AIP | Journal of Mathematical Physics

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Citation: Journal of Mathematical Physics **51**, 123517 (2010); doi: 10.1063/1.3525976
View online: http://dx.doi.org/10.1063/1.3525976
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/51/12?ver=pdfcov
Published by the AIP Publishing

The fractional Schrödinger equation for delta potentials

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(Received 16 August 2010; accepted 18 November 2010; published online 21 December 2010)

The fractional Schrödinger equation is solved for the delta potential and the double delta potential for all energies. The solutions are given in terms of Fox's H-function.
 [doi:10.1063/1.3525976]

I. INTRODUCTION

In recent years the study of fractional integrodifferential equations applied to physics and other areas has grown. For example, in Ref. 1 the authors discussed the integrodifferential equations with time-fractional integral, in Ref. 2 author has studied the partial differential equations with time-fractional derivative, and in Ref. 3 author has presented a discussion involving time- and space-fractional partial differential equations whose solutions are given in terms of Fox's H-function. Moreover, recently the fractional generalized Langevin equation is proposed to discuss the anomalous diffusive behavior of a harmonic oscillator driven by a two-parameter Mittag-Leffler noise.[4]

In this paper we discuss the fractional Schrödinger equation (FSE), as introduced by Laskin in Refs. 5 and 6. It was obtained in the context of the path integral approach to quantum mechanics. In this approach, path integrals are defined over Lévy flight paths, which is a natural generalization of the Brownian motion.[7]

There are some papers in the literature studying solutions of FSE. Some examples are Refs. 8, 9, and 10. However, recently Jeng *et al.*[11] have shown that some claims to solve the FSE have not taken into account the fact that the fractional derivation is a *nonlocal* operation. As a consequence, all those attempts based on local approaches are intrinsically wrong.

Jeng *et al.* pointed out that the only correct one they found is the one[12] involving the delta potential. However, in Ref. 12 the FSE with delta potential was studied only in the case of *negative* energies. The main objective of this paper is to solve the FSE for the delta potential for *all* energies and generalize this approach for the double delta potential, expressing the solutions in terms of Fox's H-function.

We organized this paper as follows. First, we study some properties of the FSE, its representation in momentum space, and the equation of continuity for the probability density. Then we solve the FSE for the delta and double delta potentials, presenting their respective solutions in terms of Fox's H-function. Some calculations and properties of Fox's H-function are given in Appendixes A and B.

II. THE FRACTIONAL SCHRÖDINGER EQUATION

The one-dimensional FSE is

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = D_\alpha(-\hbar^2\triangle)^{\alpha/2}\psi(x,t) + V(x)\psi(x,t), \tag{1}$$

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where $1 < \alpha \leq 2$, D_α is a constant, $\triangle = \partial_x^2$ is the Laplacian, and $(-\hbar^2\triangle)^{\alpha/2}$ is the Riesz fractional derivative,[13] that is,

$$(-\hbar^2\triangle)^{\alpha/2}\psi(x,t) = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} \mathrm{e}^{ipx/\hbar}|p|^\alpha\phi(p,t)\mathrm{d}p, \tag{2}$$

where $\phi(p,t)$ is the Fourier transform of the wave function,

$$\phi(p,t) = \int_{-\infty}^{+\infty} \mathrm{e}^{-ipx/\hbar}\psi(x,t)\mathrm{d}x, \qquad \psi(x,t) = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} \mathrm{e}^{ipx/\hbar}\phi(p,t)\mathrm{d}p. \tag{3}$$

The time-independent FSE is

$$D_\alpha(-\hbar^2\triangle)^{\alpha/2}\psi(x) + V(x)\psi(x) = E\psi(x). \tag{4}$$

In the momentum representation, this equation is written as

$$D_\alpha|p|^\alpha\phi(p) + \frac{(W*\phi)(p)}{2\pi\hbar} = E\phi(p), \tag{5}$$

where $(W*\phi)(p)$ is the convolution,

$$(W*\phi)(p) = \int_{-\infty}^{+\infty} W(p-q)\phi(q)\mathrm{d}q, \tag{6}$$

and $W(p) = \mathcal{F}[V(x)]$ is the Fourier transform of the potential $V(x)$.

A very interesting property that follows the FSE is the presence of a source (or sink) term in the continuity equation for the probability density. In order to see this, we need to write the Riesz fractional derivative in terms of the Riesz potential. Let $\mathcal{R}^\alpha\psi(x)$ be the Riesz potential of $\psi(x)$ of order α given in[13]

$$\mathcal{R}^\alpha\psi(x) = \frac{1}{2\Gamma(\alpha)\cos(\alpha\pi/2)}\int_{-\infty}^{+\infty}\frac{\psi(\xi)}{|x-\xi|^{1-\alpha}}\mathrm{d}\xi, \tag{7}$$

for $0 < \alpha < 1$, and $\tilde{\mathcal{R}}^\alpha\psi(x)$ be its conjugated Riesz potential given by

$$\tilde{\mathcal{R}}^\alpha\psi(x) = \frac{1}{2\Gamma(\alpha)\sin(\alpha\pi/2)}\int_{-\infty}^{+\infty}\frac{\mathrm{sign}(x-\xi)\psi(\xi)}{|x-\xi|^{1-\alpha}}\mathrm{d}\xi. \tag{8}$$

We suppose that $\psi(x)$ satisfies the appropriated conditions to guarantee the existence of these operations (see Ref. 14). Since these potentials are written in terms of convolutions, their Fourier transform can be easily calculated. First, we note that

$$\int_{-\infty}^{+\infty} \mathrm{e}^{-ipx/\hbar}\frac{1}{|x|^{1-\alpha}}\mathrm{d}x = 2\hbar^\alpha|p|^{-\alpha}\Gamma(\alpha)\cos(\pi\alpha/2), \tag{9}$$

$$\int_{-\infty}^{+\infty} \mathrm{e}^{-ipx/\hbar}\frac{\mathrm{sign}(x)}{|x|^{1-\alpha}}\mathrm{d}x = -2i\hbar^\alpha|p|^{-\alpha}\Gamma(\alpha)\sin(\pi\alpha/2)\mathrm{sign}(p). \tag{10}$$

It follows from the convolution theorem that

$$\mathcal{F}[\mathcal{R}^\alpha\psi(x)] = \hbar^\alpha|p|^{-\alpha}\phi(p), \tag{11}$$

$$\mathcal{F}[\tilde{\mathcal{R}}^\alpha\psi(x)] = -i\hbar^\alpha\mathrm{sign}(p)|p|^{-\alpha}\phi(p), \tag{12}$$

where $\phi(p)$ is the Fourier transform of $\psi(x)$. Then, we have

$$\mathcal{F}\left[\frac{\mathrm{d}}{\mathrm{d}x}\tilde{\mathcal{R}}^{1-\alpha}\psi(x)\right] = i\frac{p}{\hbar}\mathcal{F}[\tilde{\mathcal{R}}^{1-\alpha}\psi(x)] = h^{-\alpha}|p|^\alpha\phi(p), \tag{13}$$

that is,

$$\frac{\mathrm{d}}{\mathrm{d}x}\tilde{\mathcal{R}}^{1-\alpha}\psi(x) = (-\triangle)^{\alpha/2}\psi(x), \tag{14}$$

where $0 < \alpha < 1$. For $1 < \alpha < 2$ in $(-\triangle)^{\alpha/2}$, we use that in this case $0 < 2-\alpha < 1$ and then

$$\mathcal{F}\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathcal{R}^{2-\alpha}\psi(x)\right] = -\frac{p^2}{\hbar^2}\mathcal{F}[\mathcal{R}^{2-\alpha}\psi(x)] = -h^{-\alpha}|p|^{\alpha}\phi(p), \tag{15}$$

that is,

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathcal{R}^{2-\alpha}\psi(x) = (-\triangle)^{\alpha/2}\psi(x). \tag{16}$$

Let us now use this result in the FSE. If we follow the usual steps, that is, we multiply the FSE by ψ^* and subtract from this the complex conjugated FSE multiplied by ψ, we obtain that

$$\frac{\partial\rho}{\partial t} + \frac{\partial J}{\partial x} = S, \tag{17}$$

where, as usual, the probability density $\rho = \psi^*\psi$, but the expressions for the probability current J and the source term S are

$$J = 2D_\alpha\hbar^{\alpha-1}\mathrm{Re}\left[i\psi\frac{\partial}{\partial x}(\mathcal{R}^{2-\alpha}\psi^*)\right], \tag{18}$$

$$S = 2D_\alpha\hbar^{\alpha-1}\mathrm{Re}\left[i\frac{\partial\psi}{\partial x}\frac{\partial}{\partial x}(\mathcal{R}^{2-\alpha}\psi^*)\right], \tag{19}$$

respectively.

