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Solitary waves for some nonlinear Schrödinger systems

Ondes solitaires pour certains systèmes d'équations de Schrödinger non linéaires

Djairo G. de Figueiredo, Orlando Lopes *

IMECC-UNICAMP, Caixa Postal 6065, Campinas, SP-13083-859, Brasil

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Abstract

In this paper we study the existence of radially symmetric positive solutions in $H^1_{\text{rad}}(\mathbb{R}^N) \times H^1_{\text{rad}}(\mathbb{R}^N)$ of the elliptic system:

$$-\Delta u + u - (\alpha u^2 + \beta v^2)u = 0,$$
$$-\Delta v + \omega^2 v - (\beta u^2 + \gamma v^2)v = 0,$$

$N = 1, 2, 3$, where α and γ are positive constants (β will be allowed to be negative). This system has trivial solutions of the form $(\phi, 0)$ and $(0, \psi)$ where ϕ and ψ are nontrivial solutions of scalar equations. The existence of nontrivial solutions for some values of the parameters $\alpha, \beta, \gamma, \omega$ has been studied recently by several authors [A. Ambrosetti, E. Colorado, Bound and ground states of coupled nonlinear Schrödinger equations, C. R. Acad. Sci. Paris, Ser. I 342 (2006) 453–458; T.C. Lin, J. Wei, Ground states of N coupled nonlinear Schrödinger equations in R^n, $n \leqslant 3$, Comm. Math. Phys. 255 (2005) 629–653; T.C. Lin, J. Wei, Ground states of N coupled nonlinear Schrödinger equations in R^n, $n \leqslant 3$, Comm. Math. Phys., Erratum, in press; L. Maia, E. Montefusco, B. Pellacci, Positive solutions for a weakly coupled nonlinear Schrödinger system, preprint; B. Sirakov, Least energy solitary waves for a system of nonlinear Schrödinger equations in R^N, preprint; J. Yang, Classification of the solitary waves in coupled nonlinear Schrödinger equations, Physica D 108 (1997) 92–112]. For $N = 2, 3$, perhaps the most general existence result has been proved in [A. Ambrosetti, E. Colorado, Bound and ground states of coupled nonlinear Schrödinger equations, C. R. Acad. Sci. Paris, Ser. I 342 (2006) 453–458] under conditions which are equivalent to ours. Motivated by some numerical computations, we return to this problem and, using our approach, we give a more detailed description of the regions of parameters for which existence can be proved. In particular, based also on numerical evidence, we show that the shape of the region of the parameters for which existence of solution can be proved, changes drastically when we pass from dimensions $N = 1, 2$ to dimension $N = 3$. Our approach differs from the ones used before. It relies heavily on the spectral theory for linear elliptic operators. Furthermore, we also consider the case $N = 1$ which has to be treated more extensively due to some lack of compactness for even functions. This case has not been treated before.

* Corresponding author.
E-mail addresses: djairo@ime.unicamp.br (D.G. de Figueiredo), lopes@ime.unicamp.br (O. Lopes).

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Résumé

Dans cet article, on étudie l'existence des solutions positives radialement symétriques dans $H^1_{\text{rad}}(\mathbb{R}^N) \times H^1_{\text{rad}}(\mathbb{R}^N)$ du système elliptique

$$-\Delta u + u - (\alpha u^2 + \beta v^2)u = 0,$$
$$-\Delta v + \omega^2 v - (\beta u^2 + \gamma v^2)v = 0,$$

$N = 1, 2, 3$ où α et γ sont des constantes positives (il est permis que β soit négatif). Ce système a des solutions triviales de la forme $(\phi, 0)$ et $(0, \psi)$ où ϕ et ψ sont des solutions non triviales des équations scalaires. L'existence de solutions non triviales pour certaines valeurs des paramètres $\alpha, \beta, \gamma, \omega$ a été étudiée récemment par plusieurs auteurs. Pour $N = 2, 3$ peut-être le résultat le plus général d'existence a été prouvé dans [A. Ambrosetti, E. Colorado, Bound and ground states of coupled nonlinear Schrödinger equations, C. R. Acad. Sci. Paris, Ser. I 342 (2006) 453–458] sous des conditions qui sont équivalentes aux nôtres. Motivé par quelques calculs numériques on retourne à ce problème et en utilisant notre approche on donne une description plus détaillée des régions de l'espace des paramètres pour lesquels l'existence peut être prouvée. En particulier, en se basant sur des résultats numériques, on démontre que la forme de la région de l'espace des paramètres pour lesquels l'existence de solutions peut être prouvée, change drastiquement quand on passe des dimensions $N = 1, 2$ à la dimension $N = 3$. Notre approche diffère des précédentes. Elle repose fortement sur la théorie spectrale des opérateurs linéaires. De plus, on considère aussi les cas $N = 1$ qui nécessite un traitement plus détaillé dût au manque de compacité pour les fonctions paires. Ce cas n'a pas été traité avant.

Keywords: Nonlinear Schrödinger systems; Nehari manifold

1. Introduction and statement of the results

In this paper we study the existence of radially symmetric positive solutions in $H^1_{\text{rad}}(\mathbb{R}^N) \times H^1_{\text{rad}}(\mathbb{R}^N)$ of the system

$$-\Delta u + u - (\alpha u^2 + \beta v^2)u = 0,$$
$$-\Delta v + \omega^2 v - (\beta u^2 + \gamma v^2)v = 0, \tag{1.1}$$

$N = 1, 2, 3$, where α and γ are positive constants (β will be allowed to be negative). System (1.1) has motivated a large amount of research, both theoretically and numerically, due to the fact that it gives solitary waves for Schrödinger systems that govern phenomena in many physical problems, specially nonlinear optics (see [3] and [8] and the references therein).

System (1.1) has unique trivial solutions of the form $(\phi, 0)$ and $(0, \psi)$ where ϕ and ψ are radially symmetric positive (nontrivial) functions satisfying

$$-\Delta\phi + \phi - \alpha\phi^3 = 0$$

and

$$-\Delta\psi + \omega^2\psi - \gamma\psi^3 = 0.$$

By a nontrivial solution of (1.1) we mean a pair (u, v) such that $u \not\equiv 0 \not\equiv v$.

