7.22}$$

Remark 7.2 *Let us observe that*

$$a_{12}a_{21} = 4 - n + \operatorname{tr}(\psi_1 \circ \psi_2).$$

This equality is used to present the normal forms (7.20).

Proof of Proposition 7.1. Let (ψ_{1_b}, ψ_{2_b}) be a pair of transversal linear involutions on $(\mathbb{R}^n, 0)$, where

$$\psi_{1_b}(x_1, \ldots, x_n) = (-x_1, x_2 + b_{12}x_1, x_3 + b_{13}x_1, \ldots, x_n + b_{1n}x_1)$$

$$\psi_{2_b}(x_1, \ldots, x_n) = (x_1 + b_{21}x_2, -x_2, x_3 + b_{23}x_2, \ldots, x_n + b_{2n}x_2).$$

From Proposition 5.1, (ψ_1, ψ_2) and (ψ_{1_b}, ψ_{2_b}) are equivalent if, and only if, there exists an invertible matrix

$$H = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \alpha & 0 & \cdots & 0 \\ \delta_3 & \gamma_3 & \beta_{33} & \cdots & \beta_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ \delta_n & \gamma_n & \beta_{n3} & \cdots & \beta_{nn} \end{pmatrix}$$

such that

$$\begin{aligned} b_{12} &= \alpha a_{12} \\ b_{21} &= \tfrac{1}{\alpha} a_{21}, \end{aligned}$$

and, for $3 \le j \le n$,

$$\begin{aligned} b_{1j} &= -2\delta_j + \gamma_j a_{12} + \sum_{k=3}^n \beta_{jk} a_{1k} \\ b_{2j} &= \tfrac{1}{\alpha}(\delta_j a_{21} - 2\gamma_j + \sum_{k=3}^n \beta_{jk} a_{2k}). \end{aligned}$$

For $\alpha \ne 0$ fixed, let $L_\alpha : \mathbb{R}^{2n-4} \to \mathbb{R}^{2n-4}$ denote the linear operator defined by

$$\begin{aligned} L_\alpha(\delta_3, \ldots, \delta_n, \gamma_3, \ldots, \gamma_n) = \ & (-2\delta_3 + \gamma_3 a_{12}, \ldots, -2\delta_n + \gamma_n a_{12}, \\ & \tfrac{1}{\alpha}(\delta_3 a_{21} - 2\gamma_3), \ldots, \tfrac{1}{\alpha}(\delta_n a_{21} - 2\gamma_n)). \end{aligned}$$

And, for each linear isomorphism $\beta : \mathbb{R}^{n-2} \to \mathbb{R}^{n-2}$, let $v_{\alpha\beta}$ denote the vector

$$v_{\alpha\beta} = (\beta \times \frac{1}{\alpha}\beta)((a_{13}, \cdots, a_{1n}), (a_{23}, \ldots, a_{2n})) \in \mathbb{R}^{n-2} \times \mathbb{R}^{n-2} \equiv \mathbb{R}^{2n-4}.$$

With the notation above, (ψ_1, ψ_2) and (ψ_{1_b}, ψ_{2_b}) are equivalent if, and only if, $(b_{12}, b_{21}) = (\alpha a_{12}, \frac{1}{\alpha} a_{21})$ and $(b_{13}, \ldots, b_{1n}, b_{23}, \ldots, b_{2n}) \in \operatorname{Im}(T_{v_{\alpha\beta}} \circ L_\alpha)$ for some $\alpha \ne 0$ and some isomorphism β, where $T_{v_{\alpha\beta}} : \mathbb{R}^{2n-4} \to \mathbb{R}^{2n-4}$ is the translation in the $v_{\alpha\beta}$-direction.

Let us notice that L_α is an isomorphism if, and only if, $a_{12}a_{21} \neq 4$. So we now study all the orbit types by analysing first the case $a_{12}a_{21} \neq 4$ and then the case $a_{12}a_{21} = 4$.

(a) Suppose that $a_{12}a_{21} \neq 4$. In this case, $\mathrm{Im}(T_{v_{\alpha\beta}} \circ L_\alpha) = \mathbb{R}^{2n-4}$ for any $\alpha \neq 0$ and any isomorphism β. Then (ψ_1, ψ_2) is equivalent to (ψ_{1_b}, ψ_{2_b}) if, and only if, (ϕ_1, ϕ_2) is equivalent to (ϕ_{1_b}, ϕ_{2_b}), where these two last pairs of involutions on $(\mathbb{R}^2, 0)$ are given by (5.11):

$$\phi_1(x_1, x_2) = (-x_1, x_2 + a_{12}x_1),$$

$$\phi_2(x_1, x_2) = (x_1 + a_{21}x_2, -x_2)$$

and

$$\phi_{1_b}(x_1, x_2) = (-x_1, x_2 + b_{12}x_1),$$

$$\phi_{2_b}(x_1, x_2) = (x_1 + b_{21}x_2, -x_2).$$

From (6.12), we have the normal forms (7.19) and (7.20).

(b) Suppose now that $a_{12}a_{21} = 4$. In this case, for each $\alpha \neq 0$, $\dim \mathrm{Im}(L_\alpha) = n - 2$, and $(u_1, \ldots, u_{n-2}, u_{n-1}, \ldots, u_{2n-4}) \in \mathrm{Im}(L_\alpha)$ if, and only if, $(u_{n-1}, \ldots, u_{2n-4}) = \frac{-a_{21}}{2\alpha}(u_1, \ldots, u_{n-2})$. Hence, given an isomorphism β, we have that $v_{\alpha\beta} \in \mathrm{Im}(L_\alpha)$ if, and only if, $(a_{23}, \ldots, a_{2n}) = \frac{-a_{21}}{2}(a_{13}, \ldots, a_{1n})$. Then:

- If $(a_{23}, \ldots, a_{2n}) = \frac{-a_{21}}{2}(a_{13}, \ldots, a_{1n})$, then $v_{\alpha\beta} \in \mathrm{Im}(L_\alpha)$ and, therefore, $\mathrm{Im}(T_{v_{\alpha\beta}} \circ L_\alpha) = \mathrm{Im}(L_\alpha)$, for any isomorphism β. The normal form (7.21) is obtained by taking $(b_{12}, b_{21}) = (4, 1)$ and $(b_{13}, \ldots, b_{1n}, b_{23}, \ldots, b_{2n}) = (0, \ldots, 0, 0, \ldots, 0)$.

- If $(a_{23}, \ldots, a_{2n}) \neq \frac{-a_{21}}{2}(a_{13}, \ldots, a_{1n})$, then $v_{\alpha\beta} \notin \mathrm{Im}(L_\alpha)$ for any isomorphism β, and $\cup_\beta \mathrm{Im}(T_{v_{\alpha\beta}} \circ L_\alpha) = \mathbb{R}^{2n-4}$ - $\mathrm{Im}(L_\alpha)$. The normal form (7.22) is obtained by taking $(b_{12}, b_{21}) = (4, 1)$ and $(b_{13}, \ldots, b_{1n}, b_{23}, \ldots, b_{2n}) = (0, \ldots, 0, 1, \ldots, 0)$.

Proposition 7.1 gives normal forms of pairs of transversal linear involutions on $(\mathbb{R}^n, 0)$, $n \geq 3$, based on conditions on pairs of the form (4.3). We can now explicit the bifurcation mentioned in the beginning of this section. It occurs when

$$a_{12}a_{21} = 4 \quad \text{and} \quad (a_{23}, \ldots, a_{2n}) = -\frac{a_{21}}{2}(a_{13}, \ldots, a_{1n}),$$

corresponding to the normal form (7.21).