III. DELTA POTENTIAL

Let us consider the case

$$V(x) = V_0\delta(x), \tag{20}$$

where $\delta(x)$ is the Dirac delta function and V_0 is a constant. Its Fourier transform is $W(p) = V_0$ and the convolution $(W * \phi)(p)$ is

$$(W * \phi)(p) = V_0 K, \tag{21}$$

where the constant K is

$$K = \int_{-\infty}^{+\infty}\phi(q)\mathrm{d}q. \tag{22}$$

The FSE in the momentum representation (5) is

$$\left(|p|^\alpha - \frac{E}{D_\alpha}\right)\phi(p) = -\gamma K, \tag{23}$$

where

$$\gamma = \frac{V_0}{2\pi\hbar D_\alpha}. \tag{24}$$

Now we have two situations: (i) $E < 0$ and (ii) $E \geq 0$. The case $E < 0$ has been considered by Dong and Xu,[12] but we shall consider it again here for the completeness.

(i) $E < 0$ Let us write

$$\frac{E}{D_\alpha} = -\lambda^\alpha, \tag{25}$$

where $\lambda > 0$. Then Eq. (23) gives

$$\phi(p) = \frac{-\gamma K}{|p|^\alpha + \lambda^\alpha}. \tag{26}$$

Using this in Eq. (22) gives that

$$1 = -\gamma \int_{-\infty}^{+\infty} \frac{\mathrm{d}p}{|p|^\alpha + \lambda^\alpha} = -2\gamma\lambda^{1-\alpha} \int_0^{+\infty} \frac{\mathrm{d}q}{q^\alpha + 1}. \tag{27}$$

This integral gives $(\pi/\alpha)\csc\pi/\alpha$ (see formula 3.241.2, p. 322, of Ref. 15), and therefore,

$$1 = -2\gamma\lambda^{1-\alpha}\frac{\pi}{\alpha}\csc\frac{\pi}{\alpha}. \tag{28}$$

Thus, as in usual ($\alpha = 2$) quantum mechanics, bound states exist only for a delta potential well ($V_0 < 0$). There is only one bound state whose energy follows using Eq. (25), that is,

$$E = -\left(\frac{g\csc\pi/\alpha}{\alpha\hbar D_\alpha^{1/\alpha}}\right)^{\alpha/(\alpha-1)}, \tag{29}$$

where we wrote $V_0 = -g$ ($g > 0$). Note that for $\alpha = 2$ and $D_2 = 1/(2m)$, we recover the usual result $E = -mg^2/2\hbar^2$.

The wave function $\psi(x)$ is obtained by the inverse Fourier transform of $\phi(p)$, that is,

$$\psi(x) = \frac{-\gamma K}{2\pi\hbar} \int_{-\infty}^{+\infty} \frac{\mathrm{e}^{ipx/\hbar}}{|p|^\alpha + \lambda^\alpha}\mathrm{d}p. \tag{30}$$

This integral is given by Eq. (B12) in Appendix B, and the result is

$$\psi(x) = \frac{-\alpha\gamma K}{2\lambda^\alpha |x|} H_{2,3}^{2,1}\left[\left(\hbar^{-1}\lambda\right)^\alpha |x|^\alpha \left| \begin{matrix} (1,1),(1,\alpha/2) \\ (1,\alpha),(1,1),(1,\alpha/2) \end{matrix} \right.\right], \tag{31}$$

where $H_{p,q}^{m,n}[x|-]$ denotes Fox's H-function (see Appendix B).

In order to compare the wave function for different values of α, we will first, normalize them. The calculation of $\int_{-\infty}^{+\infty} |\psi(x)|^2 \mathrm{d}x$ seems to be too complicated because of the presence of Fox's H-function; however, this task can be easily done by using Parseval's theorem, that is,

$$\int_{-\infty}^{+\infty} \psi_1^*(x)\psi_2(x)\mathrm{d}x = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} \phi_1^*(p)\phi_2(p)\mathrm{d}p. \tag{32}$$

Using formula 3.241.4 (p. 322) of Ref. 15, we have

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}p}{(|p|^\alpha + \lambda^\alpha)^2} = \frac{2\pi}{\alpha}\left(\frac{\alpha - 1}{\alpha}\right)\lambda^{1-2\alpha}\csc\frac{\pi}{\alpha}, \tag{33}$$

and then for the wave function to be normalized to unity, we must have

$$(\gamma K)^2 = \alpha\hbar\lambda^{2\alpha-1}\left(\frac{\alpha}{\alpha-1}\right)\sin\frac{\pi}{\alpha}. \tag{34}$$

Finally, with the appropriated choice of the phase, we have

$$\Psi(x) = \frac{N_\alpha}{|x|} H_{2,3}^{2,1}\left[\left(\hbar^{-1}\lambda\right)^\alpha |x|^\alpha \left| \begin{matrix} (1,1),(1,\alpha/2) \\ (1,\alpha),(1,1),(1,\alpha/2) \end{matrix} \right.\right], \tag{35}$$

where

$$N_\alpha = \frac{\alpha^2}{2}\sqrt{\frac{\hbar}{\lambda(\alpha-1)}\sin\frac{\pi}{\alpha}}. \tag{36}$$

In Fig. 1 we plot this wave function for some values of α.

(ii) $E \geq 0$ In this case, we write

$$\frac{E}{D_\alpha} = \lambda^\alpha, \tag{37}$$

where $\lambda > 0$. Since $f(x)\delta(x) = f(0)\delta(x)$, the solution of Eq. (23) in this case is

$$\phi(p) = \frac{-\gamma K}{|p|^\alpha - \lambda^\alpha} + c_1\delta(p - \lambda) + c_2\delta(p + \lambda), \tag{38}$$

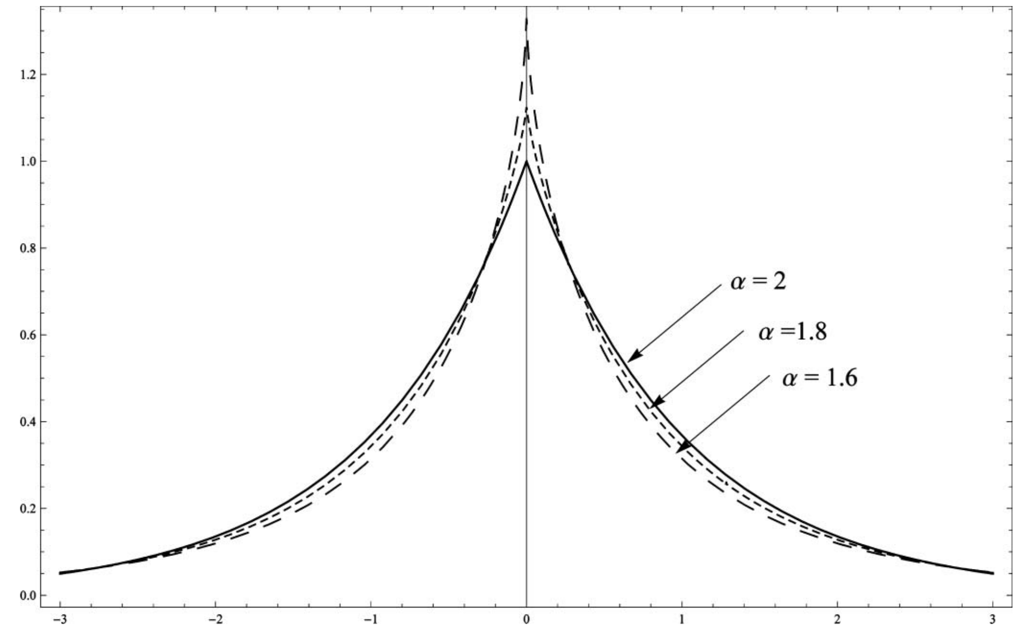

FIG. 1. Plot of $\bar{\Psi} = \Psi/L^{-1/2}$ as a function of $\bar{x} = x/L$, where $L = (\hbar^\alpha D_\alpha/E)^{1/\alpha}$, as given by Eq. (35), for different values of α.

