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Fractional Schrödinger operator with delta potential localized on circle

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We consider a system governed by the fractional Schödinger operator with a delta potential supported by a circle in $\mathbb{R}^2$. We find out the function counting the number of bound states, in particular, we give the necessary and sufficient conditions for the absence of bound state in our system. Furthermore, we reproduce the form of eigenfunctions and analyze the asymptotic behavior of eigenvalues for the strong coupling constant case. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.3691199]

I. INTRODUCTION

The models belonging to the line of research usually called Schrödinger operator with delta potential have been recently intensively studied. The Hamiltonian which governs such a system can be symbolically written as

$$-\Delta + \gamma\delta(\Gamma - x), \quad \gamma \in \mathbb{R},$$

where Δ is the Laplace operator acting in $L^2(\mathbb{R}^n)$, Γ is a submanifold of a lower dimension, $\delta(\Gamma - \cdot)$ stands for the Dirac delta supported by Γ and γ defines a coupling constant. Such models are considered as a mathematical idealization of nanostructures as, for example, "leaky quantum wires." The main question is addressed to the problem: relation between the geometry of Γ and spectral properties of the system with delta potential.[1,7–9] On the other hand, more than a decade ago Laskin introduced so-called fractional Schrödinger equation (FSE). The idea is based on Feynman path integral approach in quantum mechanics. Namely, instead of the Brownian-like trajectories Laskin considered more general framework: Lévy-like trajectories. Consequently, this generalization leads to the fractional dynamics governed by the Hamiltonian

$$H_\alpha = D_\alpha(-\Delta)^{\alpha/2} + V(x), \quad 1 < \alpha \le 2, \tag{1.1}$$

where D_α is a scaling constant. Some particular examples of (1.1) were analyzed in Refs. 11, 16–18, and 20. On the other hand, Jeng *et al*. studied the problem of nonlocality of fractional derivative and their consequences, cf. Ref. 13. The authors showed that some claims employed to find solutions of FSE have not taken into account nonlocality of fractional derivative and the only correct model they pointed out is that involving delta potential. The mentioned model was studied in Ref. 14. Further results concerning spectral and scattering properties of such a system were derived in Refs. 5 and 6. Putting in mind the latter purposes, let us recall that the one-dimensional system governed by the fractional Schrödinger operator corresponding to the expression

$$D_\alpha(-\Delta)^{\alpha/2} - \beta\delta(x), \quad \beta > 0 \tag{1.2}$$

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has exactly one bound state. The corresponding ground state energy takes the form

$$E^0_{\alpha,\beta} := -\left(\frac{\beta}{\alpha D_\alpha^{1/\alpha}\sin(\pi/\alpha)}\right)^{\alpha/(\alpha-1)}. \tag{1.3}$$

This paper is a certain continuation and extension of this line of research. We are interested in two-dimensional quantum system with Hamiltonian corresponding to the formal expression,

$$D_\alpha(-\Delta)^{\alpha/2} - \beta\delta(\cdot - C_R)\,,\ \ \beta > 0\,, \tag{1.4}$$

where C_R is a circle of radius R living in $\mathbb{R}^2$.

The main results of the paper can be formulated as follows.

• Giving a meaning of the self-adjoint operator $H_{\alpha,\,\beta}$ to the formal expression Eq. (1.4); formula (2.1).

• Construction of the resolvent of $H_{\alpha,\,\beta}$ and formulating the eigenvalues problem in the terms of resolvent poles; Theorem 2.3.

• Finding out the function counting the number of bound states of $H_{\alpha,\,\beta}$, in particular, formulating the necessary and sufficient condition for the existence of at least one bound state in the terms of parameters α, β, and R; Theorem 3.3.

• Reproducing the form of eigenfunctions of $H_{\alpha,\,\beta}$; Proposition 3.1.

• Analysis of eigenvalues asymptotics for the strong interaction case; Theorem 3.6.

In this paper we study the stationary solutions of Schrödinger equation. The time fractional Schrödinger equation was considered, for example, in Ref. 20; the general results were employed to solve the free particle and the well potential models. Moreover, in Ref. 2 and 3 the authors derived the numerical method for differential evolution equations with fractional time derivative which can be applied to Schrödinger operator as well.

Notations

• We use abbreviations $L^2 \equiv L^2(\mathbb{R}^2)$ and analogously for the Sobolev spaces $W^{2,\alpha} \equiv W^{2,\alpha}(\mathbb{R}^2)$. Let $(\cdot\,,\,\cdot)$ stand for the scalar product in L^2.

• Symbol Δ denotes two-dimensional Laplace operator acting in L^2.

• We denote $x = (x_1, x_2) \in \mathbb{R}^2$, $|x| = (x_1^2 + x_1^2)^{1/2}$ and analogously for $p \in \mathbb{R}^2$ and $|p| \in \mathbb{R}_+$. Moreover, $x_{C_R}(\phi) \equiv (R\cos\phi, R\sin\phi)$, $\varsigma(\phi,\phi') \equiv x_{C_R}(\phi) - x_{C_R}(\phi')$, where $\phi, \phi' \in [0, 2\pi]$.