To state our main existence results we need some preliminaries. We denote by ϕ_0 the unique radial positive function satisfying

$$-\Delta\phi_0 + \phi_0 - \phi_0^3 = 0 \tag{1.2}$$

and for $\eta > 0$ we define

$$-\lambda_1(\eta) = \text{principal eigenvalue of } M_0 k = -\Delta k - \eta\phi_0^2 k. \tag{1.3}$$

The behavior of the function $\lambda_1(\eta)$ for η small depends on the dimension N. In fact, as we will see later, if $N = 1, 2$ then, for $\eta > 0$, $\lambda_1(\eta)$ is positive, increasing and

$$\lim_{\eta\to 0} \lambda_1(\eta) = 0 \quad \text{and} \quad \lim_{\eta\to\infty} \lambda_1(\eta) = \infty. \tag{1.4}$$

However, if $N = 3$ then there is a $\eta_0 > 0$ such that for $0 \leqslant \eta \leqslant \eta_0$ the whole spectrum of M_0 is $[0, \infty)$ and for $\eta > \eta_0$ we have $\lambda_1(\eta) > 0$, increasing with η and

$$\lim_{\eta \to \eta_0} \lambda_1(\eta) = 0 \quad \text{and} \quad \lim_{\eta \to \infty} \lambda_1(\eta) = \infty. \tag{1.5}$$

For $N = 3$ and $0 \leqslant \eta \leqslant \eta_0$ we set $\lambda_1(\eta) = 0$. From these properties of the function $\lambda_1(\eta)$ we see that for each α and γ fixed and $N = 1, 2, 3$ there is a unique $\beta > 0$ such that $\lambda_1(\beta/\alpha) = 1/\lambda_1(\beta/\gamma)$. This value will be denoted by $\beta(\alpha, \gamma)$. Our main result is the following:

Theorem 1.1.

(i) *Suppose first $N = 1, 2$. Then*
- *if $0 \leqslant \beta < \beta(\alpha, \gamma)$ then system* (1.1) *has a nontrivial positive radially symmetric solution if*

$$\lambda_1(\beta/\alpha) < \omega^2 < \frac{1}{\lambda_1(\beta/\gamma)} \quad (\textit{region } A \textit{ in Fig. } 1); \tag{1.6}$$

- *if $\beta(\alpha, \gamma) < \beta$ then system* (1.1) *has a nontrivial positive radially symmetric solution if*

$$\frac{1}{\lambda_1(\beta/\gamma)} < \omega^2 < \lambda_1(\beta/\alpha) \quad (\textit{region } B \textit{ in Fig. } 1). \tag{1.7}$$

(ii) *Suppose now $N = 3$ and $\alpha < \gamma$ (for the case $\alpha \geqslant \gamma$ the conclusions are similar). Let $\eta_0 > 0$ be the number previously defined. Then* (1.1) *has a nontrivial positive radially symmetric solution if the parameters β and ω^2 belong either to the region $A = A_1 \cup A_2 \cup A_3$ or region B in Fig.* 2 *where $\beta_1 = \eta_0\alpha$, $\beta_2 = \eta_0\gamma$ and the regions A_1, A_2 and A_3 are defined by the following inequalities involving the parameters $\beta < \beta(\alpha, \gamma)$ and ω^2*:

$$A_1 = \{0 \leqslant \beta \leqslant \beta_1, \omega^2 > 0\}, \qquad A_2 = \left\{\beta_1 < \beta \leqslant \beta_2, 0 < \omega^2 < \frac{1}{\lambda_1(\beta/\gamma)}\right\},$$
$$A_3 = \left\{\beta_2 < \beta < \beta(\alpha, \gamma), \lambda_1(\beta/\alpha) < \omega^2 < \frac{1}{\lambda_1(\beta/\gamma)}\right\}$$

and for $\beta > \beta(\alpha, \gamma)$ the region B is described by (1.7).

It is worthwhile to remark that the numerical experiments performed by [9] for system (1.1) in the case $N = 1$ and $\alpha = \gamma = 1$ have not detected existence of positive solutions outside the regions A and B. This may be an indication that, for $N = 1, 2, 3$, the regions A and B for which our existence results hold may be optimal in the sense that, outside them, positive solutions of (1.1) with finite energy do not exist. If this is indeed the case, then Figs. 1 and 2 show how the shape of the region of existence changes when we pass from dimensions $N = 1, 2$ to dimension $N = 3$.

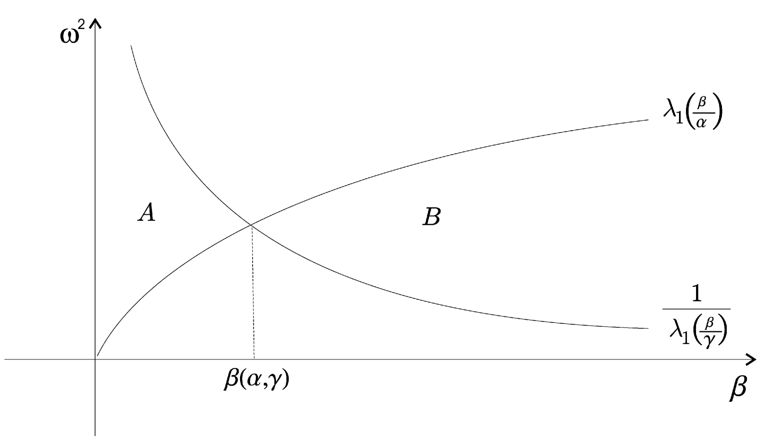

Fig. 1.

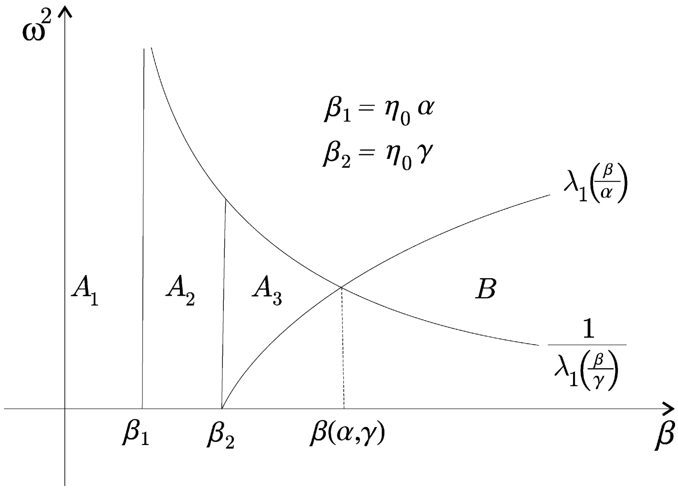

Fig. 2.

