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Magnetic monopoles without string in the Kähler–Clifford algebra bundle: A geometrical interpretation

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In substitution for Dirac monopoles with string (and for topological monopoles), "monopoles without string" have recently been introduced on the basis of a generalized potential, the sum of a vector A, and a pseudovector $\gamma_5 B$ potential. By making recourse to the Clifford bundle $\mathscr{C}(\tau M,g)$ $[(T_x M,g) = \mathbb{R}^{1,3}$; $\mathscr{C}(T_x M,g) = \mathbb{R}_{1,3}]$, which just allows adding together for each $x \in M$ tensors of different ranks, in a previous paper a Lagrangian and Hamiltonian formalism was constructed for interacting monopoles and charges that can be regarded as satisfactory from various points of view. In the present article, after having "completed" the formalism, a purely *geometrical interpretation* of it is put forth within the Kähler–Clifford bundle $\mathscr{K}(\tau^* M,\hat{g})$ of differential forms, essential ingredients being a generalized curvature and the Hodge decomposition theorem. Thus the way is paved for the extension of our "monopoles without string" to non-Abelian gauge groups. The analogy with supersymmetric theories is apparent.

I. INTRODUCTION

It is well known that, when describing the electromagnetic field $F_{\mu\nu}$ produced by a Dirac monopole[1] in terms of one single potential A_μ only, such a potential has to be singular along an arbitrary line starting from the monopole and going to infinity. This "string" has been considered—for a long time[2]—as *unphysical*, because the singularity in A_μ does not correspond to any singularity in $F_{\mu\nu}$.

It is also well known that, in the U(1) gauge theory of electromagnetism, which has as a mathematical model a Principal Fiber Bundle (PFB) $\pi: P \to M$ with group U(1), monopoles appear only if we consider a nontrivial bundle. Here, M is, in general, a four-dimensional Lorentzian manifold modeling the space time. The standard model is obtained by taking $M = \mathbb{R}^{1,3}$ and deleting from $\mathbb{R}^{1,3}$ the world line of the monopole. We then have as a model the PFB $\pi: P \to \mathbb{R}^2 \times S^2$ with group U(1) and the monopole charges appear as the Chern-numbers characterizing the PFB. These observations show that the topological theