# Magnetization, spin current, and spin-transfer torque from $SU(2)$ local gauge invariance of the nonrelativistic Pauli-Schrödinger theory

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In this Brief Report, we consider local gauge symmetries of the nonrelativistic Pauli-Schrödinger theory. From the simplest free Lagrangian density for Pauli two-component spinors, we obtain the spin interaction with a magnetic field and define the spin-current vector without invoking relativistic theory. Applying $U(1)\times SU(2)$ local gauge symmetry, and proceeding via the Noether's theorem, we are able to construct a covariant conserved spin-current density in a natural way. Our approach allow us to understand the main features of spin transport properties and suggests that $SU(2)$ is a fundamental symmetry of nonrelativistic quantum mechanics.

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The spin-based electronics, often referred to as spintronics,[1,2] is attracting growing interest due to its potential applications in nanoelectronics and information technologies. Layered magnetic structures, such as magnetic tunneling junctions, are currently used as magnetic reading heads, magnetic-field sensors, and other applications due to giant magnetoresistance effects. The phenomenon is attributed to interactions between electron spin and the local magnetization, and is induced by controlling the magnetic configuration of the electrodes with applied magnetic fields.[3,4] A novel direction in spintronics concerns with the inverse effect; i.e., how a spin-polarized current interacts with a free ferromagnetic electrode, changing the magnetization orientation via a spin-transfer torque, as first suggested by Berger[5] and Slonczewski.[6] New fundamental phenomena, such as the spin Hall effect[7,8] and electric fields induced by spin forces,[9,10] have been discovered and are currently under further study.

Some other fundamental questions are still under debate. One important point is related to the definition of the spin current itself in systems that include the spin-orbit interaction. It has been argued that a fully relativistic formulation is required to get a conserved spin-current theorem for those systems.[11] To properly define the spin-current operator, one must start with a Lagrangian or Hamiltonian density function. In nonrelativistic quantum mechanics, the electron spin is included using the two-component spinor formalism introduced in 1926 by W. Pauli. The phenomenological Pauli Hamiltonian was intended to explain the famous Stern-Gerlach experiment, and is written simply as

$$\mathcal{H} = -\mu_B \psi^\dagger \vec{\sigma}\cdot\mathbf{B}\psi, \tag{1}$$

being $\mu_B = e\hbar/2m$ the Bohr magneton, $\psi$ the Pauli spinor wave function, $\vec{\sigma}$ the Pauli matrices, and $\mathbf{B}$ the applied magnetic field.

In 1928, Dirac advanced a relativistic theory of electrons introducing the four-component spinor formalism and spin emerged quite naturally, being commonly referred to as a relativistic effect. The Dirac equation successfully predicts the $g=2$ electron gyromagnetic factor, up to self-interaction corrections and reduces to the phenomenological Pauli-Schrödinger two-component spinor equation in the nonrelativistic limit.[12,13] In Ref. 11, the authors claim the need of starting with relativistic quantum mechanics to correctly define and generalize the spin-current density. In that context, the spin-transfer torque and the spin-Hall effects are explained. Also applying relativistic theory, Wang *et al.*[14] discussed spin-current conservation laws by means of Noether's theorem.

In spite of the above results, nonrelativistic theories have been very useful to study nanomagnetism and spintronics. As an example, we mention the work developed by Sun and Xie,[15] where a nonrelativistic approach is used to define a conserved local spin-current density. It is also predicted that the spin current generates an electric field. Generally, the nonrelativistic Pauli-Schrödinger theory leads to valuable insights, but only a few papers have been devoted to study the consequences of its local gauge symmetries. A beautiful and complete review of gauge symmetries in nonrelativistic systems is given by Fröhlich and Studer,[16] showing that the electromagnetic field appears as the gauge field related to $U(1)$ invariance, while the $SU(2)$ gauge fields describe such effects as spin-orbit interaction and Thomas precession. Following the general results obtained in Ref. 16, it is the aim of this Brief Report to study the gauge symmetries of the Pauli-Schrödinger theory to get spin transport properties. The spin current is obtained as a consequence of Noether's theorem, applied to local $SU(2)$ gauge symmetry of the Lagrangian. A further step is to apply $U(1)\times SU(2)$ local gauge symmetry, recovering the results of the nonrelativistic approximation. We will obtain the interaction of the spin with a magnetic field without invoking the relativistic theory.