Next we give sufficient conditions for having the hypotheses of Theorem 1.1 satisfied in dimensions $N = 2, 3$ (the case $N = 1$ will be treated separately).

Theorem 1.2. *If*

$$\frac{\beta}{\alpha} < \min\{1, \omega^2\} \quad \text{and} \quad \frac{\beta}{\gamma} < \min\left\{1, \frac{1}{\omega^2}\right\} \tag{1.8}$$

then we are in region A of Figs. 1 *or* 2. *Hence, system* (1.1) *has a nontrivial positive solution.*

Theorem 1.3. *If*

$$\frac{\beta}{\alpha} > \max\{1, \omega^2\} \quad \text{and} \quad \frac{\beta}{\gamma} > \max\left\{1, \frac{1}{\omega^2}\right\} \tag{1.9}$$

then we are in region B of Figs. 1 *or* 2. *Hence, system* (1.1) *has a nontrivial positive solution.*

In the case $N = 1$, we have $\phi_0(x) = \sqrt{2}\,\mathrm{sech}(x)$ and the function $\lambda_1(\eta)$ is known explicitly and we have the following:

Theorem 1.4. *Suppose* $N = 1$ *and let* $\beta(\alpha, \gamma) > 0$ *be the unique solution of*

$$\frac{1}{2}\left(\sqrt{1 + \frac{8\beta}{\alpha}} - 1\right) = \frac{2}{\sqrt{1 + 8\beta/\gamma} - 1}. \tag{1.10}$$

Then system (1.1) *has a nontrivial positive solution if either* $0 < \beta < \beta(\alpha, \gamma)$ *and*

$$\frac{1}{2}\left(\sqrt{1 + \frac{8\beta}{\alpha}} - 1\right) < \omega < \frac{2}{\sqrt{1 + 8\beta/\gamma} - 1} \quad (\textit{region } A); \tag{1.11}$$

or $\beta > \beta(\alpha, \gamma)$ *and*

$$\frac{2}{\sqrt{1 + 8\beta/\gamma} - 1} < \omega < \frac{1}{2}\left(\sqrt{1 + \frac{8\beta}{\alpha}} - 1\right) \quad (\textit{region } B). \tag{1.12}$$

In the particular case $\alpha = \gamma = 1$, the existence region of Theorem 1.4 is precisely the region for which existence of positive solutions has been verified numerically in [9]. In that case, the curves intersect at $\beta = 1$ and $\omega = 1$.

Notice that if $\gamma < \alpha$ then, as we will see later, $\beta(\alpha, \gamma) > \sqrt{\alpha\gamma}$. In this case, our existence theorem applies to a region of the parameters that lies to the right of $\sqrt{\alpha\gamma}$ (see (1.20)).

Next we state a theorem which is a more geometric version of Theorem 1.1. This version applies to region A in both Figs. 1 and 2. First we recall that ϕ and ψ, respectively, are the unique positive radially symmetric decreasing functions tending to zero exponentially at infinity and satisfying the equations

$$-\Delta\phi + \phi - \alpha\phi^3 = 0 \tag{1.13}$$

and

$$-\Delta\psi + \omega^2\psi - \gamma\psi^3 = 0. \tag{1.14}$$

We define the selfadjoint operators

$$Lh = -\Delta h + \left(1 - \beta\psi^2\right)h \tag{1.15}$$

and

$$Mk = -\Delta k + \left(\omega^2 - \beta\phi^2\right)k \tag{1.16}$$

acting on functions belonging to $H^2_{\mathrm{rad}}(\mathbb{R}^N)$. Then we can state the following:

Theorem 1.5. *If $\beta > 0$ and the operators L and M are positive definite then system* (1.1) *has a nontrivial positive radially symmetric decreasing solution.*

As we will see, in the region A the trivial solutions $(\phi, 0)$ and $(0, \psi)$ have Morse index equal to one and in region B their Morse index is at least two. The case of Morse index larger than one has been considered in [5] for more general nonlinearities. Therefore, we focus our attention on region A.

The solutions of (1.1) are critical points of the functional:

$$E(u, v) = \int_{\mathbb{R}^N} \left(\frac{1}{2}|\operatorname{grad} u|^2 + \frac{1}{2}|\operatorname{grad} v|^2 + \frac{u^2}{2} + \frac{\omega^2 v^2}{2} - \frac{\alpha u^4}{4} - \frac{\beta u^2 v^2}{2} - \frac{\gamma v^4}{4}\right) \mathrm{d}x, \tag{1.17}$$

which is well defined and smooth in $H^1_{\mathrm{rad}}(\mathbb{R}^N) \times H^1_{\mathrm{rad}}(\mathbb{R}^N)$ for $N = 1, 2, 3$.

The first attempt to find nontrivial critical points of $E(u, v)$ is to minimize $E(u, v)$ on the Nehari manifold

$$\mathcal{N} = \Bigg\{(u, v) \in H^1_{\mathrm{rad}}\left(\mathbb{R}^N\right) \times H^1_{\mathrm{rad}}\left(\mathbb{R}^N\right), (u, v) \neq (0, 0):$$
$$I(u, v) \widehat{=} \int_{\mathbb{R}^N} \left(|\operatorname{grad} u|^2 + |\operatorname{grad} v|^2 + u^2 + \omega^2 v^2 - \alpha u^4 - 2\beta u^2 v^2 - \gamma v^4\right) \mathrm{d}x = 0\Bigg\}. \tag{1.18}$$

As we explain next, such a procedure does not give nontrivial critical points when the parameters lie in region A in Figs. 1 and 2.

