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Topological quantum codes on compact surfaces with genus $g \geq 2$

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In this paper we propose a construction procedure of a class of topological quantum error-correcting codes on surfaces with genus $g \geq 2$. This generalizes the toric codes construction. We also tabulate all possible surface codes with genus 2–5. In particular, this construction reproduces the class of codes obtained when considering the embedding of complete graphs K_s, for $s \equiv 1 \bmod 4$, on surfaces with appropriate genus. We also show a table comparing the rate of different codes when fixing the distance to 3–5. © *2009 American Institute of Physics.*
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I. INTRODUCTION

Shor[1] showed that quantum computers can solve complex problems faster and more efficiently, which at least, in principle, would not be possible with classical computers. Research on the construction of quantum computers and quantum gates (quantum logical devices) is increasing and several proposals have been presented. Two obstacles must be overcome to achieve the practical implementation of a quantum computer: decoherence due to interactions of the quantum system with the surrounding environment may destroy the quantum characteristic; and imperfection in quantum logic gates during the execution of a computation. However, these obstacles may be cleverly overcome by the use of quantum error-correcting codes as has been previously shown by a number of researchers.

In classical computation, errors may be corrected by using parity-check digits, a form of redundancy. This is not possible in quantum computation due to the noncloning theorem. In addition to that, there are other fundamental differences between classical and quantum informations that can be circumvented by means of quantum error correction theory, such as the existence of phase-flip errors that do not occur classically and the disturbance of the quantum coherence during syndrome measurement. Fortunately, there is a way out of this difficulty so that the construction of quantum error-correcting codes, based on the properties of classical error-correcting codes, is possible. A quantum error-correcting code may be viewed as a mapping of a k-dimensional Hilbert subspace into a n-dimensional Hilbert space, where $n > k$. The k qubits to be protected from errors are called *logical* qubits or *encoded* qubits, and the remaining $n-k$ qubits are the added redundancy.

The first quantum code was proposed by Shor.[2] It is a nine qubit quantum code capable of correcting one arbitrary quantum error. Calderbank and Shor[3] showed that there are good quantum error-correcting codes with good rate and greater error-correcting capabilities than Shor's code. Independently, Steane[4] proposed a construction yielding similar codes as those proposed by Calderbank and Shor. This led to the class of codes known as CSS codes. Gottesman[5] proposed a

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class of codes called *stabilizer codes* from which the previous codes are subclasses. The stabilizer formalism is based on group theory. The main idea is that quantum states may be better described by the operators that stabilize these quantum states.

A fundamental difference between *bits* and *qubits* is that the qubits may assume countless distinct states called *superpositions*, which are linear combinations of $|0\rangle$ and $|1\rangle$. The superposition states are so fragile that a disturbance by the surrounding environment may destroy them. A solution to this problem is to minimize the qubit interactions with the environment. However, the construction of a functional machine with a large number of very isolated qubits is a difficult problem.

An alternative solution is the topological quantum codes. The topological quantum error-correcting codes (TQCs) or surface codes, introduced by Kitaev,[6] are a special type of stabilizer codes. Each code in this class is associated with a tessellation of a surface (or a bidimensional manifold). A well-known example of a TQC is the toric code proposed by Kitaev,[6] whose qubits are in a one-to-one correspondence with the edges of the tessellation $\{4,4\}$ of the flat torus. A similar construction, however, in the projective plane $\mathbb{R}P^2$, is proposed in Ref. 7.

The central idea of TQC is to make the quantum states depend on topological properties of a physical system due to the fact that topological properties are invariant under smooth degradations. Thus, the information stored in the topology of the system makes the system resilient to the noise effects. This is a form of undertaking fault-tolerant quantum computation.

Another advantage of TQC is related to the locality of the stabilizer operators. Besides the simplicity and the effectiveness of using these operators, each one of them is involved with a few qubits in the code block. Since these qubits are close to one another, it is possible to realize measurements using only a few quantum gates.[8] These operators constitute a Hamiltonian with local interactions, whose ground state coincides with the code subspace.[6] These interactions control the intrinsic mechanism of protecting the encoded quantum states.

Within the class of topological codes, there is a subclass called *homological codes*, introduced by Bombin and Martin-Delgado.[9] In this work, homological quantum codes on surfaces of arbitrary genus is proposed, and the construction of such codes is based on graph theory. As a consequence, the tools of homology group theory for graphs embedded on surfaces are strongly used.

The aim of this paper is to propose a construction of TQCs on surfaces with genus $g \geq 2$, making use of the concepts of hyperbolic geometry. The motivation comes from the results shown in Refs. 10 and 11, where the performance of a communication system using signal constellations (digital modulation) in spaces with constant curvature $K < 0$, or equivalently, on surfaces with genus $g \geq 2$, is better, in terms of the error probability, than the signal constellations in spaces with constant curvature $K \geq 0$. Since digital modulation may be viewed as a class of codes, we conjecture that it is possible to construct more efficient quantum error-correcting codes on surfaces with genus $g \geq 2$.

Kitaev's construction of toric codes is defined on an $l \times l$ square tessellation of the torus, a surface with genus $g = 1$. The codes presented in this paper are constructed in a similar way as the Kitaev toric codes, however, considering surfaces with genus $g \geq 2$. Since these surfaces are strongly related to hyperbolic geometry, several Euclidean geometric properties cannot be employed. To circumvent this problem we make use of the strong connection between hyperbolic tessellation and plane models of surfaces as the fundamental mathematical concept to construct the TQC codes on these surfaces.

This paper is organized as follows. In Sec. II, the basic notions and definitions of stabilizer codes and TQC codes are reviewed. In Sec. III, we introduce the important definitions and theorems of hyperbolic geometry for the purpose of this paper. In Sec. IV, the construction of TQCs on surfaces with genus $g \geq 2$ is proposed. In Sec. V, several tables showing the parameters of the constructed codes are presented. In particular, the proposed construction reproduces the results shown in Ref. 9, that is, the class of homological codes obtained by the embedding of complete graphs K_s, with $s \equiv 1 \bmod 4$, on surfaces with appropriate genus.

Part of this paper was presented in the IEEE Information Theory Workshop 2008 and published in the proceedings of this event.[12]

II. TQCs

One way of characterizing quantum error-correcting codes is by employing the stabilizer group $\mathcal{S}$, that is, an Abelian subgroup of the Pauli group P_n. The Pauli group is the tensor products of the Pauli operators I, σ_x, σ_y, and σ_z, that is, $P_n = \pm\{I, \sigma_x, \sigma_y, \sigma_z\}^{\otimes n}$, where

$$I \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_x \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

A stabilizer code $\mathcal{C}$ associated with $\mathcal{S}$ is thus the simultaneous eigenspace, with eigenvalue +1, of all the elements of $\mathcal{S}$, i.e., $\mathcal{C} = \{|\psi\rangle : \mathcal{M}|\psi\rangle = |\psi\rangle \, \forall \, \mathcal{M} \in \mathcal{S}\}$.

Each operator $\mathcal{M} \in \mathcal{S}$ is a Pauli operator. If $\mathcal{S}$ has $n-k$ generators, then the dimension of $\mathcal{C}$ is 2^k, that is, $\mathcal{C}$ encodes k qubits. The generators of $\mathcal{S}$ can be viewed as the parity-check operators of a quantum code, in the same way as the generator polynomial of the dual code and its shifts may be viewed as the parity-check matrix of a classical code. Since the stabilizer $\mathcal{S}$ is an Abelian subgroup of P_n, the operators of P_n which do not commute with every $\mathcal{M} \in \mathcal{S}$ map $\mathcal{C}$ into its orthogonal complement, and so are detectable errors. However, there are many elements of P_n that commute with every $\mathcal{M} \in \mathcal{S}$ which do not belong to $\mathcal{S}$. Operators with this characteristic preserve the coding space, by not acting trivially on it, although they may corrupt the information. The code distance is given by the least weight of $E \in P_n$ such that E commutes with every $\mathcal{M} \in \mathcal{S}$ but does not belong to $\mathcal{S}$.