Let us first review the consequences of the Dirac relativistic equation:

$$(i\gamma^\mu \partial_\mu - eA_\mu - m)\Psi = 0, \tag{2}$$

where $\gamma^\mu$ are the well-known gamma matrices, obeying the anticommuting relations, $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$, $A_\mu = (A_0, -\mathbf{A})$ is the electromagnetic four-vector potential, $m$ is the electron mass, $\Psi$ is a four-component Dirac spinor,

 

$$\Psi = \begin{pmatrix} \psi \\ \chi \end{pmatrix},$$

where $\psi$ and $\chi$ are the positive and negative-energy Pauli spinor components, respectively; units as such that $\hbar = 1$, $c = 1$. It is a straightforward matter to show that the Dirac equation reduces to the Pauli-Schrödinger equation in the nonrelativistic limit. Through the Foldy-Wouthuysen transformation, one can systematically obtain relativistic corrections in all orders of $1/m$.[12] The leading terms yield to (up to terms of the order of $1/m$ and neglecting the rest energy):

$$-\frac{1}{2m}(\nabla - ie\mathbf{A})^2\psi - \mu_B\vec{\sigma}\cdot\mathbf{B}\psi + eA_0\psi = i\frac{\partial\psi}{\partial t}, \tag{3}$$

where $\mu_B = e/2m$ is Bohr magneton, and the Pauli matrices $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ obey the following algebra:

$$[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k, \tag{4}$$

$$(\vec{\sigma}\cdot\mathbf{a})(\vec{\sigma}\cdot\mathbf{b}) = \mathbf{a}\cdot\mathbf{b} + i\vec{\sigma}\cdot(\mathbf{a}\times\mathbf{b}), \tag{5}$$

being $i,j,k \to xyz$ and $\varepsilon_{ijk}$ the Levi-Civita permutation tensor. $U(1)$ gauge transformations $\Psi' = e^{-i\zeta}\Psi$, being $\zeta$ a scalar function, leave the Dirac equation invariant with a conserved four-vector current $j^\mu = e\bar{\Psi}\gamma^\mu\Psi$. Using the Gordon current decomposition, one can easily show that $j^\mu$ is given by the sum of two terms, as follows:

$$j_1^\mu = -\frac{ie}{2m}[(\partial^\mu\bar{\Psi})\Psi - \bar{\Psi}\partial^\mu\Psi] - \frac{e^2}{2m}A^\mu\bar{\Psi}\Psi,$$

$$j_2^\mu = \frac{e}{2m}\partial_\nu(\bar{\Psi}\Sigma^{\mu\nu}\Psi),$$

where $\Sigma^{\mu\nu} \equiv i[\gamma^\mu, \gamma^\nu]/2$. The first term $j_1^\mu$ is the well-known conduction current density while the second reduces to the magnetization current $\mathbf{j}_m = \nabla\times(\psi^\dagger\vec{\sigma}\psi)$ in the nonrelativistic limit.[12] The Pauli-Schrödinger Eq. (3) can be obtained variationally, introducing a nonrelativistic Lagrangian density function $\mathcal{L}$. As usual, there are many formulations leading to Eq. (3), the difference being expressed as a total divergence. Here, we will consider only two of them. The first one corresponds to the Pauli phenomenological approach, where the term (1) has been included in an *ad hoc* way:

$$\mathcal{L} = i\psi^\dagger\frac{\partial\psi}{\partial t} - \frac{1}{2m}(\nabla + ie\mathbf{A})\psi^\dagger\cdot(\nabla - ie\mathbf{A})\psi + \mu_B\psi^\dagger\vec{\sigma}\cdot\mathbf{B}\psi - e\psi^\dagger A_0\psi. \tag{6}$$

