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LETTERS AND COMMENTS

A pulsating Gaussian wave packet

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Abstract. A pulsating Gaussian wave packet for the harmonic oscillator is explicitly constructed. The mechanism for that behaviour is pointed out.

It is well known that a wave packet describing the quantum motion of a particle under the influence of a harmonic oscillator potential does not spread. The impression that this fact is always true is conveyed by most textbooks on quantum mechanics. Nevertheless, it has recently been shown that the width of the wave packet can pulsate [1]. This result has, indeed, been known for a long time. This problem was recognized in the collection of problems on quantum mechanics by Goldman *et al* [2], but no explanation of the reason for the pulsation was given. In this letter, a Gaussian wave packet with this characteristic feature is explicitly constructed and an interpretation of the phenomenon is presented in terms of interference between the eigenstates of the harmonic oscillator.

The evolution of a wave packet is governed by the Schrödinger equation, it being possible to find the wave packet at any instant t from the initial wave packet through the relation

$$\psi(x,t) = \int_{-\infty}^{\infty} K(x,t;x',0)\psi(x',0)\,\mathrm{d}x \qquad (1)$$

where $\psi(x,0)$ is the initial wave packet and $K(x,t;x',0)$ is the propagator.

Let us consider the propagator for the harmonic oscillator [3]

$$K(x,t;x',0) = \left(\frac{m\omega}{2\pi\mathrm{i}\hbar\sin(\omega t)}\right)^{1/2} \times \exp\left(\frac{\mathrm{i}m\omega}{2\hbar\sin(\omega t)} \times\left[(x^2+x'^2)\cos(\omega t) - 2xx'\right]\right) \qquad (2)$$

and the initial wave packet

$$\psi(x,0) = \left(\frac{\alpha^2}{\pi}\right)^{1/4} \exp\left(-\frac{\alpha^2}{2}(x-A)^2\right). \qquad (3)$$

Inserting (2) and (3) into (1), we obtain

$$\psi(x,t) = \left(\frac{\alpha^2}{\pi}\right)^{1/4} \times\left(\frac{\beta^2}{\beta^2\cos(\omega t)+\mathrm{i}\alpha^2\sin(\omega t)}\right)^{1/2} \times\exp\left(-\frac{\alpha^2\beta^4[x-A\cos(\omega t)]^2}{2\left[\delta\sin^2(\omega t)+\beta^4\right]}\right) \times\exp\left(\mathrm{i}\beta^2\sin(\omega t) \times\frac{\left(x^2\delta+A^2\alpha^4\right)\cos(\omega t)-2A\alpha^4x}{2\left[\delta\sin^2(\omega t)+\beta^4\right]}\right) \qquad (4)$$

where

$$\beta = \left(\frac{m\omega}{\hbar}\right)^{1/2} \qquad (5a)$$

$$\delta = \alpha^4 - \beta^4. \qquad (5b)$$

From (4) it is straightforward to obtain the probability density

$$|\psi|^2 = \left(\frac{\alpha^2}{\pi}\right)^{1/2}\left(\frac{\beta^4}{\left[\delta\sin^2(\omega t)+\beta^4\right]}\right)^{1/2} \times\exp\left(-\frac{\alpha^2\beta^4[x-A\cos(\omega t)]^2}{\left[\delta\sin^2(\omega t)+\beta^4\right]}\right). \qquad (6)$$

The expression for the probability density permits us to conclude that its Gaussian form is maintained at any time. We can also conclude that its centroid moves as a classical particle

$$\langle x\rangle = A\cos(\omega t) \tag{7}$$

and its width pulsates according to

$$(\Delta x)^2 = \frac{\delta\sin^2(\omega t) + \beta^4}{2\alpha^2\beta^4}. \tag{8}$$

It can also be shown that the product $\Delta x\Delta p$ pulsates with a minimum value equal to $\hbar/2$, in agreement with the Heisenberg uncertainty relation.

The results presented in this letter reduce to those presented in the literature when $\delta = 0$. In that particular case $\alpha = \beta$, and the wave packet becomes the ground state solution for the harmonic oscillator. On the other hand, when $\alpha \neq \beta$ then the wave packet is not that simple stationary state but a superposition of stationary solutions for the harmonic oscillator. It is this mixing effect that makes the wave packet pulsate with a frequency given by 2ω.

References

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