To achieve error-correcting operation in a stabilizer code the eigenvalues of the stabilizer generators should be measured. The eigenvalue of each $\mathcal{M} \in \mathcal{S}$ is $(-1)^{f_{\mathcal{M}}(E)}$, where $f_{\mathcal{M}}(E)$ is 0 if $\mathcal{M}$ commutes with E or 1 if $\mathcal{M}$ anticommutes with E, whenever $E \in P_n$. This provides the error syndrome. If the code is nondegenerate, then each error has a distinct syndrome, and consequently the syndrome points out the errors. However, if the code is degenerate, then there are distinct errors yielding the same syndrome. Therefore, the set of degenerate errors is pointed out. In both cases, we have to apply the error operator to fix the state. More information on stabilizer codes may be found in Refs. 13 and 14.

The TQCs are defined as follows.

Definition 1: Let $\mathbb{M}$ be a compact surface and $\{p,q\}$ a tessellation (see Sec. IV) of $\mathbb{M}$ with E edges, V vertices, and F faces. Given a vertex $v \in V$ and a face $f \in F$, we define the operators A_v as the tensor product of σ_x corresponding to each edge having v as the common vertex and the operators B_f as the tensor product of σ_z corresponding to each edge forming the border of the face f. A TQC $\mathcal{C}$ with length $n = |E|$, with stabilizer $\mathcal{S} = \{A_v | v \in V\} \cup \{B_f | f \in F\}$, encodes $k = 2g$ qubits (if the surface has no border) and its distance is $d = min\{\delta, \delta^*\}$, where δ denotes the code distance in the tessellation $\{p,q\}$, whereas δ^* denotes the code distance in the dual tessellation $\{q,p\}$.

Example 1: The toric codes with parameters $[[2l^2, 2, l]]$ are a class of TQCs whose qubits are in a one-to-one correspondence with the edges of the tessellation {4,4} of the $l \times l$ regular polygon, that is, the flat torus, see Fig. 1. In this tessellated region there are $2l^2$ edges or, equivalently, $n = 2l^2$ qubits. The stabilizer operators of this class of codes are associated with each vertex and each face of the tessellation, such that a nonzero operator (σ_x or σ_z) acts on the four qubits directly connected to the vertex or face in consideration, whereas the identity operator acts on the remaining qubits. Thus, the toric code consists of the space which is fixed by these operators. The dimension of $\mathcal{C}$ is 4, that is, $\mathcal{C}$ encodes $k=2$ qubits. The distance of the toric code is given by the minimum number of edges in the tessellation contained either in the shortest homologically nontrivial cycle (see Sec. IV D) of the tessellation or in the shortest homologically nontrivial cycle of the dual tessellation. Since a homologically nontrivial cycle is an edge path in the tessellation that cannot be shrunk to a face, it follows that on the flat torus, the shortest homologically nontrivial cycle corresponds exactly to the orthogonal axis of the tessellation, see Fig. 2. It follows from this that $d = l$.

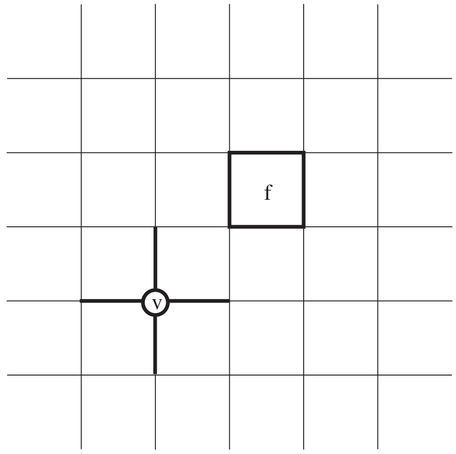

FIG. 1. {4,4} tessellation drawn on the torus.

We would like to emphasize some main aspects of Kitaev's construction.

- The parameters n and k of the code are defined as $n=|E|$ and $k=2g$, where $|E|$ denotes the number of edges of the tessellation. The number of encoded qubits is related to the number of the essential cycles of the surface. Two essential cycles (meridian and parallel) exist in the case of the torus.
- Another important aspect of this construction is the definition of the stabilizer operators,

$$A_v = \bigotimes_{j\in E_v} \sigma_x^j, \quad B_f = \bigotimes_{j\in E_f} \sigma_z^j,$$

  where E_v denotes the set of edges having v as the common vertex and E_f denotes the set of edges forming the border of f. These operators are the generators of the stabilizer group. Thus, the code is given by $\mathcal{C}=\{|\psi\rangle : A_v|\psi\rangle=|\psi\rangle, \quad B_f|\psi\rangle=|\psi\rangle \forall v,f\}$.
- A homologically trivial cycle bounds a region that may be tiled by the fundamental region of a tessellation, whereas a homologically nontrivial cycle may not be the border of any region, see Fig. 2. The distance of toric codes is defined as the minimum between the number of edges in a smallest homologically nontrivial cycle of the tessellation and of the dual

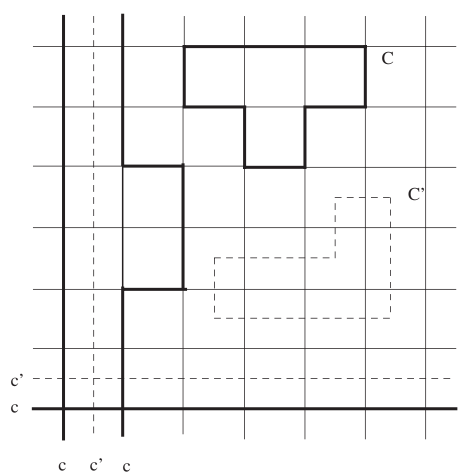

FIG. 2. The cycles C and C' are homologically trivial cycles on the tessellation and dual tessellation, respectively. The cycles c and c' are homologically nontrivial cycles on the tessellation and on the dual tessellation, respectively.

tessellation.[8] In the torus case, the smallest homologically nontrivial cycle on an $l \times l$ square tessellation coincides with the orthogonal axes of the tessellation. The same occurs with the dual tessellation. Therefore, $d=l$.

- Toric codes detect $l-1$ errors and correct $\lfloor (l-1)/2 \rfloor$ errors. These codes do not reach the quantum Hamming bound.
- As above, the usual scheme for error correction used by the stabilizer codes is brought forth by the measurement of the syndrome. In the case of toric codes the syndrome is highly ambiguous due to the code being fourfold degenerated. However, Kitaev suggested an error correction at the physical level by employing a Hamiltonian

$$H_0 = -\sum_s A_s - \sum_p B_p. \tag{1}$$

  Since the operators A_v and B_f commute, the Hamiltonian is diagonalized. Note that the ground state of the Hamiltonian coincides with the subspace protected by the code. Since the difference between the eigenvalues of A_v and B_f is equal to 2, all excited states are separated by an energy gap greater than or equal to 2.[6] Thus, they may be distinguished and corrected.
- Kitaev showed that the toric codes may be implemented by particles called *anyons*. These particles exist only in a bidimensional world and have the necessary mathematical properties for the implementation of the toric codes.[6]

III. HYPERBOLIC GEOMETRY

Since our goal is to generalize Kitaev's code construction by use of surfaces with genus $g \geq 2$, it follows that the geometry to be considered is the hyperbolic geometry. Therefore, in this section we revise some basic concepts of hyperbolic geometry necessary for the development of this paper. For a thorough revision of this subject, we refer the reader to Refs. 15 and 16.

We consider as models of the hyperbolic geometry the *upper-half plane*, $\mathbb{H}^2=\{z \in \mathbb{C}: Im(z) > 0\}$, and the *Poincaré disk*, $\mathbb{D}^2=\{z \in \mathbb{C}: |z| < 1\}$. The border of $\mathbb{H}^2$ is given by $\partial\mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$ and the border of $\mathbb{D}^2$ is $\partial\mathbb{D}^2=\{z \in \mathbb{C}: |z|=1\}$. Both are called *circle at infinity*.

The space $\mathbb{H}^2$ with the metric

$$ds = \frac{\sqrt{dx^2+dy^2}}{y}$$

is known as *hyperbolic* or *Lobachevski plane*.