• Standardly, we denote the Fourier transform as $\mathcal{F} : L^2 \mapsto L^2$, i.e., $\mathcal{F}f(p) = \frac{1}{2\pi}\int_{\mathbb{R}^2} f(x)\mathrm{e}^{ipx}\mathrm{d}x$.

• $E^0_{\alpha,\beta} := -(\frac{\beta}{\alpha D_\alpha^{1/\alpha}\sin(\pi/\alpha)})^{\alpha/(\alpha-1)}$.

• Symbols J_k, for $k \in \mathbb{Z}$ stands for the Bessel functions of kth order.

II. PRELIMINARIES

By means of the Fourier transform we define operator $(-\Delta)^{\alpha/2} : D((-\ \Delta)^{\alpha/2}) \mapsto L^2$ acting as

$$\mathcal{F}(-\Delta)^{\alpha/2} f(p) = |p|^\alpha \mathcal{F}f(p)\,,\ \ \alpha \in (1, 2\rangle,$$

where the domain $D((-\ \Delta)^{\alpha/2})$ coincides with the Sobolev space $W^{2,\alpha} := \{f \in L^2 : (|p|^2 + 1)^{\alpha/2}\mathcal{F}f(p) \in L^2\}$. To find a self-adjoint realization of (1.4) we employ the form sum method. (This technics is a standard tool in analysis of usual Laplacian with delta interaction.) For f, $g \in D((-\ \Delta)^{\alpha/4}) = W^{2,\,\alpha/2}$ consider the sequilinear form

$$\mathcal{E}_{\alpha,\beta}(f,g) := D_\alpha\int_{\mathbb{R}^2}(-\Delta)^{\alpha/4}f(-\Delta)^{\alpha/4}\overline{g}\mathrm{d}x - \beta\int_{\mathbb{R}^2} f\overline{g}\mathrm{d}\mu_{C_R}(x)\,, \tag{2.1}$$

where μ_{C_R} is the Dirac measure in $\mathbb{R}^2$ with support on C_R, i.e., $\mathrm{d}\mu_{C_R}(x) := \delta(\cdot - C_R)\mathrm{d}x = \delta(r-R)r\mathrm{d}r\mathrm{d}\phi$, where (r, ϕ) are radial coordinates.

By means of μ_{C_R} we define the space $L^2_\mu := L^2(\mathbb{R}^2, \mu_{C_R})$. Then the "delta perturbation term" in (2.1) is determined by the scalar $(\cdot\,,\,\cdot)_\mu$ in L^2_μ, i.e., $(f,g)_\mu := \int_{\mathbb{R}^2} f\overline{g}\mathrm{d}\mu_{C_R}(x)$.

Remark 2.1: Note that the form $\mathcal{E}_{\alpha,\beta}$ is well defined on $W^{2,\alpha/2}$. This comes directly from the trace theorem which says that the trace map $I_\mu : W^{2,s} \mapsto L^2_\mu$, $s > 1/2$ is bounded.

Hamiltonian. Let $H_{\alpha,\beta}$ stand for the operator associated to $\mathcal{E}_{\alpha,\beta}$, i.e., $(H_{\alpha,\beta} f, g) = \mathcal{E}_{\alpha,\beta}(f, g)$. Using again the trace map theorem we conclude that there exist a constant $C > 0$ such that

$$\|f\|^2_\mu = \int_{\mathbb{R}^2} |f|^2 \mathrm{d}\mu_{C_R}(x) \le C\|(1+|p|^2)^{\alpha/4}\mathcal{F} f\|^2 .$$

Moreover, for any $a > 0$ there exists $b > 0$ so that

$$\|f\|^2_\mu \le a \int_{\mathbb{R}^2} |(-\Delta)^{\alpha/4} f|^2 \mathrm{d}x + b\|f\|^2 . \tag{2.2}$$

The above inequality follows from Theorem 2.1 of Refs. 24 and the following convergence:

$$\lim_{\lambda\to\infty} \sup_{x\in\mathbb{R}^2} \int_{\mathbb{R}^2} G_\alpha(-\lambda^\alpha; x - x')\mathrm{d}\mu_{C_R}(x') = 0 , \tag{2.3}$$

where $G_\alpha(z; x - x')$ is the kernel of $((-\Delta)^{\alpha/2} - z)^{-1}$, $z \in \mathbb{C} \setminus [0, \infty)$. For $\alpha = 2$ Eq. (2.3) is a consequence of the fact that μ_{C_R} belongs to the Kato class. On the other hand, for $\alpha \in (1, 2)$ and $|x - x'| \to 0$ we have

$$G_\alpha(-\lambda^\alpha; x - x') = \mathcal{O}(|x - x'|^{\alpha-2}) , \tag{2.4}$$

moreover, for any $\delta > 0$

$$\sup_{|x-x'|>\delta} |x - x'|^{2+\alpha} G_\alpha(-\lambda^\alpha; x - x') \to 0 \quad \text{if} \quad \lambda \to \infty . \tag{2.5}$$

Equations (2.4) and (2.5) imply (2.3). (Formulae (2.4) and (2.5) can be obtained using, for example, properties of the Fox functions derived in the further discussion.)