First notice that if $(u, v) \in \mathcal{N}$ is a critical point of E restricted to $\mathcal{N}$, then (u, v) is a critical point of E because the Lagrange multiplier is zero. Moreover, the Morse index of a minimizer of $E(u, v)$ on $\mathcal{N}$ is at most one because it is a minimizer of a functional under one constraint (actually, in the present case, the Morse index of the minimizer is exactly one because if (u, v) solves (1.1) then $E''(u, v)((u, v), (u, v)) < 0$). For values of the parameters in the region B in Fig. 1 and in Fig. 2, the Morse index of each trivial solution $(\phi, 0)$ and $(0, \psi)$ is at least two (see [1]). Therefore, in this case, the minimizer of $E(u, v)$ on $\mathcal{N}$ gives indeed a nontrivial solution of (1.1).

However, in region A the Morse index of each trivial solutions $(\phi, 0)$ and $(0, \psi)$ is exactly one (see [1]). In this case, it is not clear that the minimization procedure produces a nontrivial solution of (1.1). In fact, for values of the parameters in region A, the level of the nontrivial solution is higher than the level of the trivial solutions $(\phi, 0)$ and $(0, \psi)$. Hence, such solution cannot be obtained by the minimization of $E(u, v)$ on $\mathcal{N}$.

In particular, in the narrower region $0 < \beta < \sqrt{\alpha\gamma}$, it is not difficult to show that the Morse index of any eventual nontrivial solution of (1.1) is at least two. Therefore, any method used to prove the existence of nontrivial solutions has to take this fact in account.

The existence of positive solutions of (1.1) has been also considered in [3] and [4] for $N = 2, 3$. Existence and nonexistence results are also proved in [8]. In both papers, the method of proving consists of minimizing the energy $E(u, v)$ on the double Nehari manifold given by the elements (u, v), $u \not\equiv 0$, $u \not\equiv 0$ satisfying:

$$\int_{\mathbb{R}^3} \left(|\operatorname{grad} u|^2 + u^2 - \alpha u^4 - \beta u^2 v^2\right) \mathrm{d}x = 0,$$
$$\int_{\mathbb{R}^3} \left(|\operatorname{grad} v|^2 + \omega^2 u^2 - \beta u^2 v^2 - \gamma v^4\right) \mathrm{d}x = 0. \tag{1.19}$$

In order to show that the minimizer is indeed a solution of (1.1) the assumption

$$\beta^2 < \alpha\gamma \tag{1.20}$$

is used in those papers. Here in this paper (and in [1] as well) such a condition is not needed. In fact, at least in the case $N = 1$, we can show existence of solution for values of β that are to the right of $\sqrt{\alpha\gamma}$.

As we have said before, our approach was motivated by the numerical experiments of [9] for $N = 1$ and $\alpha = \gamma = 1$. According to them, positive nontrivial solutions exist if either (1.11) or (1.12) with $\alpha = \gamma = 1$ hold. Since we are focusing our attention to region A, the first thing is to give a geometric meaning for condition (1.11). As we will see, it turns out that if (1.11) holds then the trivial solutions $(\phi, 0)$ and $(0, \psi)$ are local minimizers of the functional $E(u, v)$ restricted to the manifold $\mathcal{N}$. In other words, if we consider $E(u, v)$ as a map from $\mathcal{N}$ into $\mathbb{R}$, then under the condition (1.9), the functional $E(u, v)$ has a mountain pass geometry on the manifold $\mathcal{N}$.

In the general case, the region A in Theorem 1.1 is precisely the region where the functional $E(u, v)$ has the mountain pass geometry. Under certain conditions which are equivalent to ours, such mountain pass geometry has already been observed in [1] and, for $N = 2, 3$, the existence of solutions is proved there.

The case $N = 1$ requires a different approach because certain compactness argument in the space of radial functions fails in the one-dimensional case. Our approach consists in showing that the problem in a finite interval $(-n, n)$ with Dirichlet boundary conditions has a nontrivial solution if n is large and we pass to the limit as n tends to infinity. Careful estimates are needed to show that the limit is not trivial.

Remark 1.1. If $N = 2, 3$ and $-\sqrt{\alpha\gamma} < \beta < 0$ then we can prove the existence of positive nontrivial solutions but we do not know if they are decreasing. Due to this technical difficulty, our proof for the case $N = 1$ does not work for $\beta < 0$ in that range.

This paper is organized as follows:

- Section 2: proof of Theorem 1.5 for $N = 2, 3$;
- Section 3: proof of Theorem 1.5 in the case $N = 1$;
- Section 4: proof of Theorem 1.1;
- Section 5: proofs of Theorems 1.2, 1.3 and 1.4.

2. Proof of Theorem 1.5 for $N = 2, 3$

In this section we prove Theorem 1.5 in the case $N = 2, 3$. Under the assumptions of Theorem 1.5, it is easy to show that the trivial solutions $(\phi, 0)$ and $(0, \psi)$ are strict local minimizers of the functional $E(u, v)$ restricted to the manifold $\mathcal{N}$. Therefore a mountain pass argument can be used. If we define the quantities:

$$\gamma_1^2 = \inf \frac{\int_{\mathbb{R}^N} (|\operatorname{grad} k|^2 + \omega^2 k^2)\,\mathrm{d}x}{\int_{\mathbb{R}^N} \phi^2 k^2\,\mathrm{d}x}, \qquad \gamma_2^2 = \inf \frac{\int_{\mathbb{R}^N} (|\operatorname{grad} h|^2 + \omega^2 h^2)\,\mathrm{d}x}{\int_{\mathbb{R}^N} \psi^2 h^2\,\mathrm{d}x}$$

where ϕ and ψ are defined by (1.13) and (1.14) respectively, then it is also easy to see that L and M to be positive definite is equivalent to say that $\beta < \min(\gamma_1^2, \gamma_2^2)$. But, under these conditions, the mountain pass argument has been carried out in [1] and we omit it. Therefore, Theorem 1.5 is proved for $N = 2, 3$.