There exists an isometry $f:\mathbb{H}^2 \to \mathbb{D}^2$ given by

$$f(z) = \frac{zi+1}{z+i},$$

whose metric induced by f is a metric in $\mathbb{D}^2$. This allows us to work in either model according to the necessity.

Thus, $\mathbb{D}^2$, with the metric

$$ds = \frac{|2dw|}{1-|w|^2},$$

is also the *hyperbolic plane*.

Since Poincaré disk is a bounded subset of the Euclidean plane, this model is more convenient for the visualization of the hyperbolic plane. On the other hand, the upper-half plane model allows us to make use of the Cartesian coordinates in the calculus.

Let $\sigma:[a,b] \to \mathbb{H}^2$ be a piecewise differentiable path. Then the *hyperbolic length* of σ is defined by

$$h(\sigma) = \int_a^b \frac{\left|\frac{dz}{dt}\right|}{y(t)} dt.$$

The *hyperbolic distance* between any two given points $z,w \in \mathbb{H}^2$ is given by $d(z,w) = \inf h(\sigma)$, where the infimum is considered over the set of all paths σ connecting z to w in $\mathbb{H}^2$.

Definition 2: Geodesics are paths with the least hyperbolic length joining two distinct points.

Any two points $z,w \in \mathbb{H}^2$ may be connected by a unique geodesic, and the hyperbolic distance between them is equal to the hyperbolic length of the unique geodesic that connects them. Geodesics in $\mathbb{H}^2$ are semicircles and straight lines both orthogonal to the real axis $\mathbb{R}$, whereas the geodesics in $\mathbb{D}^2$ are segments of Euclidean circles orthogonal to $\partial\mathbb{D}^2$ and its diameters.[15]

The *hyperbolic angle* between two geodesics in $\mathbb{H}^2$ with intersection point z is the (Euclidean) angle between the tangent vectors to the geodesics.

The Gauss–Bonnet theorem shows that the hyperbolic area of a hyperbolic triangle depends only on its angles. Recall that the sum of the internal angles of a hyperbolic triangle is less than π.

Theorem 1: (Gauss–Bonnet) Let Δ be a hyperbolic triangle with internal angles α,β,γ. Then the area of Δ is given by

$$\mu(\Delta) = \pi - \alpha - \beta - \gamma.$$

The next definitions are important due to the fact that the edge pairings of a regular hyperbolic polygon (plane model of surfaces, see Sec. IV A) is realized by the elements of a Fuchsian group.

Consider the multiplicative group of the 2×2 real matrices,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

with $a,b,c,d \in \mathbb{R}$ and $\det(A)=1$. This group is known as the *unimodular group* and it is denoted by $SL(2,\mathbb{R})$.

A *Möbius transformation* is a mapping $T:\mathbb{C}\to\mathbb{C}$ defined by $T(z)=(az+b)/(cz+d)$, where $a,b,c,d \in \mathbb{R}$ such that $ad-bc=1$. The *projective special linear* group, denoted by $\mathrm{PSL}(2,\mathbb{R})$, is the multiplicative group of Möbius transformations, equivalently, $\mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{SL}(2,\mathbb{R}/\langle \pm I_2\rangle)$.

The group $\mathrm{PSL}(2,\mathbb{R})$ is a subgroup of the group of all isometries of $\mathbb{H}^2$, denoted by $I\,\mathrm{som}(\mathbb{H}^2)$. Consequently, any transformation in $\mathrm{PSL}(2,\mathbb{R})$ takes geodesic into geodesic. Furthermore, Möbius transformations are *conformal transformations*, that is, they preserve angles. From this it follows that the hyperbolic area is invariant under all transformations in $\mathrm{PSL}(2,\mathbb{R})$.

Definition 3: A Fuchsian group is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$.

Consider a metric space X and a group G of homeomorphisms of X. We say that a group G acts properly discontinuously in X if the G-orbit of any point $x \in X$, $G(x)=\{T(x):T\in G\}$, is locally finite. Fuchsian groups are characterized in the following theorem.

Theorem 2: (Reference 15) $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ is a Fuchsian group if and only if Γ acts properly discontinuously on $\mathbb{H}^2$.

Definition 4: Let X be a metric space and Γ a group of homeomorphisms acting properly discontinuously on X. A closed subset $\tilde{F} \subset X$ with nonempty interior is called a *fundamental region* of Γ if

(i) $\bigcup_{T\in\Gamma} T(\tilde{F}) = X$,

(ii) $\mathrm{int}\,\tilde{F} \cap T(\mathrm{int}\,\tilde{F}) = \varnothing, \quad \forall T \in \Gamma - \{Id\}$,

where $\mathrm{int}\,\tilde{F}$ is the set of interior points of $\tilde{F}$.

The family $\{T(\tilde{F}):T\in\Gamma\}$ is called a *tessellation* of X.

The area of a fundamental region, if finite, is a numeric invariant of the group.

Let Γ be a Fuchsian group and $z_1 \in \mathbb{H}^2$ such that $T(z_1) \neq z_1$ for all $T \in \Gamma \setminus \{id\}$, then $D_{z_1}(\Gamma) = \{z \in \mathbb{H}^2 : d(z, z_1) \leq d(z, T(z_1)), \forall T \in \Gamma\}$ is a fundamental region of Γ, called the *Dirichlet region*.

IV. CONSTRUCTION OF TQCs

To generalize the toric code construction we consider compact surfaces with genus $g \geq 2$. The construction of these codes consists in selecting a regular hyperbolic polygon (plane model P' of the surface) and its possible tessellations $\{p, q\}$.

Before considering the plane models of surfaces and the associated tessellations, we review some basic topological concepts. For a thorough revision of this subject we refer the reader to Refs. 17 and 18.

A *hyperbolic polygon* P' with p' edges, or a p'-gon, is a convex closed set consisting of p' hyperbolic geodesic segments. The intersection of two geodesics is called *vertex* of the polygon. A p'-gon whose edges have the same length and the internal angles are equal is called a *regular* p'-gon.

A *regular tessellation* of the Euclidean or hyperbolic plane is a covering of the whole plane by regular polygons, all with the same number of edges, without superposition of such polygons, meeting completely only on edges or vertices. We denote a regular tessellation by $\{p, q\}$, where q regular polygons with p edges meet in each vertex. In particular, if $p = q$ the tessellation is said to be *self-dual*.

The regular hyperbolic polygon, denoted by P' (plane model of the surface), is the polygon associated with the fundamental region of the tessellation $\{p', q'\}$, that is, P' is a polygon with p' edges where q' polygons with p' edges meet in each vertex. On the other hand, the tessellation $\{p, q\}$ of P' has as fundamental region a regular hyperbolic polygon, denoted by P, with p edges where q polygons with p edges meet in each vertex. The area of these two polygons are related by Eq. (4).

An important topological invariant that we use in this paper is the *Euler characteristic*. Given a compact region X, we may tessellate this region with a finite number of a given polygon. The Euler characteristic of X, denoted by $\chi(X)$, is given by

$$\chi(X) = V - E + F,$$

where V denotes the number of vertices of this tessellation, E denotes the number of edges, and F denotes the number of faces (that is, the number of polygons). Another way of establishing the Euler characteristic is by use of the genus g of X, and so $\chi(X)$ is given by

$$\chi(X) = 2 - 2g.$$

Note that, in the case of a regular tessellation, if we count the q edges in each one of the V vertices, we have counted each edge of the tessellation twice. Analogously, if we count all the p edges corresponding to the border of each one of the F faces of the tessellation, we have counted each edge of the tessellation twice. Therefore, the following equalities hold:

$$qV = 2E = pF. \tag{2}$$

For example, if we consider only one face of the tessellation, that is, a polygon P with p edges, then $F = 1$, $V = p/q$, and $E = p/2$.

A. Plane models of surfaces

In this subsection we show that the hyperbolic polygon P' has a twofold characteristic: it is used to construct surfaces by edge-pairing identifications, and it is a fundamental region for groups.

A compact topological surface $\mathbb{M}$ may be obtained from a polygon P' by pairwise edge identifications, once the length and angle conditions are satisfied.

