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Fractional wave-diffusion equation with periodic conditions

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We study a time-space fractional wave-diffusion equation with periodic conditions using Laplace transforms and Fourier series and presenting its solution in terms of three-parameter Mittag-Leffler functions. As a particular case we recover a recent result. We also present some graphics associated with particular values of the parameters. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4769270]

I. INTRODUCTION

There exist in nature phenomena that cannot be discussed by means of ordinary calculus, that is, using normal partial differential equations, because they depend on the so-called memory effect. In order to account for such dependence one must use a fractional differential equation which is to be studied within the context of the calculus of arbitrary order, more popularly known as fractional calculus. The theory of fractional calculus is an old one but has only recently been used to discuss phenomena involving memory effects, particularly in rheology,[1,2] diffusive systems,[3–5] quantum mechanics[6,7] and other fields.[8] An important feature of fractional calculus is that in a convenient limit—a particular value of the parameter involved—the results obtained by means of ordinary calculus must be recovered, a characteristic that reveals its nonlocal character.

It is well known that the solutions of a normal (integer order) partial differential equation with constant coefficients can be expressed in terms of exponential functions. In fractional calculus this role is played by the Mittag-Leffler functions, i.e., they can be used to express the solutions of fractional differential equations with constant coefficients.

The classical way to study fractional differential equations is by means of integral transforms, namely, by applying the Laplace transform to the time variable and the Fourier transform to the space variables. Besides, as we have more than one definition for the fractional derivative,[9,10] we must also choose a convenient one. We believe that the so-called Caputo derivative is the most convenient one in discussing problems in which the initial conditions are given, because it can be interpreted in the usual way. On the other hand, the so-called Riesz-Fourier derivative is also appropriate to discuss equations associated with certain particular boundary conditions. We also recall that, unlike the Fourier transform, the methods of Fourier series expansion and Laplace transform, which can be easily computed, are convenient when dealing with periodic function.

The paper is organized as follows: in Sec. II we define the derivative in the sense of Caputo and Riesz, present some properties associated with its Laplace transform and introduce the so-called three-parameter Mittag-Leffler function. In Sec. III we obtain the solution of a general fractional equation associated with a wave-diffusion equation by means of Fourier series expansion and Laplace transform; it is then expressed in terms of a three-parameter Mittag-Leffler function, a result that generalizes a recent result by Zhang and Liu.[11] We close the paper showing the graphics of this solution for a few particular values of the parameters and pointing out directions for future studies.

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II. PRELIMINARIES

In order to introduce fractional derivatives some definition is needed. The most popular one is the Riemann-Liouville derivative, which is introduced by means of the so-called Riemann-Liouville fractional integral operator.[10]

Caputo used a modified form of that definition in which the integral and the derivative in the Riemann-Liouville fractional operator are exchanged. This is the so-called fractional derivative in the Caputo sense.

An important difference between the Riemann-Liouville and the Caputo formulations is that the Caputo derivative of a constant is equal to zero while the corresponding Riemann-Liouville derivative is different from zero. This is the reason why we discuss our problem using the Caputo derivative. We use the so-called Weyl fractional derivative to introduce the Riesz fractional derivative. We close the section with the definition of the three-parameter Mittag-Leffler function as introduced by Prabhakar,[12] presenting also some of its properties.

A. Caputo and Riesz derivatives

In order to discuss a fractional differential equation in two independent variables, we introduce the fractional derivative in the Caputo sense in the time variable and the Riesz fractional derivative in the space variable.

We define the fractional derivative of order μ in the Caputo sense by

$$ {}_*D_t^{\mu} f(t) = \begin{cases} \frac{1}{\Gamma(n-\mu)} \displaystyle\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\mu+1-n}}\, \mathrm{d}\tau, \; n-1 < \mu < n, \\[2ex] \dfrac{\mathrm{d}^n}{\mathrm{d}t^n} f(t), \quad \mu = n, \end{cases} \tag{1} $$

where n is a positive integer such that $n - 1 < \mu < n$ and $\dfrac{\mathrm{d}^n}{\mathrm{d}x^n}$ is the nth order derivative operator.