The Euler-Lagrange equation for $\psi^\dagger$ reduces to the correct Pauli-Schrödinger Eq. (3). Notice that a $U(1)$ gauge transformation $\psi' = \psi + \delta\psi$, with an infinitesimal phase variation ($\delta\psi = -i\varepsilon\psi$) leaves the Lagrangian density invariant and Noether's theorem assures a conserved quantity, i.e., the conduction current

$$j_0 = e\psi^\dagger\psi, \tag{7}$$

$$\mathbf{j} = \frac{-ie}{2m}[\psi^\dagger\nabla\psi - (\nabla\psi^\dagger)\psi]. \tag{8}$$

The second form of a Lagrangian density we consider is

$$\mathcal{L} = i\psi^\dagger\frac{\partial\psi}{\partial t} - \frac{1}{2m}[\psi^\dagger(\overleftarrow{\nabla} + ie\mathbf{A})\cdot\vec{\sigma}][\vec{\sigma}\cdot(\vec{\nabla} - ie\mathbf{A})\psi] - e\psi^\dagger A_0\psi, \tag{9}$$

which comes as a limit from the Dirac relativistic theory and yields a conserved Noether current

$$j_0 = e\psi^\dagger\psi, \tag{10}$$

$$\mathbf{j} = \frac{-ie}{2m}[\psi^\dagger\nabla\psi - (\nabla\psi^\dagger)\psi] + \frac{e}{2m}\nabla\times(\psi^\dagger\vec{\sigma}\psi), \tag{11}$$

which makes explicit the magnetization current, with $\mathbf{M} = \mu_B(\psi^\dagger\vec{\sigma}\psi)$. The charge continuity equation ($\partial_\mu j^\mu = 0$) is not affected by the particular choice of the Lagrangian because the divergence of the magnetization current vanishes identically, as expected from classical electromagnetism.

Notice that the Lagrangian densities (6) and (9), and the results that follow, were inferred from a phenomenological point of view or from a previous knowledge of Dirac equation and its nonrelativistic limit. In the context of gauge-field theories, it is more satisfactory to obtain the Pauli-Schrödinger equation from gauge symmetries of a given Lagrangian, introducing covariant derivatives and gauge fields. Let us consider the simplest Schroedinger Lagrangian density for a free Pauli spinor and study its local gauge symmetries:

$$\mathcal{L} = i\psi^\dagger\frac{\partial\psi}{\partial t} - \frac{1}{2m}\nabla\psi^\dagger\cdot\nabla\psi, \tag{12}$$

being $\psi$ a two-component or Pauli spinor. The above Lagrangian is $U(1)\times SU(2)$ invariant under global gauge transformations. Weyl recognized already in 1928 that the nonrelativistic Schroedinger equation possesses invariance under local $U(1)$ phase transformations connected to electromagnetism,[16] which results in an abelian gauge field $A^\mu$. The first nonabelian gauge field was introduced in 1954 by Yang and Mills[17] as an attempt to explain isospin as a local gauge symmetry.[18] Nonabelian field theories are also considered to study the nonrelativistic behavior of nonabelian quantum fluids,[19] as well as to explain the electromagnetism of magnons,[20] in close connection with the present work. In what follows, we will explore the implications of the $SU(2)$ gauge symmetry only, applying Noether's theorem to define the spin-density current as a conserved quantity. Following the procedure described in Ref. 16, we impose a local $SU(2)$ gauge transformation of the form:

$$\psi' = \exp\left(-i\frac{\vec{\sigma}\cdot\vec{\Lambda}(\mathbf{x})}{2}\right)\psi, \quad \psi'^\dagger = \psi^\dagger\exp\left(i\frac{\vec{\sigma}\cdot\vec{\Lambda}(\mathbf{x})}{2}\right), \tag{13}$$

being $\vec{\Lambda}(\mathbf{x})$ a vector function. In order to keep the Lagrangian density (12) invariant under the above transformation, the covariant derivative $D_\mu$ must obey the following rule:

$$D_\mu \psi' = \exp\left(-i\frac{\vec{\sigma}\cdot\vec{\Lambda}(\mathbf{x})}{2}\right) D_\mu \psi. \tag{14}$$

