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Solution of the Urban Traffic Problem with Fixed Demand Using Inexact Restoration

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Abstract

Congested traffic has become a part of the day-to-day for the residents of big metropolitan centers. From an economic viewpoint, this problem has been causing huge financial damage and strategic measures must be taken to tackle it. An alternative means of solving the problem is the inclusion of toll charges on routes with a view to de-congesting the road network. The mathematical formulation of this alternative involves the solving of an optimization problem with equilibrium constraints (MPEC). This work proposes an algorithm for the solution of this problem based on the strategy of inexact restoration.

Keywords: Optimization Problem with Equilibrium Constraints, Urban Traffic Problem, Inexact Restoration

1 Introduction

Congested traffic has become a part of the day-to-day for the residents of big metropolitan areas. With the growth of these areas the construction of new roads and highways to handle the increased traffic flow is not always possible. From the economic viewpoint, according to research in the USA [5], about 48 billion dollars are consumed annually with this type of problem and approximately 65% of the drivers of the American cities analyzed have suffered this problem.

In Brazil, the same problem occurs in its big cities, specially, in the country's biggest city. We are, of course, referring to São Paulo. To minimize the impacts of congested traffic, it's necessary that the responsible authorities make decisions aiming to control the problem. One alternative would be to charge taxes or tolls (fees) on some routes (roads or avenues) in an attempt to divert the traffic during some periods of the day.

The principal aim of this study is to approach the SBTPP Problem as one of Mathematical Programming with Equilibrium Constraints(MPEC) [16] so as to solve it through an algorithm based upon Inexact Restoration.

2 Mathematical Programming with Equilibrium Constraints (MPEC)

The problem of Mathematical Programming with Equilibrium Constraints consists of:

$$\begin{array}{ll} \underset{x,y}{\text{Minimize}} & f(x,y) \\ \text{s.a} & (x,y) \in \Omega, \\ & \langle G(x,y), y-z \rangle \leq 0 \ \forall\, z \in D(x), \\ & y \in D(x), \end{array} \tag{1}$$

where $\Omega \equiv X \times Y \subset I\!R^n \times I\!R^m$, $D(x) = \{y \in I\!R^m : h(x,y) = 0$ and $g(x,y) \geq 0\}$, $f : I\!R^{n\times m} \to I\!R$, $h_i(x,.) : I\!R^m \to I\!R$ are akin to $i \in \{1,...,q\}$, $x \in X$, $g_i(x,.) : I\!R^m \to I\!R$ $i \in \{1,...,l\}$ para todo $x \in X$ e $G : I\!R^n \times I\!R^m \to I\!R^m$.

2.1 The SBTPP Problem with Fixed Demand (SBTPPF)

In the literature, the problem of determining tolls with the aim of reducing congestion is frequently denominated a Problem of Congestion Cost, which, in this section, we will refer to as CPP, from the term in English, Congestion Pricing Problem.

In general, the CPP problem can be divided into two classes. The first, known as First Best Toll Pricing Problem ([5], [14] and [15]) assumes that all the arches that make up the network are subject to tolls. In this study we refer to this problem as FBTPP, notation for the expression in English First-Best Toll Pricing Problem'. The second is known as the Second-Best Toll Pricing Problem' ([16] and [18]), which case takes into account that in some of the arches no toll is levied. We will refer to this problem as SBTPP The FBTPP problem can be considered a particular type of SBTPP if we assume that the set of arches where tolls are not levied is empty.

Many researchers down the years have developed models and algorithms to solve SBTPP ([8], [11], [16], [19] and [21]). In [16], the approach proposed exploits the properties of the SBTPP problem and presents a reformulation of it based on the literature of the MPEC problem. In this same study, a solution algorithm based on the methodology of Slice Planes [6] is considered.

To facilitate the problem's formulation, we define as follows: i : index of network arches; I : set of all the arch indexes contained in the network; k index of the pair Start/Destination; K : set of all of the Start/Destination pair indexes; x^k : flow of the arch for the Start/Destination pair k; t_k : demand for the Start/Destination pair k; E_k : $E_k = e_q - e_p$ where e_i is the canonic vector; v : the aggregate flow vector, in this case $v = \sum_{k\in K} x^k$; $s_i(v)$: cost in time of a flow unit over the arch i; $A \in I\!R^{q\times n}$: flow restriction matrix; T_i : toll charged over the arch i; Y : set of indexes of the arches which are subject to a toll (fee);

In [16], the following hypotheses are assumed about the STBPP problem:

Let us consider:

$V = \{(v,t) : v = \sum_{k\in K} x^k, Ax^k = E_k t_k, x_k, t_k \geq 0 \quad \forall k \in K\}$ and we assume that V is limited;

s_i is continuous and differentiable $\forall\ i \in I$;

On the other hand, these hypotheses are amongst those required for the result of convergence of the Inexact Restoration Algorithm proposed in [3].