3. Proof of Theorem 1.5 in the case $N = 1$

We consider the ODE system:

$$-u'' + u - (\alpha u^2 + \beta v^2)u = 0,$$
$$-v'' + \omega^2 v - (\beta u^2 + \gamma v^2)v = 0 \tag{3.1}$$

and, as before, we denote by ϕ and ψ the unique nontrivial positive symmetric functions tending to zero exponentially at infinity and satisfying

$$-\phi'' + \phi - \alpha\phi^3 = 0. \tag{3.2}$$

and

$$-\psi'' + \omega^2\psi - \gamma\psi^3 = 0, \tag{3.3}$$

respectively. We also define the selfadjoint operators

$$Lh = -h'' + (1 - \beta\psi^2)h \tag{3.4}$$

and

$$Mk = -k'' + (\omega^2 - \beta\phi^2)k. \tag{3.5}$$

The proof of Theorem 1.1 for the ODE case will be done by approximation; namely, we first show that for n large the system

$$-u'' + u - (\alpha u^2 + \beta v^2)u = 0,$$
$$-v'' + \omega^2 v - (\beta u^2 + \gamma v^2)v = 0, \tag{3.6}$$
$$x \in [-n, n], \quad u(-n) = v(-n) = 0 = u(n) = v(n)$$

has a unique nontrivial positive solution (u_n, v_n) and that its limit as n tends to infinity converges to a nontrivial solution of (3.1).

Since ϕ solves (3.2) we have

$$-\frac{\phi'(x)^2}{2} + \frac{\phi^2}{2} - \frac{\alpha\phi(x)^4}{4} = 0, \tag{3.7}$$

and this implies

$$\phi(0) = \sqrt{\frac{2}{\alpha}}. \tag{3.8}$$

We denote by ϕ_n the approximate positive symmetric (unique) solution of the problem

$$-\phi_n'' + \phi_n - \alpha\phi_n^3 = 0, \qquad \phi(-n) = 0 = \phi(n). \tag{3.9}$$

If we define $a_n = \phi_n(0)$, it follows that $a_n > \phi(0) = \sqrt{2/\alpha}$. We also define $\tilde{\phi}_n$ as the extension of ϕ_n to the entire line setting it equal to zero outside the interval $[-n, n]$. $\tilde{\psi}_n$ is defined similarly.

Lemma 3.1. *As a_n converges to $\phi(0)$, the approximate solution $\tilde{\phi}_n$ converges to ϕ in the $H^1(\mathbb{R})$ norm. In particular, $\tilde{\phi}_n$ converges to ϕ uniformly on $(-\infty, \infty)$. A similar statement holds for $\tilde{\psi}_n$.*

Proof. Using ODE methods, we obtain some estimates for ϕ_n on the interval $[0, +\infty)$. By symmetry, the same estimates hold for $(-\infty, 0]$. From (3.9) we get:

$$\phi_n'^2(x) = \phi_n^2(x) - 2\alpha\phi_n^4(x) + c_n$$

where $c_n = \phi_n'^2(-n) > 0$. Since ϕ_n converges to ϕ uniformly on compact sets and $\phi(x)$ tends to zero as x tends to $+\infty$, there is are $x_1 > 0$ and n_0 such that for $x > x_1$ and $n > n_0$ we have

$$\phi_n^2(x) - 2\alpha\phi_n^4(x) + c_n \geqslant \frac{1}{4}\big(\phi_n^2(x) + c_n\big).$$

Moreover, $\phi_n'(x) < 0$, for $x > x_1$ and then

$$\frac{\mathrm{d}\phi}{\sqrt{\phi_n^2(x) + c_n}} \leqslant -\frac{\mathrm{d}x}{2}.$$

Integrating this last inequality we get

$$\log \frac{\phi_n(x) + \sqrt{\phi_n^2(x) + c_n}}{\phi_n(x_1) + \sqrt{\phi_n^2(x_1) + c_n}} \leqslant -\frac{1}{2}(x - x_1)$$

and then

$$\frac{\phi_n(x) + \sqrt{\phi_n^2(x) + c_n}}{\phi_n(x_1) + \sqrt{\phi_n^2(x_1) + c_n}} \leqslant \mathrm{e}^{-\frac{1}{2}(x - x_1)}.$$

From this last inequality we conclude that for $x_1 < x < n$ and some constant K we have

$$2\phi_n(x) \leqslant \phi_n(x) + \sqrt{\phi_n^2(x) + c_n} \leqslant K\mathrm{e}^{-\frac{1}{2}x}. \tag{3.10}$$

Clearly (3.10) implies that $\tilde{\phi}_n$ converges to ϕ uniformly on $[0, +\infty)$ and that the norm $L^p([0, +\infty))$ of $\tilde{\phi}_n$ is bounded for $1 \leqslant p \leqslant \infty$. Next, if we subtract (3.2) from (3.9), multiply the result by $\phi_n(x) - \phi(x)$ and integrate we get:

$$\int_{-n}^{n} \left[\left(\phi_n'(x) - \phi'(x)\right)^2 + \left(\phi_n(x) - \phi(x)\right)^2 \right] \mathrm{d}x = 2\alpha \int_{-n}^{n} \left(\phi_n(x) - \phi(x)\right)^2 \left(\phi_n^2(x) + \phi_n(x)\phi(x) + \phi^2(x)\right) \mathrm{d}x.$$

Since this last integral converges to zero (because $\tilde{\phi}_n$ converges to ϕ uniformly on $(-\infty, \infty)$), the lemma is proved. □

We will also need the following

Lemma 3.2. *The solution ϕ_n of* (3.9) *is unique and zero is not an eigenvalue of the linearized operator*

$$T_n h = -h'' + \left(1 - 3\phi_n^2\right)h$$

with domain $H^2(-n, n) \cap H_0^1(-n, n)$.

Proof. The uniqueness follows from the fact, that for the nonlinearity we are treating, the period of a solution of (3.9) is a decreasing function of the amplitude. There are several general results about that subject (see for instance [2] and the references therein) but in our case we can give a simple direct proof. In fact, changing the notation, if we have the equation $u'' + f(u) = 0$ and $F'(u) = f(u)$ and $u(x)$ is a solution with $u'(0) = 0$ and $u(0) = u_0$ then the "time" the solution takes to vanish is given by

$$X(u_0) = 2 \int_0^{u_0} \frac{\mathrm{d}u}{\sqrt{F(u_0) - F(u)}}.$$

In our case, $F(u) = -u^2/2 - \alpha u^4/4$ and $u_0 > \sqrt{2/\alpha}$; then

$$X(u_0) = 4 \int_0^{u_0} \frac{\mathrm{d}u}{\sqrt{(u_0^2 - u^2)(\alpha(u_0^2 + u^2) - 2)}}.$$

If we make the change of variable $u = tu_0$ we get

$$X(u_0) = 4 \int_0^{1} \frac{\mathrm{d}t}{\sqrt{(1 - t^2)(\alpha(1 + t^2)u_0^2 - 2))}}$$

and this shows that $X(u_0)$ is a decreasing function of u_0 and the uniqueness is proved.