Recall that the Caputo derivative has been used by many authors in several physical applications.[9,10] As we have already said, the Caputo derivative of a constant is equal to zero. Here we mention another important reason for its choice, namely, the initial conditions for the fractional equation are usually expressed in terms of integer order derivatives whose interpretations are the same as in the study of the corresponding integer order differential equation with integer order derivatives.

We now present two properties which will appear in our applications.

P1. When $0 < \mu < 1$ in Eq. (1), the Caputo derivative reduces to

$$ {}_*D_t^{\mu} f(t) = \frac{1}{\Gamma(1-\mu)} \int_0^t \frac{f'(\tau)\mathrm{d}\tau}{(t-\tau)^{\mu}}. \tag{2} $$

P2. The Laplace transform of the Caputo fractional derivative is given by

$$ (\mathfrak{L}[{}_*D_t^{\mu} f(t)])(s) = s^{\mu}(\mathfrak{L}[f(t)])(s) - \sum_{k=0}^{n-1} s^{\mu-k-1}(D^k f)(0). \tag{3} $$

In the particular case $0 < \mu < 1$,

$$ (\mathfrak{L}[{}_*D_t^{\mu} f(t)])(s) = s^{\mu}(\mathfrak{L}[f(t)])(s) - s^{\mu-1} f(0), \tag{4} $$

where D^k is the integer order ordinary differential operator associated with the integer order derivative.

We now introduce the fractional derivative in the Riesz sense, denoted by $D_x^{\alpha} f(x)$, by means of the Weyl fractional integrals. The Riesz fractional derivative is defined as[13]

$$ D_x^{\alpha} f(x) = -\frac{D_+^{\alpha} f(x) + D_-^{\alpha} f(x)}{2\cos(\alpha\pi/2)}, \tag{5} $$

with $0 < \alpha < 2$ and $\alpha \neq 1$. In this expression, $D_{\pm}f(x)$ are the so-called Weyl fractional derivatives, defined in terms of the Weyl fractional integrals of order μ, denoted $\mathrm{I}_{\pm}^{\mu}$, by

$$D_{\pm}^{\mu} f(x) = \begin{cases} \pm \dfrac{\mathrm{d}}{\mathrm{d}x}\, \mathrm{I}_{\pm}^{1-\mu} f(x)\,, & 0 < \mu < 1, \\[2ex] \dfrac{\mathrm{d}^2}{\mathrm{d}x^2}\, \mathrm{I}_{\pm}^{2-\mu} f(x)\,, & 1 < \mu < 2, \end{cases}$$

where

$$\mathrm{I}_{+}^{\mu} f(x) = \frac{1}{\Gamma(\mu)} \int_{-\infty}^{x} (x-\xi)^{\mu-1} f(\xi)\mathrm{d}\xi$$

and

$$\mathrm{I}_{-}^{\mu} f(x) = \frac{1}{\Gamma(\mu)} \int_{x}^{\infty} (\xi-x)^{\mu-1} f(\xi)\mathrm{d}\xi.$$

As for the Caputo fractional derivative, we mention here only two properties (particular cases) of the Riesz fractional derivative.

P1. Identity operator
For $\mu = 0$ we have $D_{\pm}^{0} f(x) = f(x)$, the identity operator; then

$$R_x^0 f(x) = -\frac{f(x) + f(x)}{2} = -f(x).$$

P2. Continuity
For $\mu = 1$ we have

$$D_{\pm}^{1} f(x) = \pm \frac{\mathrm{d}}{\mathrm{d}x} f(x) \quad \text{and} \quad D_{\pm}^{2} f(x) = \frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x).$$

Thus, $D_x^1 f(x)$ can be written as a Hilbert transform[14] and

$$D_x^2 f(x) = \frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x)$$

that is, we recover the ordinary second order derivative.