Inequality (2.2) implies that the operator $H_{\alpha,\beta}$ associated to $\mathcal{E}_{\alpha,\beta}$ is self-adjoint bounded from below with the form domain $W^{2,\alpha/2}$.[22] In fact, $H_{\alpha,\beta}$ gives a mathematical meaning to (1.4). The technics employed above is, in fact, the extension of the form-sum method used for usual Schrödinger operator with delta type perturbations. For the problem of point interaction in one-dimensional system we recommend Chap. X, Example 3 of Ref. 22.

Remark 2.2: Note that the number E is an eigenvalue of $H_{\alpha,\beta}$ *if and only if* $\frac{E}{D_\alpha}$ is an eigenvalue of the Hamiltonian corresponding to (2.1) with $D_\alpha = 1$ and $\beta_\alpha := \frac{\beta}{D_\alpha}$ taking instead of β.

A. Resolvent of $H_{\alpha,\beta}$

Relying on the Remark 2.2 we assume that $D_\alpha = 1$.

We start with the resolvent of "free" fractional Hamiltonian, $R_\alpha(z) = ((-\Delta)^{\alpha/2} - z)^{-1}$, $z \in \mathbb{C} \setminus [0, \infty)$ which, in the Fourier representation, takes the form

$$R_\alpha(z) f = \mathcal{F}^{-1} \frac{1}{|p|^{\alpha/2} - z} (\mathcal{F} f)(p) . \tag{2.6}$$

In fact, operator $R_\alpha(z)$ is an integral with the kernel $G_\alpha(z)$ which takes the following form for $z = -\lambda^\alpha$:

$$G_\alpha(-\lambda^\alpha; x - x') = \frac{\lambda^{2-\alpha}}{2\pi} \int_0^\infty \frac{J_0(y\lambda|x - x'|)}{y^\alpha + 1} y\mathrm{d}y . \tag{2.7}$$

Indeed, using properties of the Bessel functions we have

$$\begin{aligned} R_\alpha(-\lambda^\alpha) f &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{\mathrm{e}^{ip(\cdot)}}{|p|^\alpha + \lambda^\alpha} (\mathcal{F} f)(p)\mathrm{d}p \\ &= \frac{R}{(2\pi)^2} \int_0^\infty \int_0^{2\pi} \int_{\mathbb{R}^2} \frac{\mathrm{e}^{i\rho|\cdot - x'|\cos\vartheta}}{\rho^\alpha + \lambda^\alpha} f(x')\rho\mathrm{d}\rho\mathrm{d}\vartheta\mathrm{d}x' = G_\alpha(-\lambda^\alpha) * f . \end{aligned}$$

In particular,

$$G_2(-\lambda^2; x - x') = \frac{1}{2\pi} K_0(\lambda|x - x'|), \tag{2.8}$$

where K_0 is the Macdonald function.[12]

Furthermore, we define the embeddings of "free resolvent" by

$$\check{R}_\alpha(z)_\mu : L^2_\mu \mapsto L^2, \quad \check{R}_\alpha(z)_\mu f = G_\alpha(z) * f\mu_{C_R},$$

where $G_\alpha(z) * f\mu_{C_R}$ stands for the convolution. The adjoint operator $\hat{R}_\alpha(z)_\mu$ is also determined by $G_\alpha(z)$ but it acts from L^2 to L^2_μ. Finally, let $R_\alpha(z)_{\mu\mu}$ stand for the integral operator with the kernel $G_\alpha(z)$ acting from L^2_μ to L^2_μ. Now we are ready to construct resolvent and to formulate the spectral condition.

Theorem 2.3: *Assume that for $z \in \mathbb{C} \setminus [0, \infty)$ operator $\Gamma(z) := I - \beta R_\alpha(z)_{\mu\mu}$ is invertible. Then the resolvent $R_{\alpha,\beta}(z):=(H_{\alpha,\beta} - z)^{-1}$ is given by*

$$R_{\alpha,\beta}(z) = R_\alpha(z) + \beta \check{R}_\alpha(z)_\mu \Gamma(z)^{-1} \hat{R}_\alpha(z)_\mu . \tag{2.9}$$

Moreover, for $\lambda > 0$ the number $-\lambda^\alpha$ states an eigenvalue of $H_{\alpha,\beta}$ if and only of

$$\ker \Gamma(-\lambda^\alpha) \neq \emptyset . \tag{2.10}$$

The corresponding eigenfunction takes the form $\check{R}_\alpha(-\lambda^\alpha) f$ where $f \in \ker \Gamma(-\lambda^\alpha)$.