2.2 The MPEC formulation of SBTPPF

Let's consider the problem of Equilibrium with Limited Flow given in [13]:

$$\begin{array}{c} \underset{v}{\text{Minimize}} \ \sum_{i=1}^{n} \int_0^{v_i} s_i(t)dt \\ \text{s.a} \quad v_i \le c_i \quad (capacity) \quad i = 1, ..., n \qquad (BFEP) \\ v \in V = \{z \in I\!R^n : Az = b \ e \ z \ge 0\} \end{array} \tag{2}$$

If c_i were known for each arch i, the SBTPPF problem would be reduced to solving the BFEP problem but, in general, it's not easy to estimate c_i. This justification is not explicit in [16], where the SBTPP problem is treated as an MPEC problem.

If we consider the optimality conditions of the BFEP (C.O.T):

$$\begin{array}{c} \langle s(v) + T, v - u \rangle \le 0 \qquad \forall \ u \in V \\ T_i(c_i - v_i) = 0 \quad i = 1, ..., n \quad (C.O.T) \\ v_i, T_i \ge 0 \quad i = 1, ..., n \end{array} \tag{3}$$

Observe that:

- if $T_i > 0$ then $c_i = v_i$;
- if $T_i = 0$ then $c_i = \infty$;

In an intuitive manner we can conclude that the variable T_i plays the role of the toll charge in arch i, with the goal of increasing or diminishing the flow in this arch. As c_i exists, but is unknown and, in practice, we are interested in finding the pair (T, v) that solves the C.O.T problem and, at the same time, minimizes the time cost $(s(v), v)$, the mathematical formulation is:

$$\begin{array}{c} \underset{v,T}{Minimize} \langle s(v), v \rangle \\ \text{s.a} \\ T_i = 0 \text{ if} \notin Y \\ T_i \ge 0 \text{ if} \in Y \\ \langle s(v) + T, v - u \rangle \le 0 \ \forall z \in V \end{array}$$

These details of this reformulation and their respective properties can be found in [16].

3 Solution via the Inexact Restoration Algorithm

Consider the SBTPPF problem with the Mathematical Programming with Equilibrium Constraints (MPEC).

$$\begin{aligned} \underset{v,T}{\text{Minimize}}\ \langle s(v), v\rangle & \\ \text{s.a} & \\ T_i = 0 \text{ se } i \in Y & \qquad\qquad \textbf{FD-MPEC} \\ T_i \geq 0 \text{ se } i \notin Y & \\ \langle s(v) + T, v - z\rangle \leq 0\ \forall z \in V & \end{aligned}$$

onde $V = \{z \in I\!R^n : Az = b \ e \ z \geq 0\}$, $s_i : I\!R^n \to I\!R$ are separable and monotonous and differentiable at least twice, for $i \in \{1, .., n\}$.

We define,

$$C(T, v, \mu, \gamma) = \begin{pmatrix} s(v) + T + A^t\mu - \gamma \\ Av - b \\ \gamma_1 v_1 \\ \vdots \\ \gamma_l v_l \end{pmatrix} \tag{4}$$

e

$$L(T, v, \mu, \gamma, \lambda) = \langle s(v), v\rangle + C(T, v, \mu, \gamma)^T\lambda, \tag{5}$$

where $(\mu, \gamma) \in \Delta \subset I\!R^q \times I\!R^n$, with Δ convex.

To find the solution to the problem, we used the method proposed in [10]. The basic idea behind this method is to solve the MPEC problem through a sequence of approximate problems in two, distinct phases: Minimization and Restoration. In the minimization phase, the function L (defined in this section) is minimized about a plane tangential to the constraint C (defined in this section) and, in the restoration phase, the problem of equilibrium is solved by using a projection algorithm [22], thus taking advantage of the characteristic of the problem of equilibrium without any reformulation. In [10], the convergence is demonstrated of the algorithm considering some of the hypotheses satisfied by the SBPTTF problem.

4 Numerical results

The following stopping criteria, input parameters and notations were used:

- $||d_{tan}^k|| \leq 10^{-4}$, $||C(s^k)|| \leq 10^{-4}$;
- Maximum number of external iterations $k = 250$;
- Restoration Constant $r = 0, 9$;
- Stopping Criterion for the Projection Algorithm:$\|R(y^k)\| \leq 10^{-4}$;
- It_{RI}: iterations of do Inexact Restoration Algorithm;
- It_{Min}: Minimization Phase iteration Counter;

4.1 Description of the Test Problems

To test and validate the algorithm proposed in this study, some tests from the literature were solved and, apart from these, new tests were created based on the literature.

- Test I We considered the problem studied in [7].
- Test II

  Test II was considered as FBTPPF in [14] and SBTPPF in [16] The parameters for the remaining tests were extracted from [13], but had never before been treated as FBTPPF or SBTPPF. The cost function was defined in the following manner:

$$s_i(v_i) = t_i(1 + 0.15(\frac{v_i}{c_i})^4) \text{ for every } i \in A \tag{6}$$

The tables containing the numerical experiments were organized in two classes, the First-Best Toll Pricing Problem and Second-Best Toll Pricing Problem. Comparing the solutions obtained in [17] and [13] in tests I and III, respectively, we can conclude that the values of the solution in the objective function are equal to those obtained by Algorithm 2.1 as per Table 2.