where c_1 and c_2 are arbitrary constants. Using this in Eq. (22) gives that

$$K = -\gamma K \int_{-\infty}^{+\infty} \frac{\mathrm{d}p}{|p|^\alpha - \lambda^\alpha} + c_1 + c_2, \tag{39}$$

where the integral is interpreted in the sense of Cauchy principal value, and it gives

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}p}{|p|^\alpha - \lambda^\alpha} = 2\lambda^{1-\alpha} \int_0^{+\infty} \frac{\mathrm{d}q}{q^\alpha - 1} = -2\lambda^{1-\alpha} \frac{\pi}{\alpha} \cot \frac{\pi}{\alpha}, \tag{40}$$

where we have used formula 3.241.3 (p. 322) of Ref. 15–see Eq. (B18). The constant K is therefore

$$K = \frac{(c_1 + c_2)\alpha\lambda^{\alpha-1}}{\alpha\lambda^{\alpha-1} - 2\pi\gamma \cot\pi/\alpha}, \tag{41}$$

and we have

$$\phi(p) = c_1\delta(p-\lambda) + c_2\delta(p+\lambda) - \frac{\gamma(c_1+c_2)\alpha\lambda^{\alpha-1}}{(\alpha\lambda^{\alpha-1} - 2\pi\gamma\cot\pi/\alpha)} \frac{1}{(|p|^\alpha - \lambda^\alpha)}. \tag{42}$$

Next, we need to calculate the inverse Fourier transform of $\phi(p)$ to obtain $\psi(x)$, that is,

$$\psi(x) = C_1 \mathrm{e}^{i\lambda x/\hbar} + C_2 \mathrm{e}^{-i\lambda x/\hbar} - \frac{\gamma(C_1+C_2)\alpha\lambda^{\alpha-1}}{(\alpha\lambda^{\alpha-1} - 2\pi\gamma\cot\pi/\alpha)} \int_{-\infty}^{+\infty} \frac{\mathrm{e}^{ipx/\hbar}}{|p|^\alpha - \lambda^\alpha} \mathrm{d}p, \tag{43}$$

where we defined $C_j = c_j/2\pi\hbar$ for $j = 1, 2$. The above integral is calculated in Appendix B and is given by Eq. (B20). Using this result and the definitions of γ in Eq. (24) and λ in Eq. (37), we can write that

$$\psi(x) = C_1 \mathrm{e}^{i\lambda x/\hbar} + C_2 \mathrm{e}^{-i\lambda x/\hbar} + \Omega_\alpha \frac{(C_1+C_2)}{2} \Phi_\alpha\left(\frac{\lambda|x|}{\hbar}\right), \tag{44}$$

where

$$\Phi_\alpha\left(\frac{\lambda|x|}{\hbar}\right)=\frac{\alpha\hbar}{\lambda|x|}\left(H^{2,1}_{2,3}\left[\left(\frac{\lambda|x|}{\hbar}\right)^\alpha\left|\begin{matrix}(1,1),(1,(2+\alpha)/2)\\(1,\alpha),(1,1),(1,(2+\alpha)/2)\end{matrix}\right.\right]\right.$$
$$\left.-H^{2,1}_{2,3}\left[\left(\frac{\lambda|x|}{\hbar}\right)^\alpha\left|\begin{matrix}(1,1),(1,(2-\alpha)/2)\\(1,\alpha),(1,1),(1,(2-\alpha)/2)\end{matrix}\right.\right]\right), \tag{45}$$

and

$$\Omega_\alpha=\left[\left(\frac{E}{U}\right)^{\frac{\alpha-1}{\alpha}}-\cot\frac{\pi}{\alpha}\right]^{-1}, \tag{46}$$

and

$$U=\left(\frac{V_0}{\alpha\hbar D_\alpha^{1/\alpha}}\right)^{\alpha/(\alpha-1)}. \tag{47}$$

Now we want to find the constants C_1 and C_2 in order to compare the wave function for different values of α, and we will do this by studying the asymptotic behavior of it. The asymptotic behavior of Fox's H-function is given, if $\Delta>0$, by Eq. (A8) or Eq. (A10) according to $\Delta^*>0$ or $\Delta^*=0$, respectively—see Eq. (A7). In $\Phi_\alpha(\lambda|x|/\hbar)$, we have the difference between two Fox's H-functions of the form

$$H^{2,1}_{2,3}\left[w^\alpha\left|\begin{matrix}(1,1),(1,\mu)\\(1,\alpha),(1,1),(1,\mu)\end{matrix}\right.\right],$$

for $\mu=(2+\alpha)/2$ and $\mu=(2-\alpha)/2$. In both cases we have $\Delta=\alpha>0$, but $\Delta^*=0$ for $\mu=(2+\alpha)/2$ and $\Delta^*>0$ for $\mu=(2-\alpha)/2$. Therefore, using Eq. (A8) when $\mu=(2-\alpha)/2$ and Eq. (A10) when $\mu=(2+\alpha)/2$, we have, respectively, that

$$H^{2,1}_{2,3}\left[w^\alpha\left|\begin{matrix}(1,1),(1,(2+\alpha)/2)\\(1,\alpha),(1,1),(1,(2+\alpha)/2)\end{matrix}\right.\right]=\frac{2w}{\alpha}\sin w+o(1),\quad |w|\to\infty, \tag{48}$$

$$H^{2,1}_{2,3}\left[z^\alpha\left|\begin{matrix}(1,1),(1,(2-\alpha)/2)\\(1,\alpha),(1,1),(1,(2-\alpha)/2)\end{matrix}\right.\right]=o(1),\qquad |w|\to\infty, \tag{49}$$

and then

$$\Phi_\alpha\left(\frac{\lambda|x|}{\hbar}\right)=2\sin\frac{\lambda|x|}{\hbar}+o\big(|x|^{-1}\big),\quad |x|\to\infty. \tag{50}$$

The behavior of the wave function $\psi(x)$ for $x\to\pm\infty$ is, therefore,

$$\psi(x)=C_1\mathrm{e}^{i\lambda x/\hbar}+C_2\mathrm{e}^{-i\lambda x/\hbar}\pm\Omega_\alpha(C_1+C_2)\sin\frac{\lambda x}{\hbar}+o\big(x^{-1}\big),\quad x\to\pm\infty, \tag{51}$$

or

$$\psi(x)=A\mathrm{e}^{i\lambda x/\hbar}+B\mathrm{e}^{-i\lambda x/\hbar}+o\big(x^{-1}\big),\quad x\to-\infty, \tag{52}$$

$$\psi(x)=C\mathrm{e}^{i\lambda x/\hbar}+D\mathrm{e}^{-i\lambda x/\hbar}+o\big(x^{-1}\big),\quad x\to+\infty, \tag{53}$$

where we defined

$$A=C_1+i(C_1+C_2)\Omega_2/2,\quad B=C_2-i(C_1+C_2)\Omega_2/2, \tag{54}$$

$$C=C_1-i(C_1+C_2)\Omega_2/2,\quad D=C_2+i(C_1+C_2)\Omega_2/2. \tag{55}$$

Now let us consider the situation of particles coming from the left and scattered by the delta potential. In this case $D=0$ (no particles coming from the right) and $B=rA$ and $C=tA$, where the reflexion