Proof: The first step of proof is to check that $R_{\alpha,\beta}(z)$ satisfies

$$\mathcal{E}_{\alpha,\beta}(f, R_{\alpha,\beta}(z)g) - z(f, R_{\alpha,\beta}(z)g) = (f, g). \tag{2.11}$$

This, in view of definition of $H_{\alpha,\beta}$, means that $R_{\alpha,\beta}(z)$ is the inverse of $H_{\alpha,\beta} - z$. Combining (2.1) and (2.9) we have

$$\begin{aligned}
&\mathcal{E}_{\alpha,\beta}(f, R_{\alpha,\beta}(z)g) - z(f, R_{\alpha,\beta}(z)g) \\
&= (f, (-\Delta)^{\alpha/2} R_{\alpha,\beta}(z)g) - z(f, R_{\alpha,\beta}(z), g) - \beta(f, R_{\alpha,\beta}(z)g)_\mu \\
&= (f, g) + \beta(f, \Gamma(z)^{-1} \hat{R}_\alpha(z)_\mu g)_\mu - \beta(f, \hat{R}_\alpha(z)_\mu g) \\
&-\beta^2(f, R_\alpha(z)_{\mu\mu} \Gamma(z)^{-1} \hat{R}_\alpha(z)_\mu g)_\mu = (f, g).
\end{aligned}$$

This completes the proof of (2.11). To show the remaining statements assume that $g \in D((-\Delta)^{\alpha/2})$ and $t := ((-\Delta)^{\alpha/2} + \lambda^\alpha)g$. Then for any $f \in W^{2,\alpha/2}$ the equation is valid

$$\mathcal{E}_{\alpha,\beta}(f, g) + \lambda^\alpha (f, g) = (f, t) - \beta(f, R_\alpha(-\lambda^\alpha)_{\mu\mu} I_\mu t)_\mu = 0,$$

if and only if (2.10) is satisfied. ■

Remark 2.4: The self-adjointness of $H_{\alpha,\beta}$ can be equivalently obtained following the treatment derived by Posilicano.[23] For this aim it suffices to check that $\Gamma(\cdot)$ satisfies the pseudo-resolvent equivalence. Moreover, we have

$$\operatorname{Ran} \chi \cap L^2 = \{0\},$$

where $\chi : L^2_\mu \mapsto W^{2,-\alpha/2}$ acts as $\chi f = f\mu_{C_R}$. Since the proofs of this facts can be obtained by repeating the arguments from Refs. 8 and 23 we omit here details.

III. EIGENVALUES OF $H_{\alpha,\beta}$

To recover eigenvalues of $H_{\alpha,\beta}$ we will relay on (2.10). Therefore, we start with analysis of $R_\alpha(-\lambda^\alpha)_{\mu\mu}$. Using (2.7) we conclude that the kernel of $R_\alpha(-\lambda^\alpha)_{\mu\mu}$ is

$$G(\lambda, \alpha; \varsigma(\phi, \phi')) = \frac{\lambda^{2-\alpha}}{2\pi} \int_0^\infty \frac{y J_0(y\lambda|\varsigma(\phi, \phi')|)}{y^\alpha + 1} \mathrm{d}y, \tag{3.1}$$

where $x_{C_R}(\phi) = (R\cos\phi, R\sin\phi)$ and $\varsigma(\phi, \phi') := x_{C_R}(\phi) - x_{C_R}(\phi')$.

Due to the symmetry of system operator $R_\alpha(-\lambda^\alpha)_{\mu\mu}$ is diagonal in the orthonormal basis $\{\xi_k\}_{k\in\mathbb{Z}} \in L^2_\mu$, where $\xi_k = \frac{1}{\sqrt{2\pi R}}\mathrm{e}^{ik(\cdot)}$. Thus,

$$G(\lambda, \alpha; \varsigma(\phi, \phi')) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} S_k(\lambda, \alpha)\mathrm{e}^{ik(\phi-\phi')}\,,$$

where

$$S_k(\lambda, \alpha) = (\xi_k, G(\lambda, \alpha)\xi_k)_\mu = \frac{R}{2\pi}\int_0^{2\pi}\int_0^{2\pi} G(\lambda, \alpha; \varsigma(\phi, \phi'))\mathrm{e}^{-ik(\phi-\phi')}\mathrm{d}\phi\mathrm{d}\phi'\,.$$

With this notation the spectral condition formulated in Theorem 2.3 after scaling (see Remark 2.2) reads as follows.

Proposition 3.1: Given $k \in \mathbb{Z}$ the number $-\lambda_k^\alpha = -(\lambda_k)^\alpha$, $\lambda_k > 0$ is an eigenvalue of $H_{\alpha,\beta}$ if and only if λ_k is a solution of

$$1 = \beta_\alpha S_k(\lambda, \alpha)\,. \tag{3.2}$$

The corresponding eigenfunction takes the form

$$\breve{R}_\alpha(-\lambda_k^\alpha)\xi_k = \int_0^{2\pi}\int_0^\infty \frac{J_0(y\lambda_k|(\cdot - x_{C_R}(\phi)|)}{y^\alpha + 1}\mathrm{e}^{ik\phi}\mathrm{d}\phi y\mathrm{d}y\,.$$

In fact, the eigenfunction can be expressed by means of the Fox functions which we study in further discussion. Moreover, note that $S_k(\lambda, \alpha) = S_{-k}(\lambda, \alpha)$. This implies that all eigenvalues apart from the ground state have double degeneracy.

To study the condition (3.2) we need the following technical lemma with the proof postponed to part A of this section.