The pairwise edge-identification operation is formally defined as an oriented edge-pairing transformation. An oriented edge-pairing transformation of a hyperbolic polygon P', with equal length edges, is an isometry $\gamma \neq Id$ of an orientation preserving isometry group Γ, taking an edge s of P' to another edge $\gamma(s)=s'$ of P'. Furthermore, $\gamma^{-1} \in \Gamma \setminus \{Id\}$ takes $\gamma(s)=s'$ to s. Thus, we say that the edges s and s' are paired. If s is identified with s' and s' is identified with s'', then s is identified with s''. Such a chain of identifications may also occur with vertices, and so we call a maximal set $\{v_1, v_2, \ldots, v_k\}$ of identified vertices a *vertex cycle*.

An edge pairing of P' defines an *identification space* $S_{P'}$. This identification space has a distance function agreeing with the hyperbolic distance for sufficiently small regions in the interior of P', making it a hyperbolic surface when the angles of each vertex cycle adds up to 2π.

Note that the number of edges of P' is even, since the edges are identified in pairs. If P' is a 2-gon, then there are only two possibilities for edge identification. One of them yields a sphere and the other one the projective plane. Similarly, if P' is a 4-gon, the possible edge identifications give a sphere, projective plane, torus, or Klein bottle, and these surfaces can be realized geometrically as Euclidean surfaces. All other compact surfaces can be realized geometrically as hyperbolic surfaces.

It can be shown that any topological surface $S_{P'}$ not homeomorphic to a sphere is homeomorphic to a surface S_{P^*} for which P^* has a single vertex cycle. Thus, we can assume that P' has a particularly simple edge pairing. Therefore, any compact surface can be realized geometrically, see Ref. 17.

According to the Killing–Hopf theorem (see Ref. 17) any complete and connected hyperbolic surface is of the form $\mathbb{H}^2/\Gamma$, where Γ acts properly discontinuously on $\mathbb{H}^2$, that is, Γ is a Fuchsian group.

From the Killing–Hopf theorem, since $S_{P'}$ is complete, that is, each line segment in $S_{P'}$ can be extended indefinitely, we may express $S_{P'}$ as a quotient $\mathbb{S}^2/\Gamma$, $\mathbb{R}^2/\Gamma$, or $\mathbb{H}^2/\Gamma$. It can be shown that $S_{P'}$ is complete if P' is compact, see Ref. 17. Thus, identification spaces of compact polygons may be realized by geometric surfaces.

On the other hand, a compact surface $\mathbb{S}^2/\Gamma$, $\mathbb{R}^2/\Gamma$, or $\mathbb{H}^2/\Gamma$ is the identification space of a polygon in the corresponding geometry. The spherical $(\mathbb{S}^2/\Gamma)$ and the Euclidean $(\mathbb{R}^2/\Gamma)$ are the simplest cases among the identification spaces.

The hyperbolic surfaces $\mathbb{H}^2/\Gamma$ obtained as identification spaces of polygons are those for which Γ is finitely generated. Since Γ is generated by edge-pairing transformations and a polygon P' has only finitely many edges, Γ is finitely generated when P' is a fundamental region for Γ. The converse is true, it suffices to construct a polygonal fundamental region for a given finitely generated Γ.

Thus, a compact hyperbolic surface $\mathbb{H}^2/\Gamma$ is the identification space of a polygon if the polygon is a fundamental region for Γ. A necessary and sufficient condition for a polygon to be a fundamental region is the following.

Edge and Angle conditions (Reference 17). If a compact polygon P' is the fundamental region for an orientation preserving isometry group Γ of $\mathbb{S}^2$ (sphere surface), $\mathbb{R}^2$ (Euclidean plane), or $\mathbb{H}^2$ (hyperbolic plane).

(i) *For each edge s of P' there exists a unique edge s' of P' such that $s'=\gamma(s)$, for $\gamma \in \Gamma$.*

(ii) *Given edge pairings of P', for each set of the identified vertices, the sum of the angles has to be equal to 2π. This set is a vertex cycle.*

Theorem 3: [Poincaré (Reference 17)] A compact polygon P' satisfying the edge and angle conditions is a fundamental region for the group Γ generated by the edge-pairing transformations of P', and Γ is a Fuchsian group.

For surfaces with a Fuchsian group Γ the structure of the fundamental region P' is provided by the presentation of Γ, that is, the generators of Γ, denoted by $a_1, b_1, a_2, b_2, \ldots, a_g, b_g$, with a defining relation, also known as the "word of a surface," $a_1 b_1 a_1^{-1} b_1^{-1}, \ldots, a_g b_g a_g^{-1} b_g^{-1} = e$, with e being the identity of Γ, so that the Fuchsian group is a discrete group of transformation preserving this "word."

1. Topological classification of surfaces

All geometric surface is of the form $\mathbb{S}^2/\Gamma$, $\mathbb{R}^2/\Gamma$, or $\mathbb{H}^2/\Gamma$, thus the problem of classifying surfaces is replaced by the problem of classifying groups Γ. One way of distinguishing hyperbolic surfaces is to make use of topology. Surfaces may be distinguished topologically by its genus.

It is known that a compact topological surface may be obtained as an identification space $S_{P'}$ of a polygon P', and if P' is not homeomorphic to $\mathbb{S}^2$, then P' may be chosen to have a single vertex cycle. We use this fact to classify the nonspherical surfaces $S_{P'}$ by showing that all such surfaces are obtainable from the *normal form* polygons. The surfaces with different normal forms are nonhomeomorphic.

Orientable surfaces are such that any two directly opposite identified edges in the border of P' are used to define a normal form polygon.

We denote the identified edges in the border of P' by the same letter, for example, a. When the edges are directly opposite, we denote them by a, a^{-1}.

In the normal form each compact orientable surface is homeomorphic to an identification space $S_{P'}$ of a polygon P' with border of the form aa^{-1} or $a_1b_1a_1^{-1}b_1^{-1}, \ldots, a_gb_ga_g^{-1}b_g^{-1}$. In the first case, $S_{P'}$ is said to be of genus 0, in the latter case $S_{P'}$ has genus g. The genus g can be informally defined as the number of "handles" or "holes" because each segment $a_ib_ia_i^{-1}b_i^{-1}$ in the border of P' gives rise to a handle. The genus is the invariant which distinguishes topologically the surfaces. A more detailed construction of the normal form can be found in Ref. 17.

2. Geometric classification of surfaces

When constructing geometrically orientable surfaces in the normal form, any surface of genus 0 becomes a sphere (elliptic), any surface of genus 1 becomes a torus (Euclidean), and surfaces of genus ≥ 2 become g-tori (hyperbolic). Conversely all compact orientable hyperbolic surfaces are of genus ≥ 2, or equivalently, its Euler characteristic is a negative even number.

Let $\mathbb{M} = \mathbb{H}^2/\Gamma$ be a compact hyperbolic surface. $\mathbb{M}$ can be realized as the identification space $S_{P'}$ of a convex polygon P' (the Dirichlet region for the group Γ). The area of this polygon is a geometric property distinguishing such surfaces. In fact, let P' be a polygon with p' edges, or a p'-gon, that is, the fundamental region of the tessellation $\{p', q'\}$. From the Gauss–Bonnet theorem, the area of $\mathbb{M}$ is given by

$$
\begin{aligned}
\mu(\mathbb{M}) &= \mu(P') \\
&= p'\left(\pi - \frac{2\pi}{p'} - \frac{2\pi}{q'}\right) \\
&= p'\left(\frac{\pi(p'q' - 2p' - 2q')}{p'q'}\right) \\
&= -2\pi\left(\frac{p'}{q'} - \frac{p'}{2} + 1\right) = -2\pi(V - E + F) \\
&= -2\pi\chi(S) = -2\pi(2 - 2g) \\
&= 4\pi(g-1),
\end{aligned}
\tag{3}
$$

where $V = p'/q'$ and $E = p'/2$.