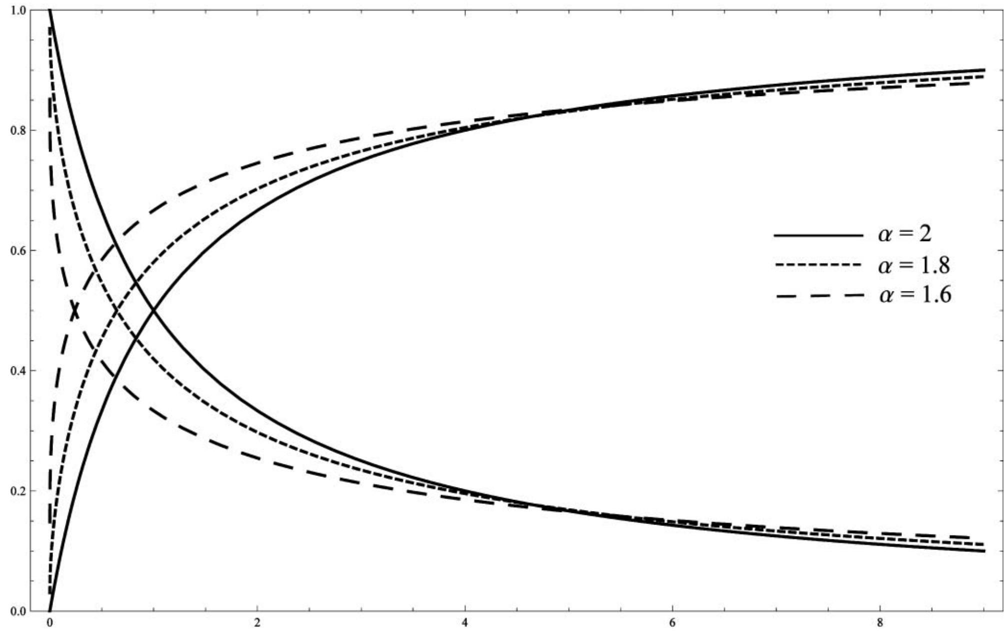

FIG. 2. Reflection and transmission coefficients as a function of E/U, as given by Eq. (57), for different values of α.

R and transmission T coefficients are given by $R = |r|^2$ and $T = |t|^2$ (see, for example, Ref. 16). The result is

$$r = \frac{-i\Omega_\alpha}{1 + i\Omega_\alpha}, \quad t = \frac{1}{1 + i\Omega_\alpha}, \tag{56}$$

and then

$$R = \frac{\Omega_\alpha^2}{1 + \Omega_\alpha^2}, \quad T = \frac{1}{1 + \Omega_\alpha^2}. \tag{57}$$

In Fig. 2 we show the behavior of these coefficients for different values of α. In Fig. 3 we show the probability distribution $|\psi(x)|^2$ for this situation when $E/U = 2$.

IV. DOUBLE DELTA POTENTIAL

Now let the potential be given by

$$V(x) = V_0[\delta(x + R/2) + \mu\delta(x - R/2)]. \tag{58}$$

When $V_0 < 0$ this potential can be seen as a model for the one-dimensional limit of the molecular ion H_2^+.[17] The parameter R is interpreted as the internuclear distance, and the coupling parameters are V_0 and μV_0. Its Fourier transform is

$$W(p) = V_0 \mathrm{e}^{ipR/2\hbar} + V_0\mu \mathrm{e}^{-ipR/2\hbar} \tag{59}$$

and for the convolution

$$(W * \phi)(p) = V_0 \mathrm{e}^{ipR/2\hbar} K_1(R) + V_0\mu \mathrm{e}^{-ipR/2\hbar} K_2(R), \tag{60}$$

where $K_1(R)$ and $K_2(R)$ are constants given by

$$K_1(R) = K_2(-R) = \int_{-\infty}^{+\infty} \mathrm{e}^{-iRq/2\hbar}\phi(q)\mathrm{d}q. \tag{61}$$

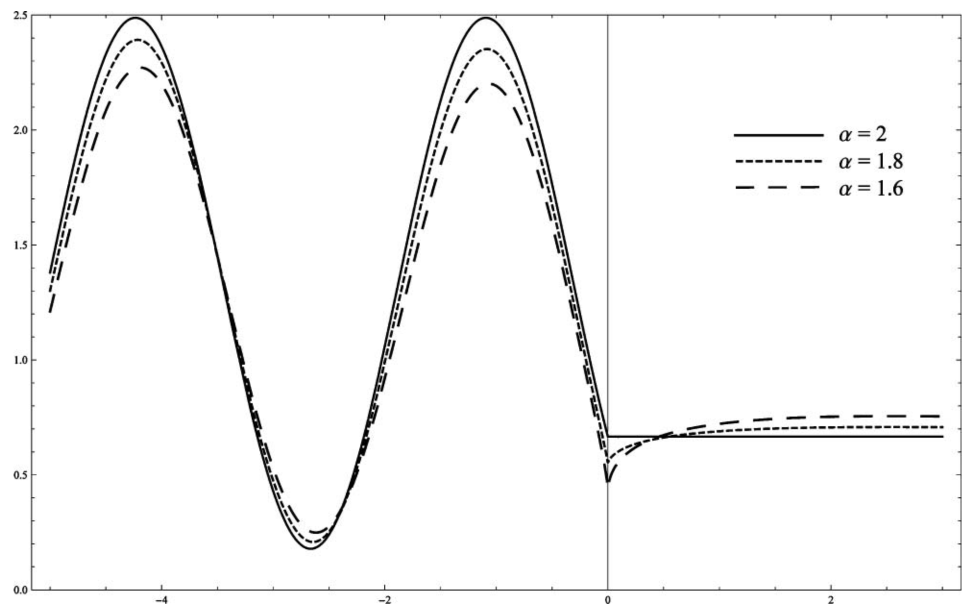

FIG. 3. Probability distribution $|\psi(x)|^2$ (in units of A^2) as a function of $x/(\hbar\lambda^{-1})$, as given by Eq. (44), for different values of α, when $E/U = 2$, and r and t given by Eq. (56).

The FSE in momentum space is

$$\left(|p|^\alpha - \frac{E}{D_\alpha}\right)\phi(p) = -\gamma e^{iRp/2\hbar}K_1(R) - \gamma\mu e^{-iRp/2\hbar}K_2(R), \tag{62}$$

where we used the notation introduced in Eq. (24). Here again we need to consider two separated cases: (i) $E < 0$ and (ii) $E > 0$.

(i) $E < 0$ If we use λ as in Eq. (25), we can write the solution of Eq. (62) in the form

$$\phi(p) = -\frac{\gamma e^{iRp/2\hbar}K_1(R)}{|p|^\alpha + \lambda^\alpha} - \frac{\gamma\mu e^{-iRp/2\hbar}K_2(R)}{|p|^\alpha + \lambda^\alpha}. \tag{63}$$

Now we use this expression for $\phi(p)$ in the definition of constants $K_1(R)$ and $K_2(R)$ in Eq. (61), and we obtain that

$$K_1(R) = -2\pi\gamma\lambda^{1-\alpha}I(0)K_1(R) - \mu 2\pi\gamma\lambda^{1-\alpha}I(R\lambda/\hbar)K_2(R), \tag{64}$$

$$K_2(R) = -2\pi\gamma\lambda^{1-\alpha}I(R\lambda/\hbar)K_1(R) - \mu 2\pi\gamma\lambda^{1-\alpha}I(0)K_2(R), \tag{65}$$

where we have defined

$$I(w) = \frac{1}{\pi}\int_0^{+\infty}\frac{\cos wy}{y^\alpha + 1}dy. \tag{66}$$

The system (64) and (65) has nontrivial solutions only when

$$(2\pi\gamma\lambda^{1-\alpha}I(0) + 1)(\mu 2\pi\gamma\lambda^{1-\alpha}I(0) + 1) - \mu[2\pi\gamma\lambda^{1-\alpha}I(R\lambda/\hbar)]^2 = 0. \tag{67}$$