Lemma 3.2: We have

$$S_k(\lambda, \alpha) = R^{\alpha-1}\int_0^\infty \frac{|J_k(y)|^2}{(\lambda R)^\alpha + y^\alpha}y\mathrm{d}y\,. \tag{3.3}$$

Moreover, for $\alpha \in (1, 2)$

$$\lim_{\lambda\to 0} S_k(\lambda, \alpha) = R^{\alpha-1}\omega_k(\alpha)\,, \tag{3.4}$$

where

$$\omega_k(\alpha) := \frac{1}{2^{\alpha-1}}\frac{\Gamma(\alpha-1)}{[\Gamma(\alpha/2)]^2}\frac{\Gamma(1-\alpha/2+k)}{\Gamma(\alpha/2+k)}\,, \quad \text{for} \quad k = 0\,, 1\,2... \tag{3.5}$$

and $\omega_{-k}(\alpha) = \omega_k(\alpha)$

Combining the above results we get.

Theorem 3.3: *Hamiltonian $H_{\alpha,\beta}$ has at least one eigenvalue if and only if*

$$\beta_\alpha R^{\alpha-1}\omega_0(\alpha) > 1\,. \tag{3.6}$$

Moreover, the next eigenvalues appear if and only if

$$\beta_\alpha R^{\alpha-1}\omega_k(\alpha) > 1\,, \ \text{for} \ k \in \mathbb{Z}\setminus\{0\}\,. \tag{3.7}$$

Consequently, the number of eigenvalues (including degeneracy) is given by

$$\sharp\sigma_{\mathrm{d}}(H_{\alpha,\beta}) = \sharp\{k \in \mathbb{Z} \,:\, (22)\ is\ satisfied\}\,.$$

Proof: To show the statement we employ the Birman–Schwinger technics. Note that given $k \in \mathbb{Z}$ and $\alpha \in (1, 2)$ function $S_k(\,\cdot\,, \alpha)$ is continuous and monotonically decreasing. Moreover, using (3.3) we get the limit $\lim_{\lambda\to\infty} S_k(\lambda, \alpha) = 0$. Relying on (3.4) we conclude that there exists a solution of

(3.2) *if and only if* (3.7) is satisfied. Furthermore, the sequence $\omega_k(\alpha)$ is decreasing with respect to $k = 0\,,\,1\,,\,2....$ Indeed, denote

$$a_k := \frac{\Gamma(1-\frac{\alpha}{2}+k)}{\Gamma(\frac{\alpha}{2}+k)}\,.$$

Using the equivalence $\Gamma(z\,+\,1) = z\Gamma(z)$ we get

$$a_{k+1} = \frac{1-\frac{\alpha}{2}+k}{\frac{\alpha}{2}+k}a_k < a_k$$

and $a_1 < a_0$. This, in view of (3.2) implies the claim of theorem. ■

A. Proof of Lemma 3.2

Proof. Using the Mellin-Barnes integral representation of $J_0(\,\cdot\,)$, that is,

$$J_0(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left(\frac{x}{2}\right)^{-2s}\frac{\Gamma(s)}{\Gamma(1-s)}\mathrm{d}s\,,\ \text{ with }\ 0<\gamma<3/4\,,$$

and

$$\int_0^\infty \frac{x^{\mu-1}}{x^\nu+1}\mathrm{d}x = \frac{1}{\nu}\Gamma(\mu/\nu)\Gamma(1-\mu/\nu)\,,\ \text{ for }\ \mathrm{Re}\,\nu > \mathrm{Re}\,\mu\,,$$

we obtain from (3.1) that

$$G(\lambda,\alpha,\varsigma(\phi,\phi')) = \frac{\lambda^{2-\alpha}}{2\pi\alpha}H^{2,1}_{1,3}\left[\left(\frac{\lambda|\varsigma(\phi,\phi')|}{2}\right)^2\middle|\begin{matrix}(1-\frac{2}{\alpha},\frac{2}{\alpha})\\(0,1),(1-\frac{2}{\alpha},\frac{2}{\alpha}),(0,1)\end{matrix}\right], \tag{3.8}$$

where we have used the definition of the Fox's H-function given by Eq. (A3).

Using the equivalence $|\varsigma(\phi,\phi')|^2 = 4R^2\sin^2\theta/2$, where $\theta = \phi\,-\,\phi'$ and the Mellin–Barnes representation of the Fox's H-function, we have

$$S_k(\lambda,\alpha) = \frac{R\lambda^{2-\alpha}}{\alpha}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left(\frac{1}{2\pi}\int_0^{2\pi}(\lambda R\sin\theta/2)^{-2s}\,\mathrm{e}^{-ik\theta}\mathrm{d}\theta\right)$$
$$\cdot\frac{\Gamma(s)\Gamma(2(1-s)/\alpha)\Gamma(1-2(1-s)/\alpha)}{\Gamma(1-s)}\mathrm{d}s,$$

where $\gamma = \mathrm{Re}\,s > 1-2/\alpha$. On the other hand, using formula 3.892.1 of Ref. 12, that is,

$$\int_0^\pi \mathrm{e}^{i\beta x}\sin^{\nu-1}x\mathrm{d}x = \frac{\pi\,\mathrm{e}^{i\beta\pi/2}\Gamma(\nu)}{2^{\nu-1}\Gamma((\nu+\beta+1)/2)\Gamma((\nu-\beta+1)/2)},\quad(\mathrm{Re}\,\nu>-1),$$