The procedure being proposed for the construction of TQCs takes into consideration polygons P' of the type $4g$-gon ($\{4g, 4g\}$) as the plane models of the corresponding surfaces. In these polygons the edge-pairing transformations are defined differently from the usual normal form, $\gamma: S \to S$, by $\gamma(s_i) = s_{i+2g}$, where $S = \{s_1, \ldots, s_{4g}\}$ is the set of edges of P', $i = 1, 2, \ldots, 4g$, and the

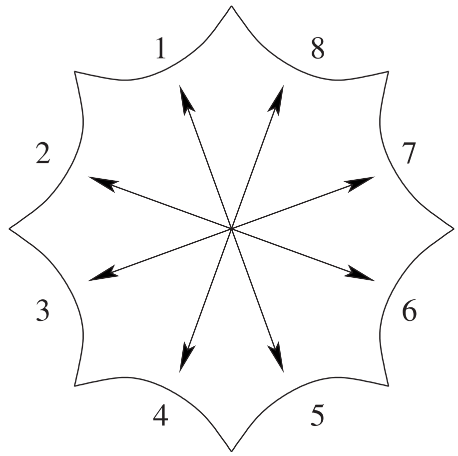

FIG. 3. Edge-pairing transformation $\gamma(s_i)=s_{i+2g}$.

sum of the subscripts of s is realized modulo $4g$. An isometry γ realizes the pairings of opposite edges of P', see Fig. 3. The selection of these edge-pairing transformations leads to a code distance having the greatest hyperbolic distance between the identified edges of P'. Since $p'=q'=4g$, the unique cycle of vertices obtained from these edge-pairing transformations has the sum of the internal angles equal to $(p'/q')(2\pi)=2\pi$, and so satisfying the necessary and sufficient conditions for P' to be a fundamental region for the group of these edge-pairing transformations Γ.

We call the attention to the fact that, for each fixed value of g, polygons different from $\{4g,4g\}$, for example, $\{4g+2,2g+1\}$, $\{8g-4,4\}$, $\{12g-6,3\}$, generate surfaces with the same genus. For example, the Klein group (see Ref. 17) is a surface of genus 3. The Klein group is a 14-gon ($\{14,7\}$) in the hyperbolic plane where the edges are connected by the relation $s_{2i+1}\mapsto s_{2i+6}$, and the sum of the subscripts of s is realized modulo 14 (Fig. 4). Note that, since this 14-gon

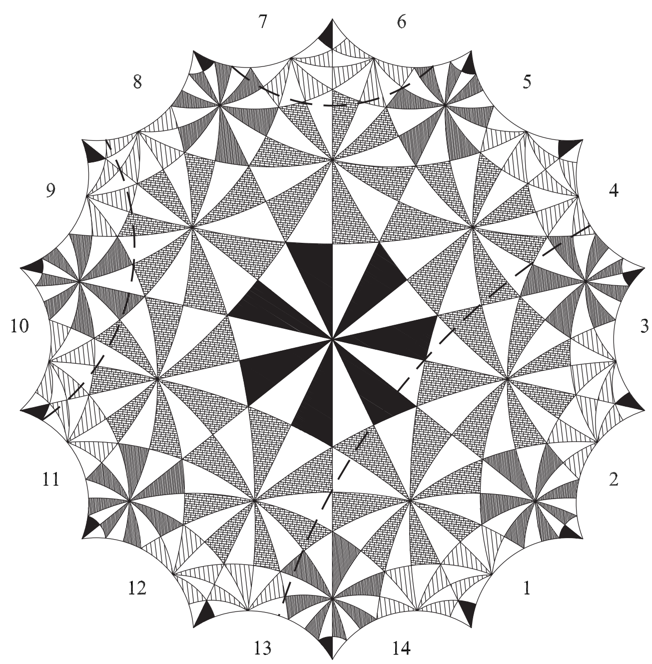

FIG. 4. Klein group - a 14 - gon with tessellation {7,3}.

satisfy Poincaré's theorem, it is a fundamental region for Γ, where Γ is the group consisting of the edge-pairing transformations. Theoretically, any polygon which generates a compact surface can be employed in the construction of such codes. Nevertheless, not all of these polygons with this property satisfy Poincaré's theorem when considering the opposite edge pairings. Hence, among all these possibilities the $4g$-gons are chosen as those achieving the greatest minimum distance of the code.

B. Tessellation $\{p,q\}$

Similarly to Kitaev's construction, we shall determine all possible tessellation $\{p,q\}$ of the polygon P'. For this purpose we have to find the solutions of equation

$$\mu(P') = n_f \mu(P), \tag{4}$$

where the hyperbolic tessellations must satisfy the following constraint $(p-2)(q-2)>4$. In (4) $\mu(P')$ denotes the area of the polygon P' associated with the fundamental region of the tessellation $\{p',q'\}$, $\mu(P)$ denotes the area of the polygon associated with the fundamental region of the tessellation $\{p,q\}$, and n_f is a positive integer. Note that, given a tessellation $\{p,q\}$, the dual tessellation $\{q,p\}$ has to satisfy the same previous conditions.

The tessellations obtained as the solutions of Eq. (4) are, in fact, all the possible tessellations of P' because they satisfy the following theorem.

Theorem 4: (Reference 19) Let $\mathbb{M}$ be a closed surface and let p,q,V,E,F be positive integers such that

$$V - E + F = \chi(\mathbb{M}), \tag{5}$$

$$pF = 2E = qV. \tag{6}$$

Then the following hold.

- *Existence.* There exist *a* $\{p,q\}$-pattern on $\mathbb{M}$ consisting of F p-sided faces, E edges, and V vertices each of valence q; except when $\mathbb{M}$ is the projective plane, $\{p,q\}=\{3,3\}$, $V=F=2$, and $E=3$.
- *Geometrization.* A $\{p,q\}$-pattern on $\mathbb{M}$ can be made geometric.
- *Classification.* A $\{p,q\}$-pattern on the sphere or projective plane is unique. For all other closed surfaces $\mathbb{M}$ the $\{p,q\}$-patterns on $\mathbb{M}$ are classified by conjugate classes of subgroups isomorphic to the fundamental group of $\mathbb{M}$ in the extended $(p,q,2)$-triangle groups of Schwarz.

As an example, consider the Klein group. It is a 14-gon and it may be tiled with a set of 24 identical regular heptagons or alternatively with a set of 56 equilateral triangles. These two tessellations are dual to each other, in the sense that the vertices of one tessellation correspond to the faces of the other. That is, the 14-gon is tiled by the tessellation {7,3} and by its dual tessellation {3,7}. Observe that the area of the 14-gon is equal to the area of the 24 heptagons or the area of the 56 equilateral triangles.

From Eq. (3) and the Gauss–Bonnet theorem, Eq. (4) may be written as

$$4\pi(g-1) = n_f\left[(p-2)\pi - \frac{2p\pi}{q}\right]. \tag{7}$$

Hence, the number of faces of the tessellation $\{p,q\}$ of P' is given by

$$n_f = \frac{4q(g-1)}{pq-2p-2q}. \tag{8}$$

C. Operators and parameters

As in Kitaev's construction, given a vertex v of the tessellation, the vertex operator acts nontrivially on the q qubits having v as the common vertex and the identity operator acts on the remaining qubits, that is, $A_v=\otimes_{j\in E_v}\sigma_x^j$, where E_v denotes the set of edges having v as the common vertex. Similarly, given a face f of the tessellation, the face operator acts nontrivially on the p qubits forming the border of this face, and the identity operator acts on the remaining qubits of the tessellation, that is, $B_f=\otimes_{j\in E_f}\sigma_z^j$, where E_f denotes the set of edges forming the border of f. Therefore, the code is given by $\mathcal{C}=\{|\psi\rangle : A_v|\psi\rangle=|\psi\rangle, B_f|\psi\rangle=|\psi\rangle \,\forall\, v,f\}$.

From Eq. (4), we have that n_f is the number of faces of the tessellation $\{p,q\}$ of P'. Since each edge of this tessellation belongs simultaneously to two faces, we have $n=n_f p/2$ edges or qubits.