There are two solutions,

$$2\pi\gamma\lambda^{1-\alpha}I(R\lambda/\hbar) = \pm\sqrt{(2\pi\gamma\lambda^{1-\alpha}I(0) + 1)(2\pi\gamma\lambda^{1-\alpha}I(0) + \mu^{-1})}, \tag{68}$$

and this gives

$$\frac{K_2(R)}{K_1(R)} = \mp\frac{1}{\mu}F(\mu), \tag{69}$$

where

$$F(\mu)=\sqrt{\frac{2\pi\gamma\lambda^{1-\alpha}I(0)+1}{2\pi\gamma\lambda^{1-\alpha}I(0)+\mu^{-1}}}. \tag{70}$$

The expression for $\phi(p)$ is

$$\phi(p)=-\gamma K_1(R)\left[\frac{\mathrm{e}^{iRp/2\hbar}}{|p|^\alpha+\lambda^\alpha}\pm F(\mu)\frac{\mathrm{e}^{-iRp/2\hbar}}{|p|^\alpha+\lambda^\alpha}\right]. \tag{71}$$

Using this, we have for the wave function $\psi(x)$ that

$$\psi(x)=\frac{-\gamma K_1(R)}{2\pi\hbar}\left[\int_{-\infty}^{+\infty}\frac{\mathrm{e}^{ip(x+R/2)/\hbar}}{|p|^\alpha+\lambda^\alpha}\mathrm{d}p\pm F(\mu)\int_{-\infty}^{+\infty}\frac{\mathrm{e}^{ip(x-R/2)/\hbar}}{|p|^\alpha+\lambda^\alpha}\mathrm{d}p\right]. \tag{72}$$

Now, if we identify the wave function for the single delta potential as given by Eq. (30), we can write that

$$\psi(x)=A[\Psi(x+R/2)\pm F(\mu)\Psi(x-R/2)]. \tag{73}$$

where $\Psi(x)$ is given by Eq. (35) and A is a constant.

Particular case: $\boxed{\mu=1}$ Let us look at Eq. (68) in more details in this case. We can write it in the form

$$\pm I(R\lambda/\hbar)=I(0)+\frac{\lambda^{\alpha-1}}{2\pi\gamma}. \tag{74}$$

Since $|I(R\lambda/\hbar)|\leq|I(0)|$, it is easy to see that there is nontrivial solutions for λ only when $\gamma<0$, which corresponds to the case of a double delta potential well. If we write $V_0=-g$ with $g>0$ and denote

$$H=\frac{\hbar\alpha D_\alpha}{gR^{\alpha-1}},\qquad \kappa=\frac{\lambda R}{\hbar}, \tag{75}$$

then Eq. (74) can be written as

$$\pm I(\kappa)=I(0)-H\kappa^{\alpha-1}. \tag{76}$$

In Fig. 4 we have the plot of the functions $\pm I(\kappa)$ and $I(0)-\pi^{-1}\kappa^{\alpha-1}$ (that is, for $H=\pi^{-1}$) for $\alpha=2$, 1.8, and 1.6 and the identification of the corresponding eigenvalues $\kappa_\alpha^\pm$. We can see that the ground and excited states are the ones with superscript $+$ and $-$, respectively, and that the energy difference between these states diminishes as α decreases.

The wave function $\psi(x)$ in Eq. (73) can be written as

$$\Psi_\pm(x)=\mathcal{N}_\alpha[\Psi(x+R/2)\pm\Psi(x-R/2)], \tag{77}$$

where $\mathcal{N}_\alpha$ is a normalization constant given by

$$\mathcal{N}_\alpha=\left[2\pm2\left(\frac{\alpha}{\alpha-1}\right)\frac{\alpha}{\pi}\sin\frac{\pi}{\alpha}\int_0^{+\infty}\frac{\cos pR\lambda/\hbar}{(p^\alpha+1)^2}\mathrm{d}p\right]^{-1/2}. \tag{78}$$

In Fig. 5 we plot $\Psi_\pm(x)$ for some values of α.

$\boxed{\text{(ii)}\quad E>0}$ Here we use λ as in Eq. (37), and for the solution of Eq. (62) we have

$$\phi(p)=2\pi\hbar C_1\delta(p-\lambda)+2\pi\hbar C_2\delta(p+\lambda)-\frac{\gamma\mathrm{e}^{iRp/2\hbar}K_1(R)}{|p|^\alpha-\lambda^\alpha}-\frac{\mu\gamma\mathrm{e}^{-iRp/2\hbar}K_2(R)}{|p|^\alpha-\lambda^\alpha}, \tag{79}$$

where the constant $2\pi\hbar$ was introduced for later convenience. Using this expression for $\phi(p)$ in Eq. (61) of definitions of $K_1(R)$ and $K_2(R)$, we have

$$(1+2\pi\gamma\lambda^{1-\alpha}J(0))K_1(R)+\mu2\pi\gamma\lambda^{1-\alpha}J(\lambda R/\hbar)K_2(R)=2\pi\hbar C_1', \tag{80}$$

$$2\pi\gamma\lambda^{1-\alpha}J(\lambda R/\hbar)K_1(R)+(1+\mu2\pi\gamma\lambda^{1-\alpha}J(0))K_2(R)=2\pi\hbar C_2', \tag{81}$$

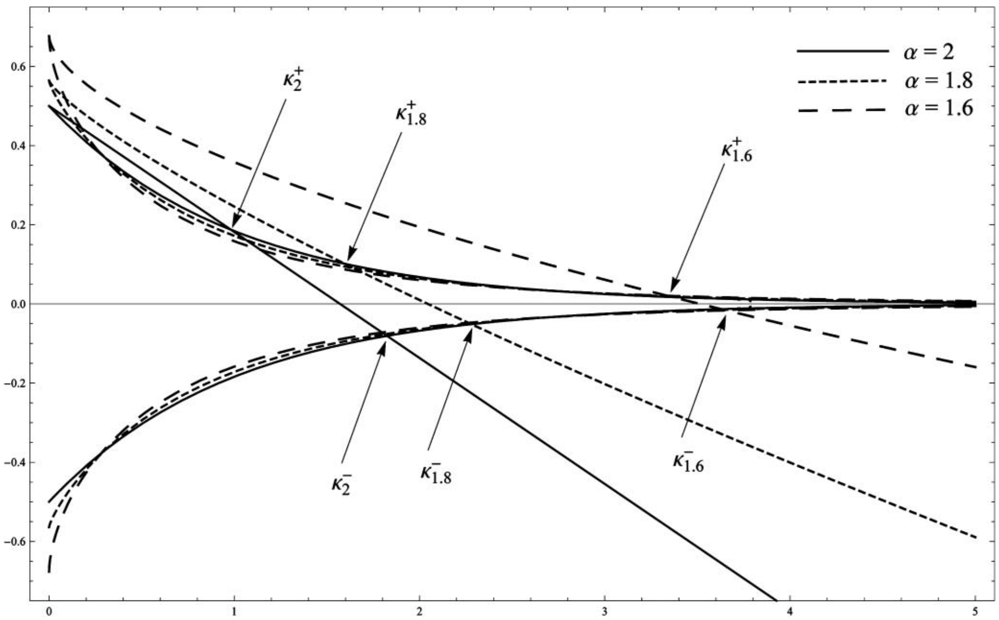

FIG. 4. Plot of the functions in Eq. (76) (with $H = \pi^{-1}$) and identification of the eigenvalues of κ for different values of α.

where

$$C_1' = C_1 e^{-iR\lambda/2\hbar} + C_2 e^{iR\lambda/2\hbar}, \qquad C_2' = C_1 e^{iR\lambda/2\hbar} + C_2 e^{-iR\lambda/2\hbar}, \tag{82}$$

and $J(w)$ is the Cauchy principal value of the integral

$$J(w) = \frac{1}{\pi}\int_0^{+\infty} \frac{\cos wq}{q^\alpha - 1} dq. \tag{83}$$