we obtain

$$S_k(\lambda,\alpha) = \frac{(-1)^kR\lambda^{2-\alpha}}{\alpha}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left(\frac{\lambda R}{2}\right)^{-2s}\Lambda(s)$$
$$\cdot\frac{\Gamma(s)\Gamma(2(1-s)/\alpha)\Gamma(1-2(1-s)/\alpha)\Gamma(1-2s)}{\Gamma(1-s)\Gamma(1-k-s)\Gamma(1+k-s)}\mathrm{d}s,$$

with $1-\alpha/2 < \gamma = \mathrm{Re}\,s < 1$. Thus, the last equation can be written in terms of Fox's H-function as

$$S_k(\lambda,\alpha) = \frac{(-1)^kR\lambda^{2-\alpha}}{\alpha}H^{2,2}_{2,5}\left[\left(\frac{\lambda R}{2}\right)^2\middle|\begin{matrix}(1-\frac{2}{\alpha},\frac{2}{\alpha}),(0,2)\\(0,1),(1-\frac{2}{\alpha},\frac{2}{\alpha}),(0,1),(k,1),(-k,1)\end{matrix}\right] \tag{3.9}$$

But we know, from tables of Mellin transforms (see, for example, formulas 4.1 (page 182) and 5.40 (page 196) of Ref. 21), that

$$\mathcal{M}^{-1}\left[\Gamma(1-2s/\alpha)\Gamma(2s/\alpha)\right] = \frac{\alpha}{2}(1+x^{\alpha/2})^{-1}, \quad (0 < \mathrm{Re}\, s < 1),$$
$$\mathcal{M}^{-1}\left[\frac{\Gamma(s)\Gamma(1-2s)}{\Gamma(1-s)\Gamma(1-k-s)\Gamma(1+k-s)}\right] = (-1)^k |J_k(2\sqrt{x})|^2, \quad (0 < \mathrm{Re}\, s < 1/2),$$

where $\mathcal{M}^{-1}$ denotes the inverse Mellin transform. Then, restricting s to $1-\alpha/2 < \mathrm{Re}\, s < 1/2$ (which holds for $1 < \alpha \leq 2$), and using the convolution theorem for Mellin transforms, we obtain that

$$S_k(\lambda,\alpha) = \frac{R\lambda^{2-\alpha}}{2}\int_0^\infty \frac{|J_k(\lambda R\sqrt{t})|^2}{1+t^{\alpha/2}}\mathrm{d}t,$$

and then Eq. (3.3).

The limit $\lim_{\lambda\to 0} S_k(\lambda,\alpha)$ can be conveniently studied in terms of the Fox's H-function representation of $S_k(\lambda,\alpha)$. Employing the series expansion Eq. (A10) we get the following asymptotics for $\lambda \to 0$

$$S_k(\lambda,\alpha) = \frac{(-1)^k R}{\alpha}\left(\frac{R}{2}\right)^{\alpha-2} h'_{20} + \mathcal{O}(\lambda^{2-\alpha}),$$

where $1 < \alpha < 2$ and

$$h'_{20} = \frac{\alpha}{2}\frac{\Gamma(1-\alpha/2)\Gamma(\alpha-1)}{\Gamma(\alpha/2)\Gamma(-k+\alpha/2)\Gamma(k+\alpha/2)},$$

Consequently, we have

$$S_k(0,\alpha) = \frac{(-1)^k}{\alpha}\frac{R^{\alpha-1}}{2^{\alpha-2}}h'_{20}.$$

Using $\Gamma(z)\Gamma(1-z) = \pi/(\sin\pi z)$ we obtain Eq. (3.4) after straightforward manipulations. ∎

Remark 3.4 Let us note that Eq. (3.8) generalizes Eq. (2.8); indeed, for $\alpha = 2$ we have

$$H^{2,1}_{1,3}\left[z^2\left|\begin{matrix}(0,1)\\(0,1),(0,1),(0,1)\end{matrix}\right.\right] = H^{0,2}_{2,0}\left[z^2\left|\begin{matrix}-\\(0,1),(0,1),-\end{matrix}\right.\right] = 2K_0(2z),$$

where the last equality follows from formula (1.128) of Ref. 19. This gives Eq. (2.8).

B. Asymptotics of eigenvalues

In this section we analyze the behavior of eigenvalues for the strong interaction case, i.e., $\beta_\alpha R^{\alpha-1}$ large. Since we may expect that in such a system eigenvalues go to $-\infty$ our first step is to study asymptotics of $S_k(\lambda,\alpha)$ for the large spectral parameter λ.