B. Three-parameter Mittag-Leffler function

We introduce the three-parameter Mittag-Leffler function as proposed by Prabhakar[12] by means of the series

$$E_{\alpha,\beta}^{\rho}(z) = \sum_{k=0}^{\infty} \frac{(\rho)_k}{\Gamma(k\alpha+\beta)} \frac{z^k}{k!}, \tag{6}$$

where $(\rho)_k$ is the Pochhammer symbol, $z \in \mathbb{C}$, $\mathrm{Re}(\rho) > 0$, $\mathrm{Re}(\alpha) > 0$, and $\mathrm{Re}(\beta) > 0$. This function generalizes the classical Mittag-Leffler $E_\alpha(z)$ and the two-parameter Mittag-Leffler function $E_{\alpha,\,\beta}(z)$. Indeed, for $\rho = \beta = 1$, we have

$$E_{\alpha,1}^{1}(z) = E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}$$

and taking $\rho = 1$ we obtain,

$$E_{\alpha,\beta}^{1}(z) = E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}.$$

We also mention that for the particular case $\alpha = \rho = \beta = 1$ we have $E_{1,1}^{1}(z) = \mathrm{e}^z$, which allows us to say that the classical Mittag-Leffler function is a generalization of the exponential function.

III. A FRACTIONAL EQUATION WITH PERIODIC CONDITIONS

In this section we present the solution of a general fractional differential equation associated with a wave-diffusion equation by means of a Fourier series expansion in the space variable (in which we have periodic conditions) and a Laplace transform in the time variable. The solution is expressed in terms of a three-parameter Mittag-Leffler function. Our result generalizes a recent result by Zhang and Liu.[11]

Let us consider the following general fractional differential equation in two variables, time t and space x, with $t, x \in \mathbb{R}$ and with parameters $\mu,\ \nu \in \mathbb{R}$,

$$\nu_* D_t^{\beta} u(x,t) + \mu_* D_t^{\gamma} u(x,t) = D_x^{\alpha} u(x,t), \tag{7}$$

with $1 < \beta \leq 2$, $0 < \gamma \leq 1$ and $1 < \alpha \leq 2$. In this equation ${}_*D_t^{\beta}$ and D_x^{α} are respectively the fractional derivative in the Caputo sense and the fractional Riesz derivative.[10] Also, suppose that the initial conditions are given by $u(x, 0) = f_1(x)$ and $u_t(x, 0) = f_2(x)$, where f_1 and f_2 are two periodic real functions which can be expressed in terms of a Fourier series.

We apply the Laplace transform to Eq. (7) in the time variable and we obtain the following differential equation:

$$\nu\{s^{\beta}\widehat{u}(x,s) - s^{\beta-1}u(x,0) - s^{\beta-2}u_t(x,0)\} + \mu\{s^{\gamma}\widehat{u} - s^{\gamma-1}u(x,0)\} = D_x^{\alpha}\widehat{u}(x,s)\,.$$

It can be written as

$$(\nu s^{\beta} + \mu s^{\gamma})\widehat{u}(x,s) - (\nu s^{\beta-1} + \mu s^{\gamma-1}) f_1(x) - \nu s^{\beta-2} f_2(x) = D_x^{\alpha}\widehat{u}(x,s), \tag{8}$$

where $f_1(x)$ and $f_2(x)$ are the initial conditions.