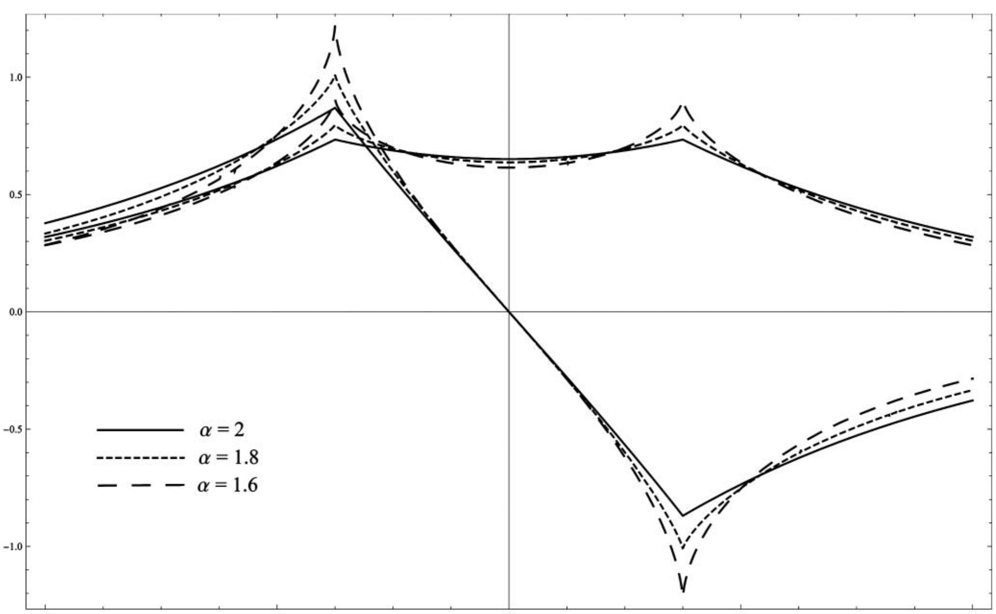

FIG. 5. Plot of $\bar{\Psi}_\pm = \Psi_\pm / L^{-1/2}$ as a function of $\bar{x} = x/L$, where $L = (\hbar^\alpha D_\alpha / E)^{1/\alpha}$, as given by Eq. (77), for different values of α.

In order to write the solution of these equations, it is convenient to define

$$\varepsilon = \frac{\lambda^{\alpha-1}}{2\pi\gamma} = \left(\frac{E}{U}\right)^{\frac{\alpha-1}{\alpha}}, \tag{84}$$

where U was defined in Eq. (47) in such a way that we have

$$K_1(R) = \frac{2\pi\hbar\varepsilon}{W}[(\varepsilon\mu^{-1} + J(0))C_1' - J(\lambda R/\hbar)C_2'], \tag{85}$$

$$K_2(R) = \frac{2\pi\hbar\varepsilon}{\mu W}[(\varepsilon + J(0))C_2' - J(\lambda R/\hbar)C_1'], \tag{86}$$

where

$$W = (\varepsilon + J(0))(\varepsilon\mu^{-1} + J(0)) - (J(\lambda R/\hbar))^2. \tag{87}$$

Using $K_1(R)$ and $K_2(R)$ in Eq. (79) gives $\phi(p)$. Then, for $\psi(x)$, we have

$$\begin{aligned}\psi(x) = {} & C_1 \mathrm{e}^{i\lambda x/\hbar} + C_2 \mathrm{e}^{-i\lambda x/\hbar} \\ & + \frac{1}{2\alpha W}[(\varepsilon\mu^{-1} + J(0))C_1' - J(\lambda R/\hbar)C_2']\Phi_\alpha\left(\frac{\lambda|x + R/2|}{\hbar}\right) \\ & + \frac{1}{2\alpha W}[(\varepsilon + J(0))C_2' - J(\lambda R/\hbar)C_1']\Phi_\alpha\left(\frac{\lambda|x - R/2|}{\hbar}\right),\end{aligned} \tag{88}$$

where we have expressed the result in terms of the function Φ_α defined in Eq. (45).

V. CONCLUSIONS

We have solved the FSE for the delta potential and the double delta potential for all energies, expressing the results in terms of Fox's H-function.

APPENDIX A: FOX'S $H-$FUNCTION

Fox's H-function, also known as H function or Fox's function, was introduced in the literature as an integral of Mellin–Barnes type.[18]

Let m, n, p, and q be integer numbers. Consider the function

$$\Lambda(s) = \frac{\prod_{i=1}^{m}\Gamma(b_i + B_i\, s)\prod_{i=1}^{n}\Gamma(1 - a_i - A_i\, s)}{\prod_{i=m+1}^{q}\Gamma(1 - b_i - B_i\, s)\prod_{i=n+1}^{p}\Gamma(a_i + A_i\, s)} \tag{A1}$$

with $1 \le m \le q$ and $0 \le n \le p$. The coefficients A_i and B_i are positive real numbers; a_i and b_i are complex parameters.

Fox's H-function, denoted by

$$H_{p,q}^{m,n}(x) = H_{p,q}^{m,n}\left(x \left| \begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\right.\right) = H_{p,q}^{m,n}\left[x \left| \begin{matrix}(a_1, A_1), \cdots, (a_p, A_p)\\(b_1, B_1), \cdots, (b_q, B_q)\end{matrix}\right.\right], \tag{A2}$$

is defined as the inverse Mellin transform, i.e.,

$$H_{p,q}^{m,n}(x) = \frac{1}{2\pi i}\int_L \Lambda(s)\, x^{-s}\, \mathrm{d}s \tag{A3}$$

where $\Lambda(s)$ is given by Eq. (A1), and the contour L runs from $L - i\infty$ to $L + i\infty$, separating the poles of $\Gamma(1 - a_i - A_i\, s)$, $(i = 1, \ldots, n)$ from those of $\Gamma(b_i + B_i\, s)$, $(i = 1, \ldots, m)$. The complex parameters a_i and b_i are taken with the imposition that no poles in the integrand coincide.

There are some interesting properties associated with Fox's H-function. We consider here the following ones.

1. Change the independent variable

Let c be a positive constant. We have

$$H_{p,q}^{m,n}\left[x\left|\begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\right.\right] = c\, H_{p,q}^{m,n}\left[x^c\left|\begin{matrix}(a_p, c\,A_p)\\(b_q, c\,B_q)\end{matrix}\right.\right]. \tag{A4}$$

To show this expression, one introduce a change of variable $s \to c\,s$ in the integral of inverse Mellin transform.

2. Change the first argument

Set $\alpha \in \mathbb{R}$. Then we can write

$$x^\alpha H_{p,q}^{m,n}\left[x\left|\begin{matrix}(a_p, A_p)\\(b_q, B_q)\end{matrix}\right.\right] = H_{p,q}^{m,n}\left[x\left|\begin{matrix}(a_p+\alpha A_p, A_p)\\(b_q+\alpha B_q, B_q)\end{matrix}\right.\right]. \tag{A5}$$

To show this expression first, we introduce the change $a_p \to a_p + \alpha A_p$ and take $s \to s - \alpha$ in the integral of inverse Mellin transform.

3. Lowering of Order

If the first factor (a_1, A_1) is equal to the last one, (b_q, B_q), we have

$$H_{p,q}^{m,n}\left[x\left|\begin{matrix}(a_1, A_1),\ldots,(a_p, A_p)\\(b_1, B_1),\ldots,(b_{q-1}, B_{q-1})(a_1, A_1)\end{matrix}\right.\right] = H_{p-1,q-1}^{m,n-1}\left[x\left|\begin{matrix}(a_2, A_2),\ldots,(a_p, A_p)\\(b_1, B_1),\ldots,(b_{q-1}, B_{q-1})\end{matrix}\right.\right]. \tag{A6}$$

To show this identity is sufficient to simplify the common arguments in the Mellin–Barnes integral.