Lemma 3.5: For $\lambda \to \infty$ we have

$$S_k(\lambda,\alpha) = \frac{1}{\lambda^{\alpha-1}}\frac{1}{\alpha\sin\pi/\alpha}\left(1-\frac{k^2-1/4}{2(R\lambda)^2}+\ldots\right). \tag{3.10}$$

Proof: Since $\Upsilon = 2$ and $\Upsilon^* = 4/\alpha$ (see definition in Eq. (A7)) we can use the asymptotic expansion given by Eq. (A8) in Eq. (3.9). Since

$$h_{10} = 0, \quad h_{21} = 0,$$

the asymptotic expansion gives for $1 < \alpha \leq 2$ the following result

$$H^{2,2}_{2,5}\left[z^2\left|\begin{matrix}(1-\frac{2}{\alpha},\frac{2}{\alpha}),(0,2)\\(0,1),(1-\frac{2}{\alpha},\frac{2}{\alpha}),(0,1),(k,1),(-k,1)\end{matrix}\right.\right] = h_{20}z^{-1} + h_{22}z^{-3} + \ldots$$

where

$$h_{20} = \frac{(-1)^k}{2\sin\pi/\alpha}, \quad h_{22} = \frac{(-1)^k(1/4 - k^2)}{2^4 \sin\pi/\alpha},$$

this implies Eq. (3.10). ■

Finally, we get the following behavior of eigenvalues of $H_{\alpha,\beta}$.

Theorem 3.6: *For $\beta_\alpha R^{\alpha-1} \to \infty$ the eigenvalues of $H_{\alpha,\beta}$ behave as*

$$E_{\alpha,\beta,k} = E^0_{\alpha,\beta} + \frac{\alpha}{\alpha-1}\left(\frac{k^2 - 1/4}{2R^2}(-E^0_{\alpha,\beta})^{(\alpha-2)/\alpha}\right)^{\frac{1}{\alpha-1}} + \dots. \tag{3.11}$$

Proof: Combining Eqs. (3.2) and (3.10) with $(1+x)^\gamma = 1 + \gamma x + \mathcal{O}(x^2)$ we get the claim. ■

Formula (3.11) shows that the dominated term for the strong interaction is determined by ground state energy $E^0_{\alpha,\beta}$ of the one dimensional system with point interaction, see Eq. (1.3).

IV. DISCUSSION

Recall that the number of eigenvalues is determined by the spectral condition

$$\beta_\alpha R^{\alpha-1}\omega_k(\alpha) > 1. \tag{4.1}$$

Therefore to get the number of bound states we have to study functions $(1, 2) \ni \alpha \mapsto \omega_k(\alpha)$, with $k \in \mathbb{Z}$ defined by Eq. (3.5). Note that $\omega_k(\cdot)$ are continuous with boundary limits

$$\lim_{\alpha\to 1}\omega_k(\alpha) = \infty, \quad \text{for all} \quad k \in \mathbb{Z}, \tag{4.2}$$

and

$$\lim_{\alpha\to 2}\omega_0(\alpha) = \infty, \quad \lim_{\alpha\to 2}\omega_k(\alpha) = \frac{1}{2|k|}, \quad \text{for} \quad k \in \mathbb{Z}\setminus\{0\}. \tag{4.3}$$

Moreover, as was already mentioned, given α the numbers $\{\omega_k(\alpha)\}_{k=0}^\infty$ makes a decreasing sequence.

- The above statements implies that given $\alpha \in (1, 2)$ and $\beta_\alpha R^{\alpha-1}$ small enough *Hamiltonian $H_{\alpha,\beta}$ has no bound state.* Precisely, this situation realizes if $\beta_\alpha R^{\alpha-1}\omega_0(\alpha) \le 1$. This stays in contrast to the case $\alpha = 2$. Namely, $H_{2,\beta}$ has always at least one bound state, cf. Ref. 10; this remains in consistency to the first limit of (4.3).
- It follows from (4.2) that given R and β the number of bound states of $H_{\alpha,\beta}$ goes to infinity for $\alpha \to 1$.
- The second limit of (4.3) implies that the necessary and sufficient condition for kth eigenvalue of $H_{2,\beta}$ reads

$$R\beta > 2|k|,$$

which again coincides with the results obtained in Ref. 10.

- Note that for the limiting case $\alpha = 2$ Eq. (3.11) gives the following asymptotics of eigenvalues:

$$\frac{-\alpha^2}{4} + \frac{k^2 - 1/4}{R^2} + \mathcal{O}(\alpha^{-2}R^{-4}),$$

which was obtained in Ref. 10.

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APPENDIX: A FOX'S H – FUNCTION

The Fox's H – function, also known as H – function or Fox's function, was introduced in the literature as an integral of Mellin-Barnes type.[19]

Let m, n, p, and q be integer numbers. Consider the function

$$\Lambda(s) = \frac{\prod_{i=1}^{m} \Gamma(b_i + B_i\, s) \prod_{i=1}^{n} \Gamma(1 - a_i - A_i\, s)}{\prod_{i=m+1}^{q} \Gamma(1 - b_i - B_i\, s) \prod_{i=n+1}^{p} \Gamma(a_i + A_i\, s)} \tag{A1}$$

with $1 \le m \le q$ and $0 \le n \le p$. The coefficients A_i and B_i are positive real numbers; a_i and b_i are complex parameters.