As we have already said, f_1 and f_2 can be represented by Fourier series:

$$f_j = \sum_{n=-\infty}^{+\infty} f_{j,n}(0)\mathrm{e}^{inx}$$

for $j = 1, 2$; the corresponding coefficients are then given by

$$f_{j,n}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_j(x)\mathrm{e}^{-inx}\mathrm{d}x\,.$$

On the other hand, we must calculate the Riesz fractional derivative of $\widehat{u}(x,s)$. We use Eq. (5) to do this. First, we suppose that

$$u(x,t) = \sum_{n=-\infty}^{+\infty} d_n(t)\mathrm{e}^{inx}\,.$$

Then, for $m - 1 < \alpha < m$ we can write[11]

$$\begin{aligned}
\mathrm{I}_+^{-\alpha}u(x,t) &= \frac{\mathrm{d}^m}{\mathrm{d}x^m}\left(\frac{1}{\Gamma(m-\alpha)}\int_{-\infty}^{x}\frac{u(\xi,t)}{(x-\xi)^{1+\alpha-m}}\mathrm{d}\xi\right)\\
&= \frac{\mathrm{d}^m}{\mathrm{d}x^m}\left(\frac{1}{\Gamma(m-\alpha)}\sum_{n=-\infty}^{+\infty} d_n(t)\int_{-\infty}^{x}\frac{\mathrm{e}^{in\xi}}{(x-\xi)^{1+\alpha-m}}\mathrm{d}\xi\right)\\
&= \sum_{n=-\infty}^{+\infty}\frac{d_n(t)}{\Gamma(m-\alpha)}\frac{\mathrm{d}^m}{\mathrm{d}x^m}\left(\mathrm{e}^{inx}\int_0^{+\infty}\frac{\mathrm{e}^{-inr}}{r^{1+\alpha-m}}\mathrm{d}r\right)\\
&= \sum_{n=-\infty}^{+\infty} d_n(t)(in)^{\alpha}\mathrm{e}^{inx}
\end{aligned}$$

and

$$
\begin{aligned}
\mathrm{I}_{-}^{-\alpha}u(x,t) &= (-1)^m \frac{\mathrm{d}^m}{\mathrm{d}x^m}\left(\frac{1}{\Gamma(m-\alpha)}\int_x^{+\infty}\frac{u(\xi,t)}{(\xi-x)^{1+\alpha-m}}\mathrm{d}\xi\right)\\
&= (-1)^m \frac{\mathrm{d}^m}{\mathrm{d}x^m}\left(\frac{1}{\Gamma(m-\alpha)}\sum_{n=-\infty}^{+\infty} d_n(t)\int_x^{+\infty}\frac{\mathrm{e}^{in\xi}}{(\xi-x)^{1+\alpha-m}}\mathrm{d}\xi\right)\\
&= (-1)^m \sum_{n=-\infty}^{+\infty}\frac{d_n(t)}{\Gamma(m-\alpha)}\frac{\mathrm{d}^m}{\mathrm{d}x^m}\left(\mathrm{e}^{inx}\int_0^{+\infty}\frac{\mathrm{e}^{-inr}}{r^{1+\alpha-m}}\mathrm{d}r\right)\\
&= \sum_{n=-\infty}^{+\infty} d_n(t)(-in)^{\alpha}\mathrm{e}^{inx}.
\end{aligned}
$$

Therefore, using Eq. (5), we obtain

$$
D_x^{\alpha}u(x,t) = -\frac{1}{2\cos\left(\frac{\alpha\pi}{2}\right)}\left\{\mathrm{I}_{+}^{-\alpha}u(x,t)+\mathrm{I}_{-}^{-\alpha}u(x,t)\right\} = \sum_{n=-\infty}^{+\infty} d_n'(t)\mathrm{e}^{inx}, \tag{9}
$$

where the coefficients are given by

$$
\begin{aligned}
d_n'(t) &= -\frac{1}{2\cos\left(\frac{\alpha\pi}{2}\right)}\left[(in)^{\alpha}+(-in)^{\alpha}\right]d_n(t)\\
&= -\frac{|n|^{\alpha}}{2\cos\left(\frac{\alpha\pi}{2}\right)}\left[i^{\alpha}+(-i)^{\alpha}\right]d_n(t)\\
&= -\frac{|n|^{\alpha}}{2\cos\left(\frac{\alpha\pi}{2}\right)}\left[\mathrm{e}^{i\alpha\pi/2}+\mathrm{e}^{-i\alpha\pi/2}\right]d_n(t).
\end{aligned}
$$

This result entails the following relation:

$$
d_n'(t) = -|n|^{\alpha}d_n(t). \tag{10}
$$

Now, taking the Laplace transform of Eq. (9) we get

$$
\mathfrak{L}[D_x^{\alpha}u(x,t)] = \sum_{n=-\infty}^{+\infty}(-|n|^{\alpha})\widehat{d}_n(s)\mathrm{e}^{inx}, \tag{11}
$$

where $\widehat{d}_n(s) = \mathfrak{L}[d_n(t)]$.