In the remaining tests, as these are new problems, we can only emphasize that the stopping criteria of the algorithm were satisfied within a reasonable

Table 1: Input Parameters

		Test II		Test III		Test IV		Test V	
Link	n.Link	t_i	c_i	t_i	c_i	t_i	c_i	t_i	c_i
A-E	1	5	12	5	10^2	5	25	5	25
A-F	7	6	18	5	16^2	5	25	5	25
B-E	8	3	35	3	35	5	25	5	25
B-F	4	9	35	9	18^2	5	25	5	25
E-F	9	9	20	9	20	5	25	5	25
E-G	2	2	11	2	11	5	25	5	25
E-I	12	8	26	8	26	5	25	5	25
F-E	10	11	4	11	20	5	25	5	25
F-H	5	6	33	11	20	5	25	5	25
F-I	11	7	32	7	32	5	25	5	25
G-C	3	3	25	3	25	5	25	5	25
G-D	18	6	24	6	24	5	25	5	25
G-H	15	9	19	9	19	5	25	5	25
H-C	17	8	39	8	39	5	25	5	25
H-D	6	6	43	6	43	5	25	5	25
H-G	16	4	36	4	36	5	25	5	25
I-G	13	4	26	4	26	5	25	5	25
I-H	14	8	30	8	30	5	25	5	25

number of iterations. Analyzing Table 2, we can compare only the solution obtained in test II. In this case, the value for the objective function obtained in reference [16] is 2455.06, while our algorithm returned 2521.8.

In relation to the others, once again a solution was reached that satisfied the algorithm's stopping criteria. For the purpose of comparative analysis, let's examine the solution obtained by Algorithm 2.1 compared with the solution obtained in [13] when considering the problem as a FBTPPF, as per table 4.

The solution v returned in this case is exactly that obtained in [13]. On the other hand, the solution T is slightly different but Ttv is exactly equal. This is expected to happen, because only one solution for v, we can have various solutions for T.

Table I contains the input parameters for the tests II, III, IV and V.

The tables bellow present the demands for Tests II, III, IV, V.

Table 2: Numerical Results FBTPPF - Algorithm 2.1

Test	$\|d_{tan}^k\|$	$\|C(s^k)\|$	It_{RI} (k)	It_{Min}	f	$\|T\|$
I	10^{-5}	10^{-5}	5	5	498	13
II	10^{-5}	10^{-5}	17	82	2253.9	14.3
III	10^{-5}	10^{-4}	30	650	2160.9	14.233
IV	10^{-5}	10^{-5}	5	70	2128.0	5.6569
V	10^{-5}	10^{-5}	6	71	1967.8	5.6569

Table 3: Numerical Results SBTPPF - Algorithm 2.1

Test	$\|d_{tan}^k\|$	$\|C(s^k)\|$	It_{RI} (k)	It_{Min}	f	$\|T\|$
I*	10^{-5}	10^{-5}	5	5	498	13.8512
I**	10^{-5}	10^{-5}	3	3	552	0
II***	10^{-5}	10^{-4}	6	6	2521.8	0.2453
III***	10^{-5}	10^{-4}	8	138	2457.4	1.052
IV***	10^{-5}	10^{-4}	3	3	2313.7	0
V***	10^{-5}	10^{-4}	14	409	2153.3	0

II	C	D	**III**	C	D	**IV**	C	D	**V**	C	D
A	10	20	A	10	20	A	25	25	A	10	20
B	30	40	B	30	40	B	25	25	B	30	40

where: (*) Arch 1 charges no toll; (**) No arch charges tolls; (***) Only arches 3 and 18 charge tolls.

Table 4: Test II Solution FBTPPF

		Solution RI		paper [13]	
Link	n.Link	v_i	T_i	v_i	T_i
A-E	1	9.41	0	9.41	0
A-F	7	20.6	0	20.6	0
B-E	8	38.34	4	38.34	4
B-F	4	31.67	0	31.67	0
E-F	9	0	0	0	0
E-G	2	21.3	10.34	21.3	11.2
E-I	12	26.44	0	26.44	0
F-E	10	0	0	0	0
F-H	5	39.47	7.2	39.47	7.2
F-I	11	12.78	0	12.78	0
G-C	3	29.6	4.86	29.6	4
G-D	18	20.76	0.86	20.76	0
G-H	15	0	0	0	0
H-C	17	19.39	0	19.39	0
H-D	6	39.24	0	39.24	0
H-G	16	0	0	0	0
I-G	13	29.06	2.34	29.06	3.2
I-H	14	10.16	0	10.16	0
$s(v)^t v$		2253.917		2253.917	
$T^t v$		887.57		887.57	

5 Conclusions

This study presented an alternative method for solving the SBTPPF problem, still unreported in the literature, based upon an algorithm of inexact restoration. The advantage of this method is that it exploits the characteristics of the problem of equilibrium and solves it without the traditional reformulations. The solution algorithm behaved very well in all the test problems.

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