As pointed out in Remark 7.2, we have that

$$a_{12}a_{21} = 4 - n + \mathrm{tr}(\psi_1 \circ \psi_2). \tag{7.23}$$

We now observe that

$$a_{21} = 0 \Leftrightarrow \text{Im}(\psi_2 - Id) \subset \text{Fix}(\psi_1) \tag{7.24}$$

and, if $a_{12}a_{21} = 4$,

$$(a_{23}, \ldots, a_{2n}) = \frac{-a_{21}}{2}(a_{13}, \ldots, a_{1n}) \Leftrightarrow \text{Im}(\psi_1 - Id) = \text{Im}(\psi_2 - Id). \tag{7.25}$$

The right-hand side of the three statements above are invariant under linear simultaneous conjugacy. With this fact we can rewrite Proposition 7.1 in terms of conditions on general pairs of transversal linear involutions. Then we have:

Theorem 7.3 *Let (φ_1, φ_2) be a pair of transversal linear involutions on $(\mathbb{R}^n, 0)$, $n \geq 3$. Consider the group $\Lambda_2 = [\varphi_1, \varphi_2]$.*
(a) If Λ_2 is Abelian, then (φ_1, φ_2) is equivalent to the canonical pair $(\varphi_1^0, \varphi_2^0)$, where

$$\varphi_1^0(x_1, \ldots, x_n) = (-x_1, x_2, x_3, \ldots, x_n), \quad \varphi_2^0(x_1, \ldots, x_n) = (x_1, -x_2, x_3, \ldots, x_n). \tag{7.26}$$

(b) Let now Λ_2 be non-Abelian. We have to consider two cases:
(b1) Suppose that $\text{tr}(\varphi_1 \circ \varphi_2) \neq n$. If $\text{Im}(\varphi_2 - Id) \subset \text{Fix}(\varphi_1)$, then (φ_1, φ_2) is equivalent to $(\bar{\psi}_1, \bar{\psi}_2)$ given in (7.19). If $\text{Im}(\varphi_2 - Id) \not\subset \text{Fix}(\varphi_1)$, then (φ_1, φ_2) is equivalent to $(\bar{\psi}_1, \bar{\psi}_2)$ given by

$$\begin{aligned} \bar{\psi}_1(x_1, \ldots, x_n) &= (-x_1, x_2 + (4 - n + \text{tr}(\varphi_1 \circ \varphi_2))x_1, x_3, \ldots, x_n), \\ \bar{\psi}_2(x_1, \ldots, x_n) &= (x_1 + x_2, -x_2, x_3, \ldots, x_n). \end{aligned} \tag{7.27}$$

(b2) Suppose that $\text{tr}(\varphi_1 \circ \varphi_2) = n$. If $\text{Im}(\varphi_1 - Id) = \text{Im}(\varphi_2 - Id)$, then (φ_1, φ_2) is equivalent to $(\bar{\psi}_1, \bar{\psi}_2)$ given in (7.21). If $\text{Im}(\varphi_1 - Id) \neq \text{Im}(\varphi_2 - Id)$, then (φ_1, φ_2) is equivalent to $(\bar{\psi}_1, \bar{\psi}_2)$ given in (7.22).

7.2 Normal forms of divergent diagrams of folds

We now move on to the diagrams of folds. The classification theorem is as follows:

Theorem 7.4 *Let $(f_1, f_2) : (\mathbb{R}^n, 0) \to (\mathbb{R}^n \times \mathbb{R}^n, 0)$ be a divergent diagram of folds associated with a pair (φ_1, φ_2) of transversal linear involutions on $(\mathbb{R}^n, 0)$, $n \geq 3$. Consider the group $\Lambda_2 = [\varphi_1, \varphi_2]$.*
(a) If Λ_2 is Abelian, then (f_1, f_2) is equivalent to the canonical diagram (f_1^0, f_2^0), where

$$f_1^0(x_1, \ldots, x_n) = (x_1^2, x_2, x_3, \ldots, x_n), \quad f_2^0(x_1, \ldots, x_n) = (x_1, x_2^2, x_3, \ldots, x_n). \tag{7.28}$$

(b) Let Λ_2 be non-Abelian.