The operators A_v and B_f are the stabilizer operators of this code. Besides being Pauli operators, A_v and B_f are mutually commutative. Furthermore, each edge belongs either to the border of two faces or have two vertices as end points, which implies that each edge is counted twice when considering the product of all the vertex or face operators. Thus,

$$\prod_v A_v = 1 \quad \text{and} \quad \prod_f B_f = 1. \tag{9}$$

Each vertex and face operator can be expressed as the product of the others such operators. We conclude from this fact that there are n_f-1 independent face operators and n_v-1 independent vertex operators, where n_v is the number of vertices of the tessellation $\{p,q\}$ of P'. Thus, we have n_f+n_v-2 generators of the stabilizer group. Consequently, the number of qubits to be encoded is $k=n-(n_f+n_v-2)$.

Note that the number of vertices of the tessellation coincides with the number of faces of the dual tessellation, then $n_v=n_f p/q$. From (7), we have

$$2(2g-2) = n_f p - 2n_f - 2n_f\frac{p}{q},$$

$$2g-2 = n_f\frac{p}{2} - n_f - n_f\frac{p}{q},$$

$$2g-2 = n - n_f - n_v,$$

$$2g = n - n_f - n_v + 2.$$

Therefore, the number of encoded qubits is $k=n-n_f-n_v+2=2g$, and the dimension of the code $\mathcal{C}$ is $2^{2g}=4^g$.

Substituting $n_f=2n/p$ in (8), and for each tessellation $\{p,q\}$, the asymptotic code rate k/n is given by $k/n \to (pq-2p-2q)/pq$ when $g\to\infty$.

D. Code distance

Before defining the distance of this class of codes, we have to introduce some concepts from homology theory. For that we refer the reader to Ref. 20. 1-*chain* is an application that fixes an element of $\mathbb{Z}_2$ either to each edge of the tessellation or to the set of all edges that are fixed to the value of 1 by this application. Analogously, 0-*chain* and 2-*chain* are defined as the corresponding applications that fix an element of $\mathbb{Z}_2$ to the vertices and to the faces of the tessellation, respectively.

The *border* of a face is the sum of its edges, and the border of an edge is the sum of its two vertices. For example, let v_0, v_1, v_2 be the vertices of a triangle, the border of the edge that joins v_0 to v_1 is defined by $\partial(v_0,v_1)=v_1-v_0$, whereas the border of the face of this triangle is defined by $\partial(v_0,v_1,v_2)=\partial(v_0,v_1)+\partial(v_1,v_2)+\partial(v_2,v_0)$. Thus, a *cycle* may be defined as a chain whose border

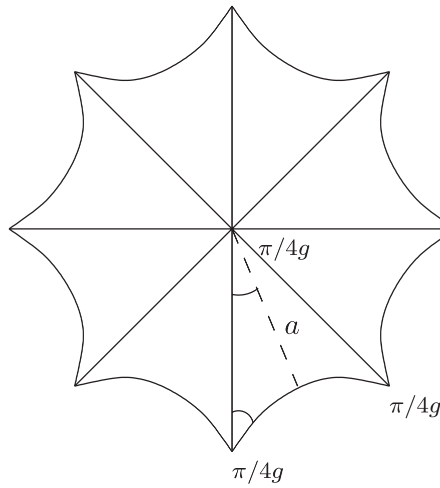

FIG. 5. Hyperbolic length $(2a)$ of the minimum geodesic of P'.

is trivial. A cycle is said to be *homologically trivial* if it may be written as the border of a 2-chain. Otherwise, the cycle is called *homologically nontrivial*.

Remember that the distance of a stabilizer code is the weight of the minimal-weight Pauli operator that preserves the code subspace and acts nontrivially within the code subspace. In terms of a toric code, the distance is the number of edges of the tessellation contained in the shortest homologically nontrivial cycle either on the tessellation or on the dual tessellation.[8]

For TQC on surfaces of genus $g \geq 2$, the distance is similar to the distance of a toric code. We are looking for the shortest homologically nontrivial cycle either on the tessellation or on the dual tessellation.

We call the attention to the fact that the shortest homologically nontrivial cycle in a p'-gon is given by the geodesics of least length that connect the identified edges of P'. In terms of the edges of the tessellation of P', the shortest homologically nontrivial cycle is an edge path that is closest to the geodesic with shortest length. Thus, the code distance is the minimum number of edges between the shortest homologically nontrivial cycle of the tessellation and the shortest homologically nontrivial cycle of the dual tessellation.

From the elements of hyperbolic trigonometry, it is possible to determine the distance between the edge pairings of P'. This distance, denoted by d_h, is the hyperbolic length of the orthogonal geodesic common to this two edges (Fig. 5), and it is given by

$$d_h = 2a = 2 \operatorname{arccosh}\left[\frac{\cos(\pi/4g)}{\sin(\pi/4g)}\right]. \tag{10}$$

This equation is obtained from the relation $\cosh a \sin \beta = \cos \alpha$ (see Ref. 16), where in this case $\alpha = \beta = \pi/4g$.

Since the minimum distance of the TQC, denoted by d_{TQC}, is a function of the number of edges of the tessellation $\{p,q\}$ of P', we derive a lower bound on this distance as the ratio of d_h by the edge length $l(p,q)$ of the fundamental region of the tessellation $\{p,q\}$, that is, $d_{\mathrm{TQC}} \geq d_h / l(p,q)$, where the edge length of the tessellation is given by

$$l(p,q) = \operatorname{arccosh}\left[\frac{\cos^2(\pi/q) + \cos(2\pi/p)}{\sin^2(\pi/q)}\right]. \tag{11}$$

This expression may be obtained by the cosine rule, that is, $\cosh c = (\cos \alpha \cos \beta + \cos \gamma)/\sin \alpha \sin \beta$, where c corresponds to the edge of the tessellation $\{p,q\}$ (Fig. 6).

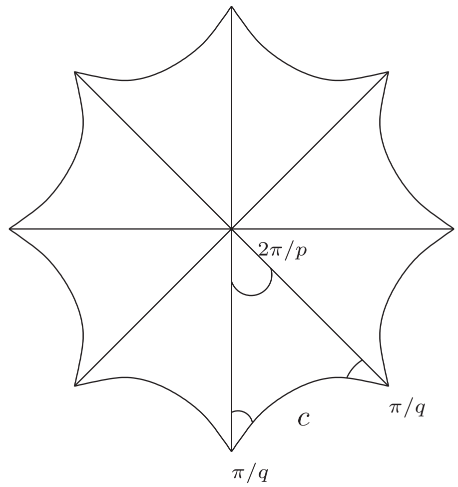

FIG. 6. Edge length (c) of the tessellation $\{p,q\}$.

Since the values of $l(p,q)$ are invariant, due to the type of identification considered (opposite edge pairings), it follows that $d_h/l(p,q)\to\infty$ when $g\to\infty$. Note that, $n=n_f p/2\to\infty$ when $g\to\infty$. However, the ratio k/d_{TQC} does not necessarily go to infinity when $g\to\infty$ because d_{TQC} increases more rapidly than k, and k is not bounded.

V. TABLES OF TQCs

The lower bound $d_h/l(p,q)$ on the minimum distance of a TQC on a surface with genus g depends on the selection of the tessellation $\{p,q\}$ of P'. The smallest the hyperbolic edge length of the fundamental region of the tessellation is the greater will be the lower bound.

From (4) or (8) we may determine all possible tessellations $\{p,q\}$ of P'. Although we have an extensive list of possible codes, we tabulate the most important TQCs on surfaces with genus 2–5.

TABLE I. Tessellations and parameters of the TQCs for $g=2$ and $d_h=3.057\,14$.