Then, substituting these results in Eq. (8) we obtain

$$
\begin{aligned}
&(\nu s^{\beta}+\mu s^{\gamma})\sum_{n=-\infty}^{+\infty}\widehat{d}_n(s)\mathrm{e}^{inx} - (\nu s^{\beta-1}+\mu s^{\gamma-1})\sum_{n=-\infty}^{+\infty} f_{1,n}(0)\mathrm{e}^{inx}\\
&-\nu s^{\beta-2}\sum_{n=-\infty}^{+\infty} f_{2,n}(0)\mathrm{e}^{inx} = \sum_{n=-\infty}^{+\infty}(-|n|^{\alpha})\widehat{d}_n(s)\mathrm{e}^{inx}.
\end{aligned}
$$

This implies that

$$
(\nu s^{\beta}+\mu s^{\gamma})\widehat{d}_n(s) - (\nu s^{\beta-1}+\mu s^{\gamma-1})f_{1,n}(0) - \nu s^{\beta-2}f_{2,n}(0) = -|n|^{\alpha}\widehat{d}_n(s).
$$

This equality is an algebraic equation whose solution is given by

$$
\begin{aligned}
\widehat{d}_n(s) &= \frac{\nu s^{\beta-1}+\mu s^{\gamma-1}}{\nu s^{\beta}+\mu s^{\gamma}+|n|^{\alpha}}f_{1,n}(0) + \frac{\nu s^{\beta-2}}{\nu s^{\beta}+\mu s^{\gamma}+|n|^{\alpha}}f_{2,n}(0)\\
&= \widehat{d}_{n1}(s)f_{1,n}(0) + \widehat{d}_{n2}(s)f_{1,n}(0) + \widehat{d}_{n3}(s)f_{2,n}(0).
\end{aligned} \tag{12}
$$

We then proceed to calculate the corresponding inverse Laplace transform. To this end we employ the relation[15]

$$\mathfrak{L}^{-1}\left\{\frac{s^{\rho-1}}{s^{\alpha_1}+As^{\alpha_2}+B}\right\}=\sum_{j=0}^{+\infty}(-A)^j t^{(\alpha_1-\alpha_2)j+\alpha_1-\rho}E^{j+1}_{\alpha_1,(\alpha_1-\alpha_2)j+\alpha_1+1-\rho}(-Bt^{\alpha_1}), \tag{13}$$

in which A and B are real numbers and $\mathrm{Re}(\alpha_1)>\mathrm{Re}(\alpha_2)>0$.

We have for $\widehat{d}_{n1}(s)$:

$$\widehat{d}_{n1}(s)=\frac{\nu s^{\beta-1}}{\nu s^\beta+\mu s^\gamma+|n|^\alpha}=\frac{s^{\beta-1}}{s^\beta+\frac{\mu}{\nu}s^\gamma+\frac{|n|^\alpha}{\nu}}\,, \tag{14}$$

