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AN EXTENSION OF MARCHIORO'S BOUND ON THE GROWTH OF A VORTEX PATCH TO FLOWS WITH L^p VORTICITY*

M. C. LOPES FILHO† AND H. J. NUSSENZVEIG LOPES†

Abstract. We observe that C. Marchioro's cubic-root bound in time on the growth of the diameter of a patch of vorticity [*Comm. Math. Phys*, 164 (1994), pp. 507–524] can be extended to incompressible two-dimensional Euler flows with compactly supported initial vorticity in L^p, $p > 2$, and with a distinguished sign.

Key words. L^p vorticity, two-dimensional incompressible flow, support of vorticity

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Let $\omega_0 \in L^p_c(\mathbb{R}^2)$, $p \geq 1$, be a compactly supported function. It was first shown by A. Majda (see [2] and references therein) that a weak solution to the two-dimensional inviscid, incompressible vorticity equation with ω_0 as initial data exists if $p > 4/3$. An extension of this result to all $p \geq 1$ is a consequence of the work of J.-M. Delort in [1], as was observed by S. Schochet in [4]. There are, however, very few results on weak solutions to two-dimensional Euler beyond existence.

The purpose of this note is to extend to a nonnegative initial vorticity ω_0 in L^p_c, $p > 2$, the $\mathcal{O}(t^{1/3})$ bound on the diameter of the support of $\omega(\cdot, t)$ obtained previously by C. Marchioro [3] for $\omega_0 \in L^\infty$. The exponent $p = 2$ is precisely the exponent for which velocity is no longer a priori bounded.

Our strategy is as follows. Fix $p > 2$ and $\omega_0 \in L^p_c$ a nonnegative, compactly supported function. Assume that $\operatorname{supp}(\omega_0) \subset\subset B_{R_0}$, the ball of radius R_0, centered at the origin. Let $\omega_0^\varepsilon \in C_c^\infty$ be a sequence of nonnegative functions, obtained by regularizing ω_0, so that $\operatorname{supp}(\omega_0^\varepsilon) \subset\subset B_{R_0}$. Let $\omega^\varepsilon = \omega^\varepsilon(x,t)$ be the sequence of smooth solutions of the two-dimensional vorticity equation, with initial data ω_0^ε. By the results proven in [1], [2], [4] there exists a subsequence of ω^ε converging weakly to a weak solution. Let $\omega = \omega(x,t)$ be a global weak solution obtained as the weak limit of such a subsequence. We will show that the support of ω^ε is contained in the disk of radius $r_t = (R_0^3 + b_1 t)^{1/3}$ for some positive constant b_1, independent of ε, thereby implying that the support of ω is contained in the same disk. Our result has the nature of an a priori estimate on any weak solution obtained by the process of regularization of initial data as described above. Hereafter we will omit the superscript ε.

Let $u = u(x,t)$ be the incompressible velocity field induced by the vorticity ω, given by $u = K * \omega$, the Biot–Savart law. We show below that, although the velocity field is only locally $W^{1,p}$, a simple estimate gives a global L^∞ bound. We will denote $p' = p/(p-1)$, the conjugate Lebesgue exponent, throughout.

LEMMA 1. *We have* $\|u\|_{L^\infty} \leq C_p \|\omega_0\|_{L^p} + (2\pi)^{-1} \int \omega_0$.

Proof. We estimate directly

$$|u(x,t)| \leq \int_{|x-y|\leq 1} (2\pi)^{-1}|x-y|^{-1}\omega(y,t)dy + \int_{|x-y|>1} (2\pi)^{-1}|x-y|^{-1}\omega(y,t)dy$$

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†Departamento de Matematica, IMECC-UNICAMP, Caixa Postal 6065, Campinas, SP 13081-970, Brasil (mlopes@ime.unicamp.br, hlopes@ime.unicamp.br).

$$\leq \|\omega(\cdot,t)\|_{L^p(B_1(x))} \, \| \, |y|^{-1} \, \|_{L^{p'}(B_1(0))} + (2\pi)^{-1} \int \omega_0.$$

Take $C_p = \| \, |y|^{-1} \, \|_{L^{p'}(B_1(0))} < \infty$, since $p > 2$. The estimate follows, since the L^p-norm of vorticity is conserved. ∎

Before we state and prove our theorem, some comments on Marchioro's proof of the L^∞ result are in order. The basic issue in the proof of Theorem 2.1 in [3] is to carefully estimate the radial velocity field at a point far from the center of motion. This is performed by decomposing velocity into the portions generated by neighboring vorticity, referred to as near-field velocity, and by vorticity remaining near the center of motion, the far-field velocity. The difficult part of this problem is to estimate the near-field.

The heart of Marchioro's argument is to obtain exponential decay of the mass of vorticity relevant for the near-field estimate; this is encoded in [3, eq. (2.64)]. Nevertheless, this estimate is still a far-field calculation, which means that it is insensitive to the unboundedness of vorticity. Finally, the near-field estimate is performed using the technique in [3, eq. (2.29)]. It is this final step that needs modification in order to extend Marchioro's result to unbounded vorticities. We will use Lemma 2 to estimate the near-field. As in [3], $m_t(r)$ denotes the mass of vorticity outside the disk of radius r at time t.