$\{p,q\}$	n_f	$l(p,q)$	$d_h/l(p,q)$	$[[n,k,d_{TQC}]]$
{3,7}	**28**	**1.0906**	**2.8033**	
{7,3}	**12**	**0.5663**	**5.3989**	[[**42,4,3**]]
{3,8}	16	1.5286	2	
{8,3}	6	0.7270	4.2049	[[24,4,2]]
{3,9}	12	1.8551	1.6480	
{9,3}	4	0.8192	3.7319	[[18,4,2]]
{3,10}	10	2.1226	1.4403	
{10,3}	3	0.8792	3.4773	[[15,4,2]]
{3,12}	8	2.5534	1.1973	
{12,3}	2	0.9516	3.2125	[[12,4,2]]
{4,5}	**10**	**1.2537**	**2.4384**	
{5,4}	**8**	**1.0613**	**2.8806**	[[**20,4,3**]]
{4,6}	6	1.7628	1.7343	
{6,4}	4	1.3170	2.3214	[[12,4,2]]
{4,8}	4	2.4485	1.2486	
{8,4}	2	1.5286	2	[[8,4,2]]
{5,5}	4	1.6850	1.8144	[[10,4,2]]
{6,6}	2	2.2924	1.3336	[[6,4,2]]

TABLE II. Tessellations and parameters of the TQCs for $g=3$ and $d_h=3.9833$.

$\{p,q\}$	n_f	$l(p,q)$	$d_h/l(p,q)$	$[[n,k,d_{TQC}]]$
{3,7}	**56**	**1.0906**	**3.6526**	
{7,3}	**24**	**0.5663**	**7.0347**	[[**84,6,4**]]
{3,8}	**32**	**1.5286**	**2.6059**	
{8,3}	**12**	**0.7270**	**5.4788**	[[**48,6,3**]]
{3,9}	**24**	**1.8551**	**2.1472**	
{9,3}	**8**	**0.8192**	**4.8625**	[[**36,6,3**]]
{3,10}	20	2.1226	1.8767	
{10,3}	6	0.8792	4.5307	[[30,6,2]]
{3,12}	16	2.5534	1.5600	
{12,3}	4	0.9516	4.1857	[[24,6,2]]
{3,14}	14	2.8982	1.3744	
{14,3}	3	0.9928	4.0123	[[21,6,2]]
{3,18}	12	3.4382	1.1585	
{18,3}	2	1.0359	3.8453	[[18,6,2]]
{4,5}	**20**	**1.2537**	**3.1771**	
{5,4}	**16**	**1.0613**	**3.7533**	[[**40,6,4**]]
{4,6}	**12**	**1.7628**	**2.2597**	
{6,4}	**8**	**1.3170**	**3.0246**	[[**24,6,3**]]
{4,8}	8	2.4485	1.6269	
{8,4}	4	1.5286	2.6059	[[16,6,2]]
{4,12}	6	3.3258	1.1977	
{12,4}	2	1.6629	2.3954	[[12,6,2]]
{5,5}	**8**	**1.6850**	**2.3640**	[[**20,6,3**]]
{5,6}	6	2.1226	1.8767	
{6,5}	5	1.8764	2.1228	[[15,6,2]]
{5,10}	4	3.2338	1.2318	
{10,5}	2	2.1226	1.8767	[[10,6,2]]
{6,6}	4	2.2924	1.7376	[[12,6,2]]
{6,9}	3	3.1614	1.2600	
{9,6}	2	2.4887	1.6006	[[9,6,2]]
{8,8}	2	3.0571	1.3030	[[8,6,2]]

These are shown in Tables I–IV. The parameters considered in these tables are the tessellation $\{p,q\}$, the number of faces of the tessellation, n_f, the edge length of the tessellation, $l(p,q)$, the lower bound on the code distance, $d_{\mathrm{TQC}}=\lceil d_h/l(p,q)\rceil$, and the parameters of the code $[[n,k,d_{\mathrm{TQC}}]]$, where n denotes the number of qubits, k the number of encoded qubits, and d_{TQC} the minimum distance of the code.

The best lower bounds achieved by the code distance are associated with the tessellations {3,7} and {7,3}. The minimum distance of this code is given by the minimum between $d_h/l(3,7)$ and $d_h/l(7,3)$. Consequently, the tessellation {3,7} will limit the minimum distance of the resulting code. Unfortunately, this code has low rate. On the other hand, the tessellation {4,5} and its dual {5,4} have, in general, the same minimum distance. As a consequence, the code generated by the tessellation {4,5} and its dual is the best code in terms of achieving a greater minimum distance while having a good rate. The self-dual tessellations, although having a good rate, achieve smaller values on the code distance. On the other hand, the codes defined by self-dual tessellations have smaller computational complexity, and as shown by Kitaev,[6] these codes may be implemented by use of anyons.

The distance of the toric codes grows by increasing the edge length of the flat torus, whereas the distance of the surface codes with genus $g \geq 2$ grows by increasing the genus of the surface.

TABLE III. Tessellations and parameters of the TQCs for $g=4$ and $d_h=4.596$.

$\{p,q\}$	n_f	$l(p,q)$	$d_h/l(p,q)$	$[[n,k,d_{TQC}]]$
{3,7}	**84**	**1.0906**	**4.2144**	
{7,3}	**36**	**0.5663**	**8.1165**	[[**126,8,5**]]
{3,8}	**48**	**1.5286**	**3.0067**	
{8,3}	**18**	**0.7270**	**6.3215**	[[**72,8,4**]]
{3,9}	**36**	**1.8551**	**2.4775**	
{9,3}	**12**	**0.8192**	**5.6104**	[[**54,8,3**]]
{3,10}	**30**	**2.1226**	**2.1653**	
{10,3}	**9**	**0.8792**	**5.2276**	[[**45,8,3**]]
{3,12}	24	2.5534	1.8000	
{12,3}	6	0.9516	4.8295	[[36,8,2]]
{3,15}	20	3.0486	1.5076	
{15,3}	4	1.0070	4.5639	[[30,8,2]]
{3,18}	18	3.4382	1.3367	
{18,3}	3	1.0359	4.4368	[[27,8,2]]
{3,24}	16	4.0374	1.1384	
{24,3}	2	1.0638	4.3204	[[24,8,2]]
{4,5}	**30**	**1.2537**	**3.6658**	
{5,4}	**24**	**1.0613**	**4.3306**	[[**60,8,4**]]
{4,6}	**18**	**1.7628**	**2.6073**	
{6,4}	**12**	**1.3170**	**3.4899**	[[**36,8,3**]]
{4,7}	**14**	**2.1408**	**2.1469**	
{7,4}	**8**	**1.4491**	**3.1717**	[[**28,8,3**]]
{4,8}	12	2.4485	1.8771	
{8,4}	6	1.5286	3.0067	[[24,8,2]]
{4,10}	10	2.9387	1.5640	
{10,4}	4	1.6169	2.8424	[[20,8,2]]
{4,12}	9	3.3258	1.3819	
{12,4}	3	1.6629	2.7639	[[18,8,2]]
{4,16}	8	3.9225	1.1717	
{16,4}	2	1.7073	2.6919	[[16,8,2]]
{5,5}	**12**	**1.6850**	**2.7277**	[[**30,8,3**]]
{5,10}	6	3.2338	1.4212	
{10,5}	3	2.1226	2.1653	[[15,8,2]]
{6,6}	**6**	**2.2924**	**2.0049**	[[**18,8,3**]]
{6,12}	4	3.7556	1.2238	
{12,6}	2	2.5534	1.8000	[[12,8,2]]
{8,8}	3	3.0571	1.5034	[[12,8,2]]
{10,10}	2	3.5796	1.2839	[[10,8,2]]

As mentioned in Secs. IV C and IV D, the asymptotic rate is $k/n=(pq-2p-2q)/pq$ and $d_{\mathrm{TQC}}\to\infty$ when $g\to\infty$.

In Table V we display codes with distances $d=3$, 4, and 5 for genus $g=1$, 2, 3, 4, and 5, and we compare their code rates. Observe that there are TQCs whose encoding rates are better than the toric codes, when the distance is fixed. Moreover, the effective rate of the code derived from the tessellations {4,5} and {5,4} when $g=2$, that is [[20,4,3]], is the same as the rate of the perfect code [[5,1,3]] ($g=0$) and of the code [[10,2,3]] ($g=1$) obtained in Ref. 9. Note that there exist TQC codes with distance $d=5$ only on surfaces with genus $g\geq 4$.