Taking $\rho=\beta$, $\alpha_1=\beta$, $A=\mu/\nu$, $\alpha_2=\gamma$, and $B=\frac{|n|^\alpha}{\nu}$ in Eq. (13) we get, for $\beta>\gamma>0$,

$$\begin{aligned}\mathfrak{L}^{-1}\left[\widehat{d}_{n1}(s)\right]&=\sum_{j=0}^{+\infty}\left(\frac{-\mu}{\nu}\right)^j t^{(\beta-\gamma)j+\beta-\beta}E^{j+1}_{\beta,(\beta-\gamma)j+\beta+1-\beta}\left(-\frac{|n|^\alpha}{\nu}t^\beta\right)\\&=\sum_{j=0}^{+\infty}\left(-\frac{\mu}{\nu}\right)^j t^{(\beta-\gamma)j}E^{j+1}_{\beta,(\beta-\gamma)j+1}\left(-\frac{|n|^\alpha}{\nu}t^\beta\right).\end{aligned} \tag{15}$$

Proceeding as in Eq. (14), putting $\rho=\gamma$, $\alpha_1=\beta$, $A=\mu/\nu$, $\alpha_2=\gamma$, and $B=\frac{|n|^\alpha}{\nu}$ in Eq. (13), we can show that, for

$$\widehat{d}_{n2}(s)=\frac{\mu s^{\gamma-1}}{\nu s^\beta+\mu s^\gamma+|n|^\alpha}=\frac{\mu}{\nu}\frac{s^{\gamma-1}}{s^\beta+\frac{\mu}{\nu}s^\gamma+\frac{|n|^\alpha}{\nu}}\,, \tag{16}$$

one has

$$\mathfrak{L}^{-1}\left[\widehat{d}_{n2}(s)\right]=\frac{\mu}{\nu}\sum_{j=0}^{+\infty}\left(-\frac{\mu}{\nu}\right)^j t^{(\beta-\gamma)j+\beta-\gamma}E^{j+1}_{\beta,(\beta-\gamma)j+\beta+1-\gamma}\left(-\frac{|n|^\alpha}{\nu}t^\beta\right), \tag{17}$$

with $\beta>\gamma>0$.

Finally, we must calculate

$$\widehat{d}_{n3}(s)=\frac{\nu s^{\beta-2}}{\nu s^\beta+\mu s^\gamma+|n|^\alpha}=\frac{s^{\beta-2}}{s^\beta+\frac{\mu}{\nu}s^\gamma+\frac{|n|^\alpha}{\nu}}. \tag{18}$$

In the same way, we substitute $\rho=\beta-1$, $\alpha_1=\beta$, $A=\mu/\nu$, $\alpha_2=\gamma$, and $B=\frac{|n|^\alpha}{\nu}$ in Eq. (13) to obtain

$$\mathfrak{L}^{-1}\left[\widehat{d}_{n3}(s)\right]=\sum_{j=0}^{+\infty}\left(-\frac{\mu}{\nu}\right)^j t^{(\beta-\gamma)j+1}E^{j+1}_{\beta,(\beta-\gamma)j+2}\left(-\frac{|n|^\alpha}{\nu}t^\beta\right) \tag{19}$$

with $\beta>\gamma>0$.

Using Eqs. (15), (17), and (19), we can write

$$\begin{aligned}d_n(t)&=\mathfrak{L}\left[\widehat{d}_n(s)\right]=\mathfrak{L}\left[\left(\widehat{d}_{n1}(s)+\widehat{d}_{n2}(s)\right)f_{1,n}(0)+\widehat{d}_{n3}(s)f_{2,n}(0)\right]\\&=\sum_{j=0}^{+\infty}\left(-\frac{\mu}{\nu}\right)^j t^{(\beta-\gamma)j}E^{j+1}_{\beta,(\beta-\gamma)j+1}\left(-Kt^\beta\right)f_{1,n}(0)+\\&+\frac{\mu}{\nu}\sum_{j=0}^{+\infty}\left(-\frac{\mu}{\nu}\right)^j t^{(\beta-\gamma)j+\beta-\gamma}E^{j+1}_{\beta,(\beta-\gamma)j+\beta+1-\gamma}\left(-Kt^\beta\right)f_{1,n}(0)+\\&+\sum_{j=0}^{+\infty}\left(-\frac{\mu}{\nu}\right)^j t^{(\beta-\gamma)j+1}E^{j+1}_{\beta,(\beta-\gamma)j+2}\left(-Kt^\beta\right)f_{2,n}(0),\end{aligned}$$

where $E^{\gamma}_{\mu,\nu}(z)$ is a three-parameter Mittag-Leffler function and where we have introduced the notation $K = |n|^{\alpha}/\nu$.