LEMMA 2. *Let $\beta \in (p'/2, 1)$. Then there exists a constant C such that*

$$\left| \int_{|y|\geq R} K(x-y)\,\omega(y,t)dy \right| \leq C(m_t(R))^{1-\beta} \|\omega_0\|_{L^p}^{\beta}.$$

Moreover, C depends only on β, p, and the Lebesgue measure of the support of the initial vorticity, $|supp\ (\omega_0)|$. In addition, $C = \mathcal{O}(1/(p-2))$ as p approaches 2.

Proof. Let B_R^c denote the set $\{|y| \geq R\}$. Write $\omega = \omega^{1-\beta}\omega^\beta$ and estimate directly

$$\left| \int_{B_R^c} K(x-y)\,\omega(y,t)dy \right| \leq (m_t(R))^{1-\beta} \left(\int_{B_R^c} |K(x-y)|^{1/\beta}\,\omega(y,t)dy \right)^{\beta}$$

$$\leq (m_t(R))^{1-\beta} \|\omega_0\|_{L^p}^{\beta} \left(\int_{\text{supp }(\omega(\cdot,t))} |K(x-y)|^{p'/\beta} dy \right)^{\beta/p'}.$$

Above, we have used Hölder's inequality twice and the conservation of the L^p-norm of vorticity.

The proof is concluded once we find an upper bound for

$$\left(\int_{\text{supp }(\omega(\cdot,t))} |K(x-y)|^{p'/\beta} dy \right)^{\beta/p'}. \tag{1}$$

The condition $p'/2 < \beta < 1$ is used to guarantee that (1) is finite.

By incompressibility, $|\text{supp }(\omega(\cdot,t))|$ is constant and equal to $|\text{supp }(\omega_0)|$. We adapt the idea in [3, eqs. (2.29),(2.30)] to obtain

$$(1) \leq \left(\frac{1}{2\pi} \int_{B_\eta} \left(\frac{1}{|z|} \right)^{p'/\beta} dz \right)^{\beta/p'}.$$

The radius η is chosen so that $\pi\eta^2 = |\text{supp }(\omega_0)|$. Denote $q = 2\beta(p-1) - p$.

Hence, (1) is bounded above by

$$\eta^{q/p}\left(\frac{\beta(p-1)}{q}\right)^{\beta/p'} \equiv C(\beta, p, |\text{supp }(\omega_0)|),$$

and, clearly, this constant C is $\mathcal{O}(1/(p-2))$ as $p \to 2$, as we wanted. ∎

THEOREM 3. *There exists a constant* $b_1 = b_1(p, R_0, \|\omega_0\|_{L^p}) > 0$ *such that the diameter of the support of* $\omega(\cdot, t)$ *is at most* $2(R_0^3 + b_1 t)^{1/3}$ *for* $t \geq 0$.

Proof. In this proof we will mention only those portions of the proof of Theorem 2.1 in [3] which need to be changed.

We begin by using Lemma 1 to ensure the existence of $t^* > 0$ so that the support of vorticity is contained in the disk of radius $r_t = (R_0^3 + b_1 t)^{1/3}$, for $0 \leq t \leq t^*$, for some positive b_1.

All subsequent arguments and estimates in Marchioro's proof are far-field estimates up to [3, eq. (2.64)] and can be adapted to the L^p case in a straightforward manner: simply substitute $K(x-y)$ by $K(x-y)\omega(y,t)$ whenever it appears.

Marchioro's argument consists of estimating the radial velocity at a point x, with $|x| = r_t$. This is done by decomposing the disk of radius r_t into a union of annuli: $\{a_{k-1} \leq |y| < a_k\}$, $1 \leq k \leq k^*$, and $\{a_{k^*} \leq |y| < r_t\}$. Here, $a_0 = 0$, $a_1 = R_0$, $a_k = 2a_{k-1}$, and k^* is chosen so that $a_{k^*+1} \leq r_t < a_{k^*+2}$. We restrict our attention to the near-field velocity, generated by vorticity outside the disk of radius a_{k^*}. Recall that $n = 2^{k^*-1} - 1$ and fix $\beta \in (p'/2, 1)$.

Substitute estimate [3, eq. (2.65)] by the following:

$$m_t(a_{k^*}) < C^n b_1^{-n} < \bar{C} n^{-2M}, \tag{2}$$

where $M = (1-\beta)^{-1}$. This is possible by choosing b_1 sufficiently large. Observe that b_1 blows up exponentially as $p \to 2$. Thus,

$$m_t(a_{k^*}) \leq C a_{k^*}^{-2M} \leq C r_t^{-2M}. \tag{3}$$

Finally, consider estimate [3, eq. (2.66)]. This is Marchioro's near-field estimate, which we substitute by Lemma 2, at $R = a_{k^*}$:

$$\left|\int_{|y| \geq a_{k^*}} K(x-y)\,\omega(y,t)dy\right| \leq C(m_t(a_{k^*}))^{1-\beta}\|\omega_0\|_{L^p}^{\beta} \leq \tilde{C} r_t^{-2M(1-\beta)} = \tilde{C} r_t^{-2}.$$

This concludes the proof. ∎

This result raises the question of what happens with more singular vorticity, such as L^p-vorticity, $1 \leq p \leq 2$ or even vortex sheets, keeping the distinguished sign restriction. Since velocity is no longer bounded, it could happen that the support of vorticity escapes to infinity instantly. This will be the object of forthcoming work.

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