To show the second part of the lemma, we multiply (3.9) by ϕ_n and integrating it we see that

$$\langle T_n\phi_n, \phi_n \rangle = -2\alpha \int_{-n}^{n} \phi_n^4 \,\mathrm{d}x < 0.$$

Therefore the principal eigenvalue of T_n is negative. Now suppose zero is an eigenvalue of T_n with eigenfunction $p(x)$. Then, besides vanishing at $-n$ and n, p has to vanish at an interior point of $(-n,n)$ because it is not the principal eigenfunction. Moreover, ϕ_n' also satisfies the linearized equation and then, by the Sturm oscillation theorem, ϕ_n' has at least two zeroes on $(-n,n)$ and this is a contradiction (because $\phi_n'(x)$ vanishes at $x=0$ only) and the lemma is proved. □

Next we show the existence of a nontrivial even solution of (3.6) if n is large. As in the three-dimensional case, this will be done by a mountain pass argument on a manifold. We work in the space $H^1_{\text{ev}}(\mathbb{R}) \times H^1_{\text{ev}}(\mathbb{R})$, where the subscript ev stands for even, and we define the truncated energy

$$E_n(u,v) = \int_{-n}^{n} \left(\frac{1}{2}u'^2 + \frac{1}{2}v'^2 + \frac{u^2}{2} + \frac{\omega^2 v^2}{2} - \frac{\alpha u^4}{4} - \frac{\beta u^2 v^2}{2} - \frac{\gamma v^4}{4}\right) \mathrm{d}x \tag{3.11}$$

on $H^1_{\text{ev},0}$ and the Nehari manifold

$$\mathcal{N}_n = \left\{(u,v) \in H^1_{\text{ev},0}(\mathbb{R}) \times H^1_{\text{ev},0}(\mathbb{R}),\ (u,v) \neq (0,0): \int_{-n}^{n} \left(u'^2 + v'^2 + u^2 + \omega v^2 - \alpha u^4 - 2\beta u^2 v^2 - \gamma v^4\right) \mathrm{d}x = 0\right\}. \tag{3.12}$$

We also define the selfadjoint operators:

$$L_n h = -h'' + \left(1 - \beta \tilde{\psi}_n^2\right) h \tag{3.13}$$

and

$$M_n k = -k'' + \left(\omega^2 - \beta \tilde{\phi}_n^2\right) k \tag{3.14}$$

with domain $H^2_{\text{ev}}(-n,n) \cap H^1_{\text{ev},0}(-n,n)$. The functions $\tilde{\phi}_n$ and $\tilde{\psi}_n$ have been defined right before Lemma 3.1. Since by assumption the operators L and M are positive definite, using Lemma 3.1 we see that there are n_0 and a constant m such that for $n \geqslant n_0$ we have

$$\langle L_n h, h\rangle \geqslant m\|h\|^2, \qquad \langle M_n k, k\rangle \geqslant m\|k\|^2 \tag{3.15}$$

for any $h, k \in H^2_{\text{ev}}(-n,n) \cap H^1_{\text{ev},0}(-n,n)$. Arguing exactly as in the proof of Theorem 1.5 for the three-dimensional case (the verification of Palais–Smale condition is trivial for the case of bounded interval) we construct a sequence of nontrivial symmetric, positive and decreasing functions (u_n, v_n) satisfying

$$\begin{aligned} &-u_n'' + u_n - \left(\alpha u_n^2 + \beta v_n^2\right) u_n = 0, \\ &-v_n'' + \omega^2 v_n - \left(\beta u_n^2 + \gamma v_n^2\right) v_n = 0, \\ &x \in [-n,n], \quad u_n(-n) = v_n(-n) = 0 = u_n(n) = v_n(n). \end{aligned} \tag{3.16}$$

Moreover, if we consider the path $(t\phi_n, (1-t)\psi_n)$ and we lift it to the manifold $\mathcal{N}_n$ defined by (3.12), on this lifted path the functional E_n is uniformly bounded because, according to Lemma 3.1, $\tilde{\phi}_n$ and $\tilde{\psi}_n$ tend to ϕ and ψ, respectively, in $H^1(\mathbb{R})$. Therefore, $E(u_n, v_n)$ is bounded by a constant independent of n and since (u_n, v_n) belongs to $\mathcal{N}_n$, we conclude that the $H^1(-n,n)$ norm of u_n and v_n are also bounded by a constant independent of n. In particular, the sequences $u_n(0)$ and $v_n(0)$ are bounded. Passing to a subsequence we can assume that $u_n(0)$ and $v_n(0)$ converge and our next proposition is:

Lemma 3.3. *$u_n(0)$ and $v_n(0)$ converge to a and b, respectively, where $a \neq 0 \neq b$.*

Proof. First we notice that both $u_n(0)$ and $v_n(0)$ cannot converge to zero because, otherwise, the $L^\infty(-n,n)$ norms of u_n and v_n would converge to zero (because u_n and v_n are decreasing) and this contradicts the definition (3.12) of $\mathcal{N}_n$. Therefore we have to exclude the possibility, say, $a \neq 0$ and $b = 0$. Suppose $a \neq 0$ and $b = 0$. Then $\|v_n\|_{L_\infty(-n,n)}$

tends to zero (because v_n is decreasing) and, multiplying the second equation of (3.16) by v_n and integrating, we see that the $H^1(-n,n)$ norm of v_n tends to zero. In particular, the extended functions $\widetilde{u_n}$, $\widetilde{v_n}$ converge to $(\phi,0)$ uniformly on compact sets. If we multiply the first equation of (3.16) by u_n and define the function

$$f_n(x) = -\frac{u_n'^2}{2} + \frac{u_n^2}{2} - \frac{\alpha u_n^4}{4} - \frac{1}{2}\beta u_n^2 v_n^2$$

we see that $f_n'(x) = -2\beta u_n^2 v_n' > 0$ on $[0,n]$. Therefore for $x \in [0,n]$ we have $f_n(x) \leqslant f_n(n) = --u_n'^2(n)/2 = -c_n$ where c_n is a positive constant. Then for $x \in [0,n]$ we have

$$-\frac{u_n'^2}{2} + \frac{u_n^2}{2} - \frac{\alpha u_n^4}{4} - \frac{1}{2}\beta u_n^2 v_n^2 \leqslant -c_n$$

and

$$u_n'^2 \geqslant \frac{u_n^2}{2} - \frac{\alpha u_n^4}{4} - \frac{1}{2}\beta u_n^2 v_n^2 + c_n.$$