In order to simplify this expression we first introduce a positive parameter defined by $\eta = \beta - \gamma$; rearranging terms we then get

$$d_n(t) = \sum_{j=0}^{+\infty} \left(-\frac{\mu}{\nu} t^{\eta}\right)^j \left\{ E^{j+1}_{\beta,\eta j+1}\left(-K t^{\beta}\right) + \frac{\mu}{\nu} t^{\eta} E^{j+1}_{\beta,\eta j+\beta+1-\gamma}\left(-K t^{\beta}\right) \right\} f_{1,n}(0)+ \\ + \sum_{j=0}^{+\infty} \left(-\frac{\mu}{\nu} t^{\eta}\right)^j t E^{j+1}_{\beta,\eta j+2}\left(-K t^{\beta}\right) f_{2,n}(0), \tag{20}$$

which is our main result.

Using Eq. (20) we can present two particular cases. First, we recover the recent result obtained by Zhang and Liu;[11] second, we present the calculation associated with the classical integer case.

Case 1. To recover the result obtained by Zhang and Liu[11] we put $\nu = 1$ and $\mu = 0$ in Eq. (20). In this case the only term that contributes is $j = 0$, i.e., the sum is cancelled. Thus,

$$d_n(t) = E^1_{\beta,1}(-K t^{\beta}) f_{1,n}(0) + t E^1_{\beta,2}(-K t^{\beta}) f_{2,n}(0)\,, \tag{21}$$

where $K = |n|^{\alpha}$ and $E^1_{\alpha,\beta}(z) = E_{\alpha,\beta}(z)$ is the two-parameter Mittag-Leffler function, a particular case of the three-parameter Mittag-Leffler function introduced in Sec. II B. We thus have

$$d_n(t) = E_{\beta,1}(-K t^{\beta}) f_{1,n}(0) + t E_{\beta,2}(-K t^{\beta}) f_{2,n}(0)\,, \tag{22}$$

which is the result obtained in Zhang and Liu's paper.[11]

Case 2. The classical integer case is obtained taking $\alpha = \beta = 2$ and $\gamma = 1$, which gives $\eta = 1$. So, using Eq. (20) we have

$$d_n(t) = \sum_{j=0}^{+\infty} \left(-\frac{\mu}{\nu} t\right)^j \left\{ E^{j+1}_{2,j+1}\left(-K t^2\right) + \frac{\mu}{\nu} t E^{j+1}_{2,j+2}\left(-K t^2\right) \right\} f_{1,n}(0)+ \\ + \sum_{j=0}^{+\infty} \left(-\frac{\mu}{\nu} t\right)^j t E^{j+1}_{2,j+2}\left(-K t^2\right) f_{2,n}(0), \tag{23}$$

where $K = |n|^2/\nu$.

IV. DIFFUSION EQUATION

In this section we discuss the fractional diffusion equation for a particular value of parameter ν. It is an interesting result that for $\nu = 0$ we have only one term contributing to the solution. In this remarkable case we have a fractional diffusion equation, different from the two cases shown before, in which we have dealt with a fractional wave equation.