The Fox's H – function, denoted by

$$H^{m,n}_{p,q}(x) = H^{m,n}_{p,q}\left(x \,\middle|\, \begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\right) = H^{m,n}_{p,q}\left[x \,\middle|\, \begin{matrix}(a_1, A_1), \cdots, (a_p, A_p)\\(b_1, B_1), \cdots, (b_q, B_q)\end{matrix}\right] \tag{A2}$$

is defined as the inverse Mellin transform, i.e.,

$$H^{m,n}_{p,q}(x) = \frac{1}{2\pi i} \int_L \Lambda(s)\, x^{-s}\, \mathrm{d}s \tag{A3}$$

where $\Lambda(s)$ is given by Eq. (A1), and the contour L runs from $L - i\infty$ to $L + i\infty$ separating the poles of $\Gamma(1 - a_i - A_i\, s)$, $(i = 1, \ldots, n)$ from those of $\Gamma(b_i + B_i\, s)$, $(i = 1, \ldots, m)$. The complex parameters a_i and b_i are taken with the imposition that no poles in the integrand coincide.

There are some interesting properties associated with the Fox's H – function. We consider here the following ones:

a. Change the independent variable

Let c be a positive constant. We have

$$H^{m,n}_{p,q}\left[x \,\middle|\, \begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\right] = c\, H^{m,n}_{p,q}\left[x^c \,\middle|\, \begin{matrix}(a_p, c\,A_p)\\(b_q, c\,B_q)\end{matrix}\right]. \tag{A4}$$

To show this expression one introduce a change of variable $s \to c\,s$ in the integral of inverse Mellin transform.

b. Change the first argument

Set $\alpha \in \mathbb{R}$. Then we can write

$$x^\alpha H^{m,n}_{p,q}\left[x \,\middle|\, \begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\right] = H^{m,n}_{p,q}\left[x \,\middle|\, \begin{matrix}(a_p + \alpha A_p, A_p)\\(b_q + \alpha B_q, B_q)\end{matrix}\right]. \tag{A5}$$

To show this expression first we introduce the change $a_p \to a_p + \alpha A_p$ and take $s \to s - \alpha$ in the integral of inverse Mellin transform.

c. Lowering of order

If the first factor (a_1, A_1) is equal to the last one, (b_q, B_q), we have

$$H^{m,n}_{p,q}\left[x \,\middle|\, \begin{matrix}(a_1, A_1), \cdots, (a_p, A_p)\\(b_1, B_1), \cdots, (b_{q-1}, B_{q-1})(a_1, A_1)\end{matrix}\right] = H^{m,n-1}_{p-1,q-1}\left[x \,\middle|\, \begin{matrix}(a_2, A_2), \cdots, (a_p, A_p)\\(b_1, B_1), \cdots, (b_{q-1}, B_{q-1})\end{matrix}\right]. \tag{A6}$$

To show this identity is sufficient to simplify the common arguments in the Mellin-Barnes integral.

d. Asymptotic expansions

The asymptotic expansions for Fox's H-functions have been studied in Ref. 4. Let Υ and Υ^* be defined as

$$\Upsilon = \sum_{i=1}^{q} B_i - \sum_{i=1}^{p} A_i, \qquad \Upsilon^* = \sum_{i=1}^{n} A_i - \sum_{i=n+1}^{p} A_i + \sum_{i=1}^{m} B_i - \sum_{i=m+1}^{q} B_i. \tag{A7}$$

If $\Upsilon > 0$ and $\Upsilon^* > 0$ we have[15]

$$H^{m,n}_{p,q}(x) = \sum_{r=1}^{n} \sum_{k=0}^{\infty} h_{rk} x^{(a_r-1-k)/A_r}, \qquad |x| \to \infty, \tag{A8}$$

where

$$h_{rk} = \frac{(-1)^k}{k! A_r} \frac{\prod_{j=1}^{m} \Gamma(b_j + (1 - a_r + k)B_j/A_r) \prod_{j=1, j\neq r}^{n} \Gamma(1 - a_j - (1 - a_r + k)A_j/A_r)}{\prod_{j=n+1}^{p} \Gamma(a_j - (1 - a_r + k)A_j/A_r) \prod_{j=m+1}^{q} \Gamma(1 - b_j - (1 - a_r + k)B_j/A_r)}. \tag{A9}$$

e. Series expansion

In Ref. 19 we can see that in some cases there is a series expansion for Fox's H-function. For example, when the poles of $\prod_{j=1}^{m} \Gamma(b_j + B_j s)$ are simple, we can write

$$H^{m,n}_{p,q}(x) = \sum_{j=1}^{m} \sum_{\nu=0}^{\infty} h'_{j\nu} x^{(b_j+\nu)/B_j}, \tag{A10}$$

where

$$h'_{j\nu} = \frac{(-1)^\nu}{\nu! B_j} \frac{\prod_{i=1, i\neq j}^{m} \Gamma(b_i - B_i(b_j + \nu)/B_j) \prod_{i=1}^{n} \Gamma(1 - a_i + A_i(b_j + \nu)/B_j)}{\prod_{i=m+1}^{q} \Gamma(1 - b_i + B_i(b_j + \nu)/B_j) \prod_{i=n+1}^{p} \Gamma(a_i - A_i(b_j + \nu)/B_j)}. \tag{A11}$$

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