(b1) Suppose that $\mathrm{tr}(\varphi_1 \circ \varphi_2) \neq n$. If $Im(\varphi_2 - Id) \subset Fix(\varphi_1)$, then (f_1, f_2) is equivalent to (g_1, g_2) with

$$
\begin{aligned}
g_1(x_1, \ldots, x_n) &= (x_1^2, x_2 + \tfrac{1}{2}x_1, x_3, \ldots, x_n), \\
g_2(x_1, \ldots, x_n) &= (x_1, x_2^2, x_3, \ldots, x_n).
\end{aligned}
\tag{7.29}
$$

If $Im(\varphi_2 - Id) \not\subset Fix(\varphi_1)$, then (f_1, f_2) is equivalent to (g_1, g_2) with

$$
\begin{aligned}
g_1(x_1, \ldots, x_n) &= (x_1^2, x_2 + \tfrac{1}{2}(4 - n + \mathrm{tr}(\varphi_1 \circ \varphi_2))x_1, x_3, \ldots, x_n), \\
g_2(x_1, \ldots, x_n) &= (x_1 + \tfrac{1}{2}x_2, x_2^2, x_3, \ldots, x_n).
\end{aligned}
\tag{7.30}
$$

(b2) Suppose that $\mathrm{tr}(\varphi_1 \circ \varphi_2) = n$. If $Im(\varphi_1 - Id) = Im(\varphi_2 - Id)$, then (f_1, f_2) is equivalent to (g_1, g_2), with

$$
\begin{aligned}
g_1(x_1, \ldots, x_n) &= (x_1^2, x_2 + 2x_1, x_3, \ldots, x_n), \\
g_2(x_1, \ldots, x_n) &= (x_1 + \tfrac{1}{2}x_2, x_2^2, x_3, \ldots, x_n).
\end{aligned}
\tag{7.31}
$$

If $Im(\varphi_1 - Id) \neq Im(\varphi_2 - Id)$, then (f_1, f_2) is equivalent to (g_1, g_2) with

$$
\begin{aligned}
g_1(x_1, \ldots, x_n) &= (x_1^2, x_2 + 2x_1, x_3, \ldots, x_n), \\
g_2(x_1, \ldots, x_n) &= (x_1 + \tfrac{1}{2}x_2, x_2^2, x_3 + \tfrac{1}{2}x_2, x_4, \ldots, x_n).
\end{aligned}
\tag{7.32}
$$

8 Divergent diagrams of folds, pairs of involutions and dynamical systems

The works in [9], [11], [14] show the usefulness of the tools of singularities of mappings in the study of the dynamics of smooth vector fields near the boundary of a manifold. In [11] it is also discussed the classification of singularities of discontinuous vector fields by means of the theory of singularities of mappings and a strong relationship between such systems and divergent diagrams of mappings is established. In Subsection 8.1 we explicit this relationship with attention to diagrams of pairs of folds associated with pairs of transversal involutions. An application of our results to discrete reversible systems is presented in Subsection 8.2. It is interesting to note that, in a special situation, divergent diagrams of folds turn out to be a link between the study of discontinuous vector fields and the study of reversible diffeomorphisms.

8.1 Discontinuous vector fields

In this subsection we elaborate the idea of how our main results can be applied as a first step towards the classification and the dynamics of a special class of discontinuous vector fields.

Let Z be a germ of a vector field on $(\mathbb{R}^{n+1}, 0)$ given by

$$Z(x_1, \ldots, x_{n+1}) = \begin{cases} X(x_1, \ldots, x_{n+1}), & \text{if } x_{n+1} > 0 \\ Y(x_1, \ldots, x_{n+1}), & \text{if } x_{n+1} < 0, \end{cases}$$

where X and Y are germs of smooth vector fields on $(\mathbb{R}^{n+1}, 0)$, with $X(0), Y(0) \neq 0$. This means that Z can have discontinuities on the hyperplane $H = \{x \in \mathbb{R}^{n+1} \; ; \; x_{n+1}{=}0 \}$. We recall that the orbits passing through points of H follow the Filippov's rule [3].