Using Eq. (12), we have

$$\widehat{h}_n(s) = \mu \frac{s^{\gamma-1}}{\mu s^{\gamma} + |n|^{\alpha}} = \frac{s^{\gamma-1}}{s^{\gamma} + \dfrac{|n|^{\alpha}}{\mu}}, \tag{24}$$

whose inverse is obtained by putting $A = 0$ in Eq. (13). In this case we have only one term, i.e., $j = 0$. Then we can write

$$\mathfrak{L}^{-1}\left\{ \frac{s^{\rho-1}}{s^{\alpha_1} + B} \right\} = t^{\alpha_1-\rho} E_{\alpha_1,\alpha_1+1-\rho}\left(-B t^{\alpha_1}\right) \tag{25}$$

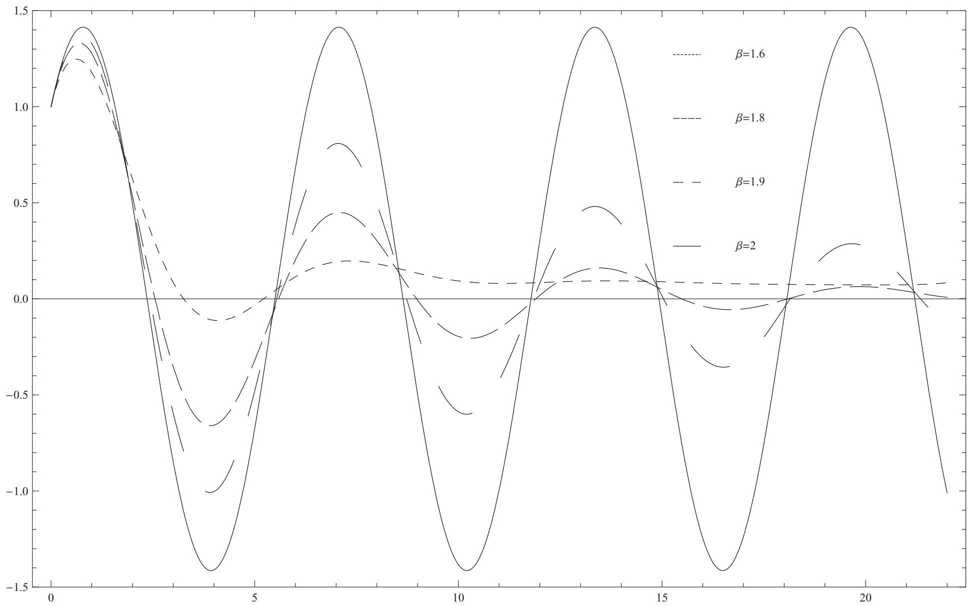

FIG. 1. $t \times u$ for $\beta = 1.6$, $\beta = 1.8$, $\beta = 1.9$ and $\beta = 2$.

and for $\alpha_1 = \gamma$, $\rho = \gamma$ and $B = \frac{|n|^\alpha}{\mu}$ we get

$$d_n(t) = t^{\gamma-\gamma} E_{\gamma,\gamma+1-\gamma}\left(-\frac{|n|^\alpha}{\mu}t^\gamma\right) f_{1,n}(0)$$

which can be written as

$$d_n = f_{1,n}(0) E_\gamma \left(-\frac{|n|^\alpha}{\mu}t^\gamma\right) \tag{26}$$

with $0 < \gamma \leq 1$ and $1 < \alpha \leq 2$.

As an example we consider Eq. (22) in order to present graphically a few particular cases of the results shown here. Taking $u(x, 0) = \sin x$ and $u_t(x, 0) = \cos x$ as initial conditions, we have for some values of β (for fixed $x = 0$) the graphics shown in Fig. 1.

V. CONCLUDING REMARKS

We close this paper pointing out that our result generalizes a recent one in which the authors[11] discussed only a particular case associated with the wave equation separately from the case associated with the diffusion equation. That's why we called our result the fractional wave-diffusion equation with periodic conditions. We note that in the limiting case, i.e., taking $\gamma = 1$ and $\beta \to 1$, we recover the diffusion equation in which parameter $\mu \to \mu + \nu$. So, the solution of the diffusion equation is the same as the one obtained in Eq. (26) with $\mu \to \mu + \nu$. Finally, we remark that this result can be extended to higher dimensions in the sense that we can consider a parameter α such that $m - 1 < \alpha < m$ and $m \in \mathbb{Z}_+$.

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