The requirement is accomplished defining it as:

$$D_\mu = \partial_\mu - ig\vec{\sigma}\cdot\mathbf{W}_\mu \equiv \partial_\mu - igW_\mu,$$

being $\mathbf{W}_\mu$ a real gauge field obeying the gauge transformation

$$\widetilde{\mathbf{W}}'_\mu = \widetilde{\mathbf{W}}_\mu + \vec{\Lambda}\times\mathbf{W}_\mu - \frac{1}{g}\partial_\mu\vec{\Lambda},$$

with $g$ being the coupling constant and $W_\mu \equiv \vec{\sigma}\cdot\mathbf{W}_\mu$ is an Hermitian matrix.

Replacing the ordinary derivative by its covariant version in (12) we have:

$$\mathcal{L} = i\psi^\dagger\left[\left(\frac{\partial}{\partial t} - igW_0\right)\psi\right] - \frac{1}{2m}[\psi^\dagger(\overleftarrow{\partial}_i - igW_i)][(\vec{\partial}_i + igW_i)\psi], \tag{15}$$

where we are using the usual convention of summing over repeated indices. The corresponding Euler-Lagrange equation can be written in covariant form as follows:

$$iD_0\psi = -\frac{1}{2m}D_iD_i\psi, \tag{16}$$

being $D_0 = \partial_t - igW_0$ and $D_i = \partial_i + igW_i$. Note that the above equation, when compared with the nonrelativistic limit of the Dirac equation, contains relativistic corrections up to terms of the order $(1/m^3)$, meaning that the proposed symmetry has a fundamental character in nonrelativistic quantum mechanics.

The nonabelian gauge field $W_\mu = \vec{\sigma}\cdot\mathbf{W}_\mu$ obeys the following Lie algebra:

$$[W_\mu, W_\nu] = 2i\vec{\sigma}\cdot(\mathbf{W}_\mu\times\mathbf{W}_\nu).$$

In order to obtain a spin coupling term of the form (1), which was introduced phenomenologically, we must specify the field $W_\mu$ in the following way:

$$W_0 = \vec{\sigma}\cdot\mathbf{W}_0 = B_k\sigma_k, \tag{17}$$

$$W_i = \vec{\sigma}\cdot\mathbf{W}_i = \frac{g'}{g}\varepsilon_{ijk}E_j\sigma_k, \tag{18}$$

where $B_k$ and $E_j$ are the components of magnetic and electric fields, respectively, according to Ref. 16. Comparing with the result obtained from the nonrelativistic approximation of Dirac equation, one finds the coupling constants to be

$$g = \mu_B = \frac{e}{2m} \quad \text{and} \quad \frac{g'}{g} = \frac{1}{2}.$$

However, there is no need of a previous knowledge of the relativistic theory to obtain the above values and correctly identify the fields, since they may be interpreted in face of experimental results. Applying Noether's theorem for an infinitesimal $SU(2)$ gauge transformation $\delta\psi = -i(\vec{\sigma}\cdot\vec{\Lambda}/2)\psi$ we find:

$$\vec{\mathbf{J}}_0 = \psi^\dagger\vec{\sigma}\psi, \tag{19}$$

$$\vec{\mathbf{J}}_i = \frac{-i}{2m}[\psi^\dagger\vec{\sigma}\partial_i\psi - (\partial_i\psi^\dagger)\vec{\sigma}\psi] + \frac{g}{m}\psi^\dagger\mathbf{W}_i\psi = \vec{\mathbf{J}}_i^S + \frac{g}{m}\psi^\dagger\mathbf{W}_i\psi, \tag{20}$$

with the definition

$$\vec{\mathbf{J}}_i^S \equiv \frac{-i}{2m}[\psi^\dagger\vec{\sigma}\partial_i\psi - (\partial_i\psi^\dagger)\vec{\sigma}\psi]. \tag{21}$$