4. Asymptotic Expansions

The asymptotic expansions for Fox's H-functions have been studied in Ref. 19. Let Δ and Δ^* be defined as

$$\Delta = \sum_{i=1}^{q} B_i - \sum_{i=1}^{p} A_i, \qquad \Delta^* = \sum_{i=1}^{n} A_i - \sum_{i=n+1}^{p} A_i + \sum_{i=1}^{m} B_i - \sum_{i=m+1}^{q} B_i. \tag{A7}$$

If $\Delta > 0$ and $\Delta^* > 0$, we have[20]

$$H_{p,q}^{m,n}(x) = \sum_{r=1}^{n}\left[h_r x^{(a_r-1)/A_r} + o\left(x^{(a_r-1)/A_r}\right)\right], \qquad |x| \to \infty, \tag{A8}$$

where

$$h_r = \frac{1}{A_r}\frac{\prod_{j=1}^{m}\Gamma(b_j+(1-a_r)B_j/A_r)\prod_{j=1,j\neq r}^{n}\Gamma(1-a_j-(1-a_r)A_j/A_r)}{\prod_{j=n+1}^{p}\Gamma(a_j-(1-a_r)A_j/A_r)\prod_{j=m+1}^{q}\Gamma(1-b_j-(1-a_r)B_j/A_r)}, \tag{A9}$$

and if $\Delta > 0$, and $\Delta^* = 0$, we have[20]

$$\begin{aligned} H_{p,q}^{m,n}(x) = &\sum_{r=1}^{n}\left[h_r x^{(a_r-1)/A_r} + o\left(x^{(a_r-1)/A_r}\right)\right] \\ &+ A x^{(\nu+1/2)/\Delta}\left(c_0 \exp[i(B + Cx^{1/\Delta})] - d_0 \exp[-i(B + Cx^{1/\Delta})]\right) \\ &+ o\left(x^{(\nu+1/2)/|\Delta|}\right), \qquad |x| \to \infty, \end{aligned} \tag{A10}$$

where

$$c_0 = (2\pi i)^{m+n-p} \exp\big[\pi i\big(\sum_{r=n+1}^{p} a_r - \sum_{j=1}^{m} b_j\big)\big],$$

$$d_0 = (-2\pi i)^{m+n-p} \exp\big[-\pi i\big(\sum_{r=n+1}^{p} a_r - \sum_{j=1}^{m} b_j\big)\pi i\big],$$

$$A = \frac{1}{2\pi i \Delta}(2\pi)^{(p-q+1)/2}\Delta^{-\nu}\prod_{r=1}^{p} A_r^{-a_r+1/2}\prod_{j=1}^{q} B_j^{b_j-1/2}\left(\frac{\Delta^\Delta}{\delta}\right)^{(\nu+1/2)/\Delta},$$

$$B = \frac{(2\nu+1)\pi}{4}, \qquad C = \left(\frac{\Delta^\Delta}{\delta}\right)^{1/\Delta},$$

$$\delta = \prod_{l=1}^{p}|A_l|^{-A_l}\prod_{j=1}^{q}|B_j|^{B_j}, \qquad \nu = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j + \frac{p-q}{2}.$$

APPENDIX B: CALCULATION OF SOME INTEGRALS

1. Calculation of the integral in Eq. (30)

Let $I(w)$ be given by

$$I(w) = \frac{1}{\pi}\int_0^{+\infty} \frac{\cos wy}{y^\alpha + 1}\mathrm{d}y. \tag{B1}$$

Then the integral in Eq. (30) can be written as

$$\int_{-\infty}^{+\infty} \frac{\mathrm{e}^{ipx/\hbar}}{|p|^\alpha + \lambda^\alpha}\mathrm{d}p = 2\pi\lambda^{1-\alpha} I(\lambda x/\hbar), \tag{B2}$$

where we remember that $\lambda > 0$. In order to calculate $I(w)$ we will take its Mellin transform, calculate the resulting integral, and then take the corresponding inverse Mellin transform. We recall that the Mellin transform pair is given by

$$\mathcal{M}[f(x)](z) = \int_0^{+\infty} x^{z-1} f(x)\mathrm{d}x, \qquad \mathcal{M}^{-1}[F(z)](x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-z}F(z)\mathrm{d}z. \tag{B3}$$

We also observe that since the Mellin transform of $I(w)$ takes only those positive values of w, and since $I(-w) = I(w)$, we only need to replace w by $|w|$ in the end of the calculation for the result to be valid for all w.

Now, since[21]

$$\mathcal{M}_w[\cos(wy)](z) = y^{-z}\Gamma(z)\cos\frac{\pi z}{2}, \tag{B4}$$

we have that

$$\mathcal{M}_w[I(w)](z) = \frac{1}{\pi}\Gamma(z)\cos\frac{\pi z}{2}\int_0^{+\infty}\frac{y^{-z}}{y^\alpha+1}\mathrm{d}y. \tag{B5}$$

This last integral is given by formula 3.241.2 (p. 322) of Ref. 15, that is,

$$\int_0^{+\infty}\frac{x^{\mu-1}}{x^\nu+1}\mathrm{d}x = \frac{\pi}{\nu}\csc\frac{\mu\pi}{\nu} = \frac{1}{\nu}B\left(\frac{\mu}{\nu}, \frac{\nu-\mu}{\nu}\right), \tag{B6}$$

where $B(a,b)$ is the beta function and $\mathrm{Re}\,\nu \geq \mathrm{Re}\,\mu > 0$. Using this result and the fact that

$$\cos\frac{\pi z}{2} = \sin\frac{\pi(1-z)}{2} = \frac{\pi}{\Gamma\left(\frac{1-z}{2}\right)\Gamma\left(1-\frac{1-z}{2}\right)}, \tag{B7}$$

we can write

$$\mathcal{M}_w[I(w)](z) = \frac{1}{\alpha}\frac{\Gamma\left(\frac{1-z}{\alpha}\right)\Gamma\left(1-\frac{1-z}{\alpha}\right)\Gamma(z)}{\Gamma\left(\frac{1-z}{2}\right)\Gamma\left(1-\frac{1-z}{2}\right)} = F_1(z). \tag{B8}$$

Then $I(w)$ is given by the inverse Mellin transform

$$I(w) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} w^{-z}F_1(z)\mathrm{d}z. \tag{B9}$$

If we compare this expression with Eqs. (A1) and (A3), we see that

$$I(w) = \frac{1}{\alpha}H^{2,1}_{2,3}\left[w\left|\begin{matrix}(1-1/\alpha, 1/\alpha), (1/2, 1/2)\\ (0, 1), (1-1/\alpha, 1/\alpha), (1/2, 1/2)\end{matrix}\right.\right]. \tag{B10}$$

Finally, if we use the properties given by Eqs. (A4) and (A5)), we have that

$$I(w) = \frac{1}{|w|}H^{2,1}_{2,3}\left[|w|^\alpha\left|\begin{matrix}(1, 1), (1, \alpha/2)\\ (1, \alpha), (1, 1), (1, \alpha/2)\end{matrix}\right.\right], \tag{B11}$$

where we used the absolute value in order to this expression to hold also for the negative values of w since $I(-w) = I(w)$. Numerical values of Fox's H-function on the RHS can be obtained by the numerical integration of $I(w)$ (the same in the next case). In this paper we have used Mathematica 7 in order to perform the numerical integrations used in all plots.