If we choose $x_2 > 0$ such that

$$\frac{u_n^2}{2} - \frac{\alpha u_n^4}{4} - \frac{1}{2}\beta u_n^2 v_n^2 + c_n \geqslant \frac{1}{4}\left(u_n^2 + c_n\right)$$

for $x > x_2$ then arguing exactly as in the proof of Lemma 3.1, we conclude that $\widetilde{u_n}$ converges to ϕ uniformly on $(-\infty,\infty)$. Then the is n_0 and a constant $m_1 > 0$ such that for $n > n_0$ we have $\langle M_n k, k\rangle \geqslant m_1\|k\|^2$ for any $k \in H^1(-\infty,\infty)$. However, if we multiply the second equation of (3.16) by v_n and integrate we get

$$\int_{-\infty}^{\infty} \left(\widetilde{v_n}'^2 + \left(\omega^2 - \beta\widetilde{u_n}^2\right)\widetilde{v_n}^2 - \gamma\widetilde{v_n}^4\right) \mathrm{d}x = 0$$

and then

$$m_1\|\widetilde{v_n}\|^2 \leqslant \int_{-\infty}^{\infty} \left(\widetilde{v_n}'^2 + \left(\omega^2 - \beta\widetilde{u_n}^2\right)\widetilde{v_n}^2\right) \mathrm{d}x \leqslant \gamma\|\widetilde{v_n}\|_{L_\infty}^2 \|\widetilde{v_n}\|^2$$

and then $v_n \equiv 0$ for n large and this is a contradiction and the lemma is proved. □

End of proof of Theorem 1.5 in the case $N = 1$. Since $\widetilde{u_n}$ and $\widetilde{v_n}$ are bounded in $H^1(-\infty,\infty)$, passing to a subsequence we can assume that $\widetilde{u_n}$ and $\widetilde{v_n}$ converge weakly in $H^1(-\infty,\infty)$ to functions u and v. Clearly (u,v) solves system (3.1) and, due to Lemma 3.3, none of them are trivial. So, Theorem 1.5 is also proved in the case $N = 1$.

4. Proof of Theorem 1.1

To begin the proof of Theorem 1.1, we recall that $-\lambda_1(\eta)$ is the principal eigenvalue of the operator

$$M_0 k = -\Delta k - \eta\phi_0^2 k \tag{4.1}$$

and we need the following proposition:

Lemma 4.1.

(i) *If $N = 1, 2$ then for $\eta > 0$ $\lambda_1(\eta)$ is positive, increasing and*

$$\lim_{\eta\to 0} \lambda_1(\eta) = 0 \quad \textit{and} \quad \lim_{\eta\to\infty} \lambda_1(\eta) = \infty. \tag{4.2}$$

(ii) *If $N = 3$ then there is a $\eta_0 > 0$ such that for $0 \leqslant \eta \leqslant \eta_0$ the whole spectrum of M_0 is $[0,\infty)$ and for $\eta > \eta_0$ we have $\lambda_1(\eta) > 0$, increasing with η and*

$$\lim_{\eta\to\eta_0} \lambda_1(\eta) = 0 \quad \textit{and} \quad \lim_{\eta\to\infty} \lambda_1(\eta) = \infty. \tag{4.3}$$

Proof. For $N = 1, 2$, the fact that $\lambda_1(\eta)$ is positive for $\eta > 0$ follows from Theorem XIII.11 (volume 4) in [6]. The other properties of the function $\lambda_1(\eta)$ are trivially verified. For $N = 3$ using Theorem XIII.9 (volume 4) in [6] we conclude that for $\eta > 0$ and small, the spectrum of M_0 is $[0, \infty)$ (recall that $\phi_0^2 \in L^{3/2}(\mathbb{R}^3)$ and then it is a Rollnik potential according to [7], Section X). Since $\lambda_1(\eta)$ is positive for $\eta > 0$ and large, the existence of η_0 is clear as well as the other properties of $\lambda_1(\eta)$. So the lemma is proved. □

Proof of Theorem 1.1. First of all we notice that, in terms of ϕ_0 defined by (1.2), the solutions ϕ and ψ of (1.13) and (1.14) are $\phi(x) = (1/\sqrt{\alpha})\phi_0(x)$ and $\psi(x) = (\omega/\sqrt{\gamma})\phi_0(\omega x)$, respectively. We define the operators

$$L_1 h = -\Delta h - \beta\psi^2 h = -\Delta h - \beta\frac{\omega^2}{\gamma}\phi_0^2(\omega x)h \tag{4.4}$$

and

$$M_1 k = -\Delta k - \beta\phi^2(x)k = -\Delta k - \frac{\beta}{\alpha}\phi_0^2(x)k. \tag{4.5}$$

Therefore

$$Lh = L_1 h + h \quad \text{and} \quad Mk = M_1 k + \omega^2 k. \tag{4.6}$$

Concerning the operator L_1, using a scaling argument we see that μ is an eigenvalue of L_1 if and only if μ/ω^2 is an eigenvalue of

$$L_2 h = -\Delta h - \frac{\beta}{\gamma}\phi_0^2(x)h. \tag{4.7}$$

Therefore, the principal eigenvalues of L_1 and M_1 are $-\lambda_2$ and $-\lambda_3$, respectively where λ_2 and λ_3 are given in terms of the function $\lambda_1(\eta)$ by

$$\lambda_2 = \omega^2\lambda_1\left(\frac{\beta}{\gamma}\right) \quad \text{and} \quad \lambda_3 = \lambda_1\left(\frac{\beta}{\alpha}\right), \tag{4.8}$$

respectively.

Consider first the case $N = 1, 2$. From the considerations above we can draw the following conclusions:

1. L is positive definite iff $1 > \omega^2\lambda_1(\beta/\alpha)$; in particular, if $1 < \omega^2\lambda_1(\beta/\alpha)$ then L has at least one negative eigenvalue.
2. M is positive definite iff $\omega^2 > \lambda_1(\beta/\gamma)$; in particular, if $\omega^2 < \lambda_1(\beta/\gamma)$ then M has at least one negative eigenvalue.