Consider the following generic situation: Let $t \mapsto \gamma_X(0, t)$ and $t \mapsto \gamma_Y(0, t)$ be the orbits of X and Y passing through 0. Assume that $\gamma_X(0) = \gamma_Y(0) = 0$, $\pi(\gamma_X'(0)) = \pi(\gamma_Y'(0)) = 0$, $\pi(\gamma_X''(0)) < 0$ and $\pi(\gamma_Y''(0)) > 0$, π being the canonical projection of $\mathbb{R}^{n+1}$ onto x_{n+1}-axis. Assume also that associated with X (resp. Y), there exists a codimension-one smooth submanifold M_X (resp. M_Y) in $(H, 0)$ defined by the points of H where X (resp. Y) is tangent to H. Suppose that M_X and M_Y are transversal at 0.

Now X (resp. Y) induces around 0 on H a smooth diffeomorphism φ_X (resp. φ_Y) defined as follows: If $(x_1, \ldots, x_n, 0) \in M_X$, then $\varphi_X(x_1, \ldots, x_n, 0) = (x_1, \ldots, x_n, 0)$; otherwise, $\varphi_X(x_1, \ldots, x_n, 0)$ is the point different from $(x_1, \ldots, x_n, 0)$ where the orbit of X passing through $(x_1, \ldots, x_n, 0)$ meets H. Similarly, we define φ_Y. We observe that φ_X and φ_Y are involution on $(H, 0)$.

Let $H_X^\perp$ and $H_Y^\perp$ be any cross sections of X and Y at 0, respectively. For each $p \in (H, 0)$, there exists a unique $t = t(p)$ in $(\mathbb{R}, 0)$ such that the orbit $t \mapsto \gamma_X(p, t)$ of X through p meets $H_X^\perp$ at a point $q = \gamma_X(p, t(p))$. Analogously, there exists a unique $\tilde{t} = \tilde{t}(p)$ in $(\mathbb{R}, 0)$ such that the orbit $t \mapsto \gamma_Y(p, t)$ of Y through p meets $H_Y^\perp$ at a point $\tilde{q} = \gamma_Y(p, \tilde{t}(p))$. So we can define the divergent diagram

$$(f_X, f_Y) : (H, 0) \quad \begin{array}{c} \overset{f_X}{\nearrow} (H_X^\perp, 0) \\[1ex] \underset{f_Y}{\searrow} \\[1ex] (H_Y^\perp, 0) \end{array}$$

where $f_X(p) = q$ and $f_Y(p) = \tilde{q}$. We notice that the singular set $\Sigma(f_X)$ (resp. $\Sigma(f_Y)$) and M_X (resp. M_Y) coincide. By taking charts, we can suppose that $H = H_X^\perp = H_Y^\perp = \mathbb{R}^n$ so that this diagram gives rise to a diagram of folds associated with the pair (φ_X, φ_Y) of involutions defined above.

8.2 Reversible diffeomorphisms

Given an involution φ on $(\mathbb{R}^n, 0)$, we say that a germ of diffeomorphism $F : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is φ-reversible if

$$\varphi \circ F = F^{-1} \circ \varphi.$$

The presence and importance of reversing symmetry has been recognized since the early days of dynamical systems by Birkhoff (for more details, see [7]). Note that for any two involutions φ_1 and φ_2 on $(\mathbb{R}^n, 0)$, the composition $F = \varphi_1 \circ \varphi_2$ is φ_1-reversible. Conversely, a germ of diffeomorphism F with an involutory reversing symmetry φ_1 can always be written as the composition of two involutions:

$$F = \varphi_1 \circ \varphi_2.$$

As a consequence, some authors have addressed the study of $F = \varphi_1 \circ \varphi_2$ to the study of the pair of involutions (φ_1, φ_2). For example, this is the approach used by Jacquemard and Teixeira in [6].