Here, $\vec{\mathbf{J}}_0$ can be identified as the spin-density vector and $\vec{\mathbf{J}}_i$ as the spin-current density. In expression (20), we have decomposed the covariant current density into a "linear" spin-current term $\vec{\mathbf{J}}_i^S$, which takes into account the translational degrees of freedom, and a term that includes the source of vector rotation (angular current), which is related to the gauge field $\mathbf{W}_i$. From our derivation, it is immediate that the "linear" spin current $\vec{\mathbf{J}}_i^S$ does not obey a continuity equation, i.e., $\partial_t\vec{\mathbf{J}}_0 + \partial_i\vec{\mathbf{J}}_i^S \neq 0$, as argued in Ref. 15, since one also has to consider the rotational degrees of freedom. This fact emerges quite naturally in our formalism, and is the most telling result of our Brief Report: from local gauge invariance, the continuity equation (conservation theorem) is written in terms of covariant derivatives instead of ordinary ones. In this case, the source terms are absorbed into the covariant definition, as given above. To pursue our argument further, we consider the covariant conservation theorem, $D_\mu\mathbf{J}^\mu = 0$, and write it down explicitly in terms of ordinary derivatives. The result is:

$$\frac{\partial\vec{\mathbf{J}}_0}{\partial t} + \partial_i\vec{\mathbf{J}}_i^S = 2g\vec{\mathbf{J}}_0\times\vec{\mathbf{B}} - \frac{g}{2m}\nabla\times(\psi^\dagger\vec{\mathbf{E}}\psi) - i\frac{g}{2m}\psi^\dagger[\vec{\mathbf{E}}\times(\overleftarrow{\nabla} - \vec{\nabla})]\times\vec{\sigma}\psi. \tag{22}$$

Notice that the calculation is straightforward, but some care must be taken when applying $D_\mu$, because it acts in a different way when applied to $\psi$ and $\psi^\dagger$, i.e., $D_\mu\psi = (\partial_\mu - igW_\mu)\psi$ and $D_\mu\psi^\dagger = \psi^\dagger(\overleftarrow{\partial}_\mu + igW_\mu)$. It is also interesting to observe that $\partial_i W_i = (\vec{\sigma}/2)\cdot\nabla\times\vec{\mathbf{E}}$. We recognize Eq. (22) as the spin continuity equation obtained in Ref. 14 using a very different approach. Now, we interpret the meaning of each term in the Eq. (22): the left-hand side yields the continuity equation of a spin current in the absence of any external field; the first term in the right-hand side (r.h.s.) is the usual torque exerted by a magnetic field on a spin 1/2 magnetic-dipole moment density; the second term corresponds to the torque caused by local changes in electron density as well as due to $\nabla\times\vec{\mathbf{E}} = -\partial\vec{\mathbf{B}}/\partial t$. The latter was incorporated phenomenologically in Ref. 21, to describe the dynamics of a spin accumulation distribution function $\mathbf{f}(\mathbf{r},t)$. The third term in the r.h.s. corresponds to the spin Hall effect, responsible for the spin accumulation at the edges of a sample, analogous to the ordinary Hall effect, when the spin and electric currents are perpendicular to each other. Remember that the latter is proportional to the vector $\mathbf{j}=-(i/2m)[\psi^{\dagger}\nabla\psi-(\nabla\psi^{\dagger})\psi]$, which is the usual conduction current obtained from $U(1)$ gauge symmetry.

In summary, in this Brief Report, we have presented the spin-torque equation as a natural consequence of $SU(2)$ local gauge symmetry in the Pauli-Schrödinger nonrelativistic theory. The spin-current density is defined naturally, and the conservation law is established via Noether's theorem. The above can be viewed as a continuity equation in terms of covariant derivatives. When expressed in terms of ordinary derivatives, one gets the proper terms in the spin-torque equation to leading orders of $g=\mu_B$. Our result suggests that $SU(2)$ is a fundamental symmetry in nonrelativistic quantum mechanics.

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