Finally, we have

$$\int_{-\infty}^{+\infty}\frac{\mathrm{e}^{ipx/\hbar}}{|p|^\alpha+\lambda^\alpha}\mathrm{d}p = \frac{2\pi\hbar}{\lambda^\alpha|x|}H^{2,1}_{2,3}\left[\left(\hbar^{-1}\lambda\right)^\alpha|x|^\alpha\left|\begin{matrix}(1, 1), (1, \alpha/2)\\ (1, \alpha), (1, 1), (1, \alpha/2)\end{matrix}\right.\right], \tag{B12}$$

which is the desired result. When $\alpha = 2$ it is not difficult to show (see Eq. (1.125) of Ref. 18) that

$$H^{2,1}_{2,3}\left[w^2\left|\begin{matrix}(1, 1), (1, 1)\\ (1, 2), (1, 1), (1, 1)\end{matrix}\right.\right] = H^{1,0}_{0,1}\left[w^2\left|\begin{matrix}-\\ (1, 2)\end{matrix}\right.\right] = \frac{|w|}{2}\exp(-|w|), \tag{B13}$$

and in such a way that we recover the well-known result

$$\int_{-\infty}^{+\infty}\frac{\mathrm{e}^{ipx/\hbar}}{p^2+\lambda^2}\mathrm{d}p = \frac{\pi}{\lambda}\exp(-\lambda|x|/\hbar). \tag{B14}$$

2. Calculation of the integral in Eq. (43)

Let $J(w)$ be given by

$$J(w) = \frac{1}{\pi}\int_0^{+\infty}\frac{\cos wy}{y^\alpha - 1}\mathrm{d}y. \tag{B15}$$

Then the integral in Eq. (31) can be written as

$$\int_{-\infty}^{+\infty}\frac{\mathrm{e}^{ipx/\hbar}}{|p|^\alpha-\lambda^\alpha}\mathrm{d}p = 2\pi\lambda^{1-\alpha}J(\lambda x/\hbar), \tag{B16}$$

where we remember that $\lambda > 0$. Taking the Mellin transform, we have that

$$\mathcal{M}_w[J(w)](z) = \frac{1}{\pi}\Gamma(z)\cos\frac{\pi z}{2}\int_0^{+\infty}\frac{y^{-z}}{y^\alpha-1}\mathrm{d}y. \tag{B17}$$

This last integral is given by formula 3.241.3 (p. 322) of Ref. 15, that is,

$$\int_0^{+\infty}\frac{x^{\mu-1}}{1-x^\nu}\mathrm{d}x = \frac{\pi}{\nu}\cot\frac{\mu\pi}{\nu}, \tag{B18}$$

where the integration is understood as the Cauchy principal value. We remember that in the inversion of the Fourier transform, the integration is to be done in the sense of the Cauchy principal value.[22]

Therefore we have

$$\mathcal{M}_w[J(w)](z) = -\frac{1}{\alpha}\Gamma(z)\sin\frac{\pi(1-z)}{2}\cot\frac{\pi(1-z)}{\alpha}. \tag{B19}$$

Using the relation $2\sin A\cos B = \sin(A+B) + \sin(A-B)$ and writing the sine function in terms of the product of gamma functions, we can write that

$$\mathcal{M}_w[J(w)](z) = -\frac{1}{2\alpha}\frac{\Gamma(z)\Gamma\left(\frac{1-z}{\alpha}\right)\Gamma\left(1-\frac{1-z}{\alpha}\right)}{\Gamma\left((1-z)\frac{(2+\alpha)}{2\alpha}\right)\Gamma\left(1-(1-z)\frac{(2+\alpha)}{2\alpha}\right)} + \frac{1}{2\alpha}\frac{\Gamma(z)\Gamma\left(\frac{1-z}{\alpha}\right)\Gamma\left(1-\frac{1-z}{\alpha}\right)}{\Gamma\left((1-z)\frac{(2-\alpha)}{2\alpha}\right)\Gamma\left(1-(1-z)\frac{(2-\alpha)}{2\alpha}\right)} = F_2(z).$$

Taking the inverse Mellin transform and using the definition of Fox's H-function, we have that

$$J(w) = -\frac{1}{2\alpha}H^{2,1}_{2,3}\left[w\left|\begin{matrix}(1-1/\alpha,1/\alpha),(1-(2+\alpha)/2\alpha,(2+\alpha)/2\alpha)\\(0,1),(1-1/\alpha,1/\alpha),(1-(2+\alpha)/2\alpha,(2+\alpha)/2\alpha)\end{matrix}\right.\right] + \frac{1}{2\alpha}H^{2,1}_{2,3}\left[w\left|\begin{matrix}(1-1/\alpha,1/\alpha),(1-(2-\alpha)/2\alpha,(2-\alpha)/2\alpha)\\(0,1),(1-1/\alpha,1/\alpha),(1-(2-\alpha)/2\alpha,(2-\alpha)/2\alpha)\end{matrix}\right.\right].$$

Using the properties given by Eqs. (A4) and (A5) and replacing w by $|w|$ since $J(-w) = J(w)$, we obtain

$$J(w) = -\frac{1}{2|w|}H^{2,1}_{2,3}\left[|w|^\alpha\left|\begin{matrix}(1,1),(1,(2+\alpha)/2)\\(1,\alpha),(1,1),(1,(2+\alpha)/\alpha)\end{matrix}\right.\right] + \frac{1}{2|w|}H^{2,1}_{2,3}\left[|w|^\alpha\left|\begin{matrix}(1,1),(1,(2-\alpha)/2)\\(1,\alpha),(1,1),(1,(2-\alpha)/2)\end{matrix}\right.\right].$$

Finally, we have

$$\int_{-\infty}^{+\infty}\frac{\mathrm{e}^{ipx/\hbar}}{|p|^\alpha-\lambda^\alpha}\mathrm{d}p = -\frac{\hbar\pi}{\lambda^\alpha|x|}\left(H^{2,1}_{2,3}\left[(\hbar^{-1}\lambda)^\alpha|x|^\alpha\left|\begin{matrix}(1,1),(1,(2+\alpha)/2)\\(1,\alpha),(1,1),(1,(2+\alpha)/\alpha)\end{matrix}\right.\right] - H^{2,1}_{2,3}\left[(\hbar^{-1}\lambda)^\alpha|x|^\alpha\left|\begin{matrix}(1,1),(1,(2-\alpha)/2)\\(1,\alpha),(1,1),(1,(2-\alpha)/2)\end{matrix}\right.\right]\right). \tag{B20}$$

Let us see what happens in the particular case $\alpha = 2$. From the definition of Fox's H-function, we can see that

$$H^{2,1}_{2,3}\left[|w|^2\left|\begin{matrix}(1,1),(1,0)\\(1,2),(1,1),(1,0)\end{matrix}\right.\right] = 0 \tag{B21}$$

and that

$$H^{2,1}_{2,3}\left[w^2\left|\begin{matrix}(1,1),(1,2)\\(1,2),(1,1),(1,2)\end{matrix}\right.\right] = H^{1,1}_{1,2}\left[w^2\left|\begin{matrix}(1,1)\\(1,1),(1,2)\end{matrix}\right.\right] = w^2H^{1,1}_{1,2}\left[w^2\left|\begin{matrix}(0,1)\\(0,1),(-1,2)\end{matrix}\right.\right].$$

But[18]

$$H^{1,1}_{1,2}\left[-z\left|\begin{matrix}(0,1)\\(0,1),(1-b,a)\end{matrix}\right.\right] = E_{a,b}(z), \tag{B22}$$

where $E_{a,b}(z)$ is the two-parameter Mittag-Leffler function. However, it is known[23] that

$$E_{2,2}(z) = \frac{\sinh\sqrt{z}}{\sqrt{z}}. \tag{B23}$$

Consequently, we have

$$H^{2,1}_{2,3}\left[w^2\left|\begin{matrix}(1,1),(1,2)\\(1,2),(1,1),(1,2)\end{matrix}\right.\right] = |w|^2E_{2,2}(-|w|^2) = |w|\sin|w|. \tag{B24}$$

Then for $\alpha = 2$, we have

$$\int_{-\infty}^{+\infty} \frac{\mathrm{e}^{ipx/\hbar}}{|p|^2 - \lambda^2} \mathrm{d}p = -\frac{\pi}{\lambda} \sin \frac{\lambda |x|}{\hbar}. \tag{B25}$$

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