As previously mentioned, our construction reproduces the results shown in Ref. 9, that is, the family of codes with parameters

TABLE IV. Tessellations and parameters of the TQCs for $g=5$ and $d_h=5.0591$.

$\{p,q\}$	n_f	$l(p,q)$	$d_h/l(p,q)$	$[[n,k,d_{TQC}]]$
{3,7}	**112**	**1.0906**	**4.6390**	
{7,3}	**48**	**0.5663**	**8.9343**	[[**168,10,5**]]
{3,8}	**64**	**1.5286**	**3.3097**	
{8,3}	**24**	**0.7270**	**6.9585**	[[**96,10,4**]]
{3,9}	**48**	**1.8551**	**2.7272**	
{9,3}	**16**	**0.8192**	**6.1757**	[[**72,10,3**]]
{3,10}	**40**	**2.1226**	**2.3835**	
{10,3}	**12**	**0.8792**	**5.7543**	[[**60,10,3**]]
{3,12}	32	2.5534	1.9813	
{12,3}	8	0.9516	5.3162	[[48,10,2]]
{3,14}	28	2.8982	1.7456	
{14,3}	6	0.9928	5.0959	[[42,10,2]]
{3,18}	24	3.4382	1.4714	
{18,3}	4	1.0359	4.8838	[[36,10,2]]
{3,22}	22	3.8576	1.3115	
{22,3}	3	1.0570	4.7861	[[33,10,2]]
{3,30}	20	4.4944	1.1257	
{30,3}	2	1.0765	4.6998	[[30,10,2]]
{3,54}	18	5.6828	0.8902	
{54,3}	1	1.0918	4.6336	[[27,10,2]]
{4,5}	**40**	**1.2537**	**4.0352**	
{5,4}	**32**	**1.0613**	**4.7670**	[[**80,10,5**]]
{4,6}	**24**	**1.7628**	**2.87**	
{6,4}	**16**	**1.3170**	**3.8415**	[[**48,10,3**]]
{4,8}	**16**	**2.4485**	**2.0662**	
{8,4}	**8**	**1.5286**	**3.3097**	[[**32,10,3**]]
{4,12}	12	3.3258	1.5212	
{12,4}	4	1.6629	3.0424	[[24,10,2]]
{4,20}	10	4.3785	1.1555	
{20,4}	2	1.7275	2.9286	[[20,10,2]]
{5,5}	**16**	**1.6850**	**3.0025**	[[**40,10,4**]]
{5,6}	**12**	**2.1226**	**2.3835**	
{6,5}	**10**	**1.8764**	**2.6961**	[[**30,10,3**]]
{5,10}	8	3.2338	1.5644	
{10,5}	4	2.1226	2.3835	[[20,10,2]]
{6,6}	**8**	**2.2924**	**2.2069**	[[**24,10,3**]]
{6,7}	7	2.6293	1.9241	
{7,6}	6	2.3884	2.1182	[[21,10,2]]
{6,9}	6	3.1614	1.6003	
{9,6}	4	2.4887	2.0328	[[18,10,2]]
{6,15}	5	4.2104	1.2016	
{15,6}	2	2.5827	1.9588	[[15,10,2]]
{7,14}	4	4.1520	1.2185	
{14,7}	2	2.8982	1.7456	[[14,10,2]]
{8,8}	4	3.0571	1.6548	[[16,10,2]]
{12,12}	2	3.9833	1.2701	[[12,10,2]]

$$\left[\left[\binom{s}{2},\binom{s}{2}-2(s-1),3\right]\right],$$

when considering the embedding of complete graphs K_s, for $s\equiv 1 \bmod 4$, on surfaces with appropriate genus.

TABLE V. Codes with distances $d=3,4$ and $d=5$.

Distance	Genus	Tessellation	k/n
$d=3$	1	{4,4}	2/18
		Ref. 9	2/10
	2	{3,7}	4/42
		{4,5}	4/20
	3	{3,8}	6/48
		{3,9}	6/36
		{4,6}	6/24
		{5,5}	6/20
	4	{3,9}	8/54
		{3,10}	8/45
		{4,6}	8/36
		{4,7}	8/28
		{5,5}	8/30
		{6,6}	8/18
	5	{3,9}	10/72
		{3,10}	10/60
		{4,6}	10/48
		{4,8}	10/32
		{5,6}	10/30
		{6,6}	10/24
$d=4$	1	{4,4}	2/32
		Ref. 9	2/17
	3	{3,7}	6/84
		{4,5}	6/40
	4	{3,8}	8/72
		{4,5}	8/60
	5	{3,8}	10/96
		{5,5}	10/40
$d=5$	1	{4,4}	2/50
		Ref. 9	2/26
	4	{3,7}	8/126
	5	{3,7}	10/168
		{4,5}	10/80

Indeed, substituting $n_f=s\equiv 1 \bmod 4$, $p=s-1$, and $k=2g=\binom{s}{2}-2(s-1)$ in (11), we conclude that $q=s-1$. Hence, resulting in a self-dual tessellation $\{p,q\}=\{s-1,s-1\}$, whenever $s\equiv 1 \bmod 4$. As a consequence, the same class of codes shown in Ref. 9 is obtained. Table VI illustrates some of these codes.

Maximum Distance Separable quantum codes with parameters $[[(2g+2),2g,2]]$, for $g=2,3,4,5,\ldots$, are shown in each one of the tables. Note that the majority of the codes, for each $g=2,3,4,5$ has minimum distance of 2. Note also that for each g, these are all the possible codes that may be constructed.

An inherent property of these TQCs is that they provide unequal error protection to each logical qubit $|0\rangle$ and $|1\rangle$, except when considering self-dual tessellations.

VI. CONCLUSIONS

One important aspect of the TQCs is that they are able to correct errors in a physical level, instead of the usual syndrome procedure. In this paper we hope to have shed some light in the way of constructing TQCs by aggregating to the inherent topology of these codes the corresponding

TABLE VI. Some codes with the same parameters as those in Ref. 9.

s	g	n_f	$\{p,q\}$	d_h	$l(p,q)$	$d_h/l(p,q)$	$[[n,k,d_TQC]]$
9	10	9	{8,8}	6.4674	3.0571	2.1155	[[36,20,3]]
13	27	13	{12,12}	8.4601	3.9833	2.1239	[[78,54,3]]
17	52	17	{16,16}	9.7716	4.596	2.1261	[[136,104,3]]
21	85	21	{20,20}	10.7546	5.0591	2.1258	[[210,170,3]]
25	126	25	{24,24}	11.5419	5.4328	2.1245	[[300,252,3]]

geometry of the surfaces with genus $g \geq 2$. From the concepts of hyperbolic geometry it is possible to generalize the notion of an associated polygon to a tessellation and the definition of the code distance in such surfaces.

The proposed construction enables us to find any topological code on compact surfaces of genus $g \geq 2$ with simple and fast calculations. Some of these codes have good distance and rate, whereas others codes do not. However, we would like to emphasize that the general construction procedure of these codes leads to codes having inherent unequal error protection. In addition, yield codes with better encoding rates than the toric codes, when the distance is fixed. Like the toric codes, these TQC codes are quasiperfect.

It is possible to find classes of good codes, that is, codes with good rates and distances, among the constructed codes. For example, the class shown in Ref. 9 and reproduced by the proposed construction.

Kitaev showed that it is possible to implement TQC in self-dual tessellations by using anyons through *string operators*. This also applies to the codes constructed in this paper from self-dual tessellations. However, since many of the tabulated codes do not belong to self-dual tessellations, it remains an open question which type of particle could be used in the implementation of such codes.

Regarding future work, as suggested by the Area Editor, we mention: (a) In Ref. 3 good quantum error corrections codes were proved to exist. However, the topological version of this result is lacking, and only constructions with $k/n \to$finite in the large n limit (physical qubits $=n$) are known; (b) since hyperbolic geometry has been introduced for TQC such that orientable and nonorientable surface codes can be treated on equal footing, it remains an interesting issue of topological codes for qubits $q=2$ in nonorientable surfaces. For qudits with $q>2$, it is not known how to construct such nonorientable codes due to the existence of the Torelli group. Any way out to this problem would be very valuable.

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