Here there is a point we want to remark: For any two pairs of involutions (φ_1, φ_2) and $(\tilde{\varphi}_1, \tilde{\varphi}_2)$ that are equivalent, the compositions $\varphi_1 \circ \varphi_2$ and $\tilde{\varphi}_1 \circ \tilde{\varphi}_2$ generate conjugate reversible systems. The converse of this property does not hold in general. For example, if we take

$$\begin{aligned}
\varphi_1(x_1, \ldots, x_n) &= (-x_1, x_2, \ldots, x_n), \\
\varphi_2(x_1, \ldots, x_n) &= (x_1 + ax_2, -x_2, x_3, \ldots, x_n),
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\varphi}_1(x_1, \ldots, x_n) &= (x_1 + bx_2, -x_2, x_3, \ldots, x_n), \\
\tilde{\varphi}_2(x_1, \ldots, x_n) &= (-x_1 - (a+b)x_2, x_2, \ldots, x_n),
\end{aligned}$$

with $a \neq 0$ and arbitrary b, then the compositions are equal, but (φ_1, φ_2) and $(\tilde{\varphi}_1, \tilde{\varphi}_2)$ are not equivalent, according to Theorem 6.2 for $n = 2$ and Theorem 7.3 for $n \geq 3$.

So we may ask what kinds of restrictions the study of the pair (φ_1, φ_2) can impose to the study of the dynamics associated to the composition $\varphi_1 \circ \varphi_2$. The classification theorems just mentioned above reveal that, up to equivalence, there is no restriction for almost all pairs (φ_1, φ_2) of transversal linear involutions. In fact, almost all normal forms are characterized by $\mathrm{tr}(\varphi_1 \circ \varphi_2)$.

Now we present the description of reversible linear diffeomorphisms on the plane given by the composition of involutions that occur in Corollary 6.3. First we observe that any rotation R_θ of angle θ, $\theta \in [0, 2\pi)$, can be written as the composition of two reflections. For $\theta \neq 0, \pi$, if R_θ is the composition of two linear involutions, then these are necessarily reflections.

Suppose that F is a linear diffeomorphism given by the composition of two involutions, $F = \varphi_1 \circ \varphi_2$, with (φ_1, φ_2) in the conditions of Corollary 6.3. Since

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$\det(F) = 1$, then $-2 < \text{tr}(F) < 2$ if, and only if, F is conjugate to a rotation. Now, by Theorem 6.2, the pair (φ_1, φ_2) is represented by the point $(2+\text{tr}(F), 1)$ on the horizontal line $a_{21} = 1$ of Fig.2, that is, we can assume that the involutions are given by

$$\varphi_1(x, y) = (-x, y + (2 + \text{tr}(F))x), \quad \varphi_2(x, y) = (x + y, -y).$$

Then, it follows that $-2 < \text{tr}(F) < 2$ gives the segment on this line of the pairs equivalent to pairs of reflections. If $\text{tr}(F) > 2$ or $\text{tr}(F) < -2$, then F corresponds to a linear hyperbolic φ_1-reversible diffeomorphism.

One last remark on this subject is concerned with a geometrical interpretation of the equivalence between two pairs of transversal planar reflections with respect to the fixed-point lines of the reflections. Recall that if φ_1 and φ_2 are reflections, then the anticlockwise angle from the line $\text{Fix}(\varphi_2)$ to the line $\text{Fix}(\varphi_1)$ is half the angle θ of rotation $R_\theta = \varphi_1 \circ \varphi_2$. So we can conclude that two pairs of transversal planar reflections are equivalent if, and only if, the angles between the lines of fixed point of each pair are equal.

The study of the planar case leads to a similar analysis of linear reversible diffeomorphisms $F = \varphi_1 \circ \varphi_2$ on $\mathbb{R}^n$, $n \geq 3$, where (φ_1, φ_2) is a pair of transversal linear involutions. In fact, all normal forms given by Theorem 7.3 except (7.22) are suspensions of the normal forms given by Theorem 6.2.

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