With those facts at hand, if we look at Fig. 1 we see that in region A the operators L and M are positive definite. Hence, the existence of solution follows from Theorem 1.5. Moreover, as we have pointed out in the introduction, in region B the Morse index of the trivial solutions $(\phi, 0)$ and $(0, \psi)$ is at least two. Hence, as we have also mentioned in the introduction, the existence of solutions follows from the minimization of $E(u, v)$ on $\mathcal{N}$. An alternative is to use the usual mountain pass theorem on the entire space $H^1_{\mathrm{rad}}(\mathbb{R}^3) \times H^1_{\mathrm{rad}}(\mathbb{R}^3)$ at the solution $(0, 0)$.

Suppose now $N = 3$ and $\alpha < \gamma$ and let us define $\beta_1 = \alpha\eta_0$ and $\beta_2 = \gamma\eta_0$. If $\beta \leqslant \beta_1 = \alpha\eta_0$, then $\beta/\alpha \leqslant \eta_0$ and $\beta/\gamma < \eta_0$. According to Lemma 4.1, this implies that the spectra of both L_1 and M_1 is $[0, \infty)$ and then L and M are positive definite. Therefore, in region A_1 Theorem 1.5 applies. If $\beta_1 \leqslant \beta \leqslant \beta_2$ then $\beta/\gamma \leqslant \eta_0$ and then the spectrum of L_1 is $[0, \infty)$ and this implies that L is positive definite. Moreover, for $\beta_1 \leqslant \beta \leqslant \beta_2$ the operator M is positive definite if $\omega^2 < 1/\lambda_1(\beta/\gamma)$. All this mean that in region A_2 Theorem 1.5 also applies. The analysis for the regions A_3 and B follows exactly as in the case $N = 1, 2$ and Theorem 1.1 is proved.

5. Proofs of Theorems 1.2, 1.3 and 1.4

Proof of Theorem 1.2. We define $f(x) = \phi(\lambda x)$ where $0 < \lambda < 1$ will be chosen later. Since $\phi(x)$ is decreasing, we have $f(x) > \phi(x)$ for $x \neq 0$. Moreover, f satisfies

$$-\Delta f + \lambda^2 f - \lambda^2\alpha f^3 = 0. \tag{5.1}$$

To impose $Mf > 0$ we use (1.20) and calculate

$$Mf = -\Delta f + (\omega^2 - \beta\phi^2)f = f(\omega^2 - \lambda^2 + \lambda^2\alpha f^2 - \beta\phi^2) > f(\omega^2 - \lambda^2 + \lambda^2\alpha\phi^2 - \beta\phi^2).$$

Therefore, $Mf > 0$ if there is a $\lambda < 1$ such that $\lambda^2 < \omega^2$ and $\beta < \lambda^2\alpha$. Clearly under the first assumption of Theorem 1.2 such λ exists. Also clearly, $f > 0$ and $Mf > 0$ implies that M is positive definite. The positivity of L is verified in a similar way and Theorem 1.2 is proved as a consequence of Theorem 1.5. □

Proof of Theorem 1.3. We define $f(x) = \phi(\lambda x)$ with $\lambda > 1$ and arguing as in the proof of Theorem 1.2 we can show that, under the assumptions of Theorem 1.3, it is possible to choose $\lambda > 1$ such that $Mf < 0$. This implies that M has at least one negative eigenvalue. According to Remark 2.1, the Morse index of the trivial solution $(0, \psi)$ is at least two. Similarly we show that the Morse index of the trivial solution $(\phi, 0)$ is at least two. Therefore, as a consequence of Theorem 1.1, Theorem 1.3 is proved. □

For the proof of Theorem 1.4 we need the following proposition taken from [9]:

Lemma 5.1. *If we define the operator*

$$P_1 h = -h'' + (\eta^2 - 2\beta\,\text{sech}^2(x))h \tag{5.2}$$

where $\beta > 0$, then the eigenvalues λ of P_1 are given by $\lambda = \eta^2 - (s-n)^2$ where

$$s = \frac{1}{2}(\sqrt{1+8\beta} - 1), \tag{5.3}$$

n is a nonnegative integer and $s - n > 0$. In particular, P_1 is positive definite if $s < \eta$.

The proof of next lemma follows immediately from the preceding lemma and a scaling argument.

Lemma 5.2. *If*

$$P_2 h = -h'' + (\eta^2 - 2\beta c^2\,\text{sech}^2(cx))h \tag{5.4}$$

then P_2 is positive definite if $s < \eta/c$ where s is define by (5.3).

Proof of Theorem 1.4. The solution of $-v'' + \omega^2 v - \gamma v^3 = 0$ is $\psi(x) = \sqrt{2/\gamma}\,\omega\,\text{sech}(\omega x)$ and then $-\beta\psi^2(x) = -2(\beta/\gamma)\omega^2\,\text{sech}^2(\omega x)$. Therefore, according to Lemma 5.2, the operator $Lh = -h'' + (1 - \beta\psi^2(x))h$ is positive definite if

$$\frac{1}{2}\left(\sqrt{1 + \frac{8\beta}{\gamma}} - 1\right) < \frac{1}{\omega}.$$

Similarly, M is positive definite if

$$\frac{1}{2}\left(\sqrt{1 + \frac{8\beta}{\alpha}} - 1\right) < \omega$$

and Theorem 1.4 is proved. □

Remark 5.1. An elementary calculation shows that

$$\frac{\sqrt{1+8y} - 1}{2} < \frac{2}{\sqrt{1+8/y} - 1} \quad \text{for } 0 < y < 1.$$

Therefore, if $\beta^2 = \alpha\gamma$ and we define $y = \beta/\alpha = \gamma/\beta$ and $y < 1$ (that is, $\gamma < \beta < \alpha$) we see that $\beta(\alpha, \gamma) > \sqrt{\alpha\gamma}$, where $\beta(\alpha, \gamma)$ is defined in the statement of Theorem 1.4. This means that part of the region of the parameters for which Theorem 1.1 applies lies to the right of $\sqrt{\alpha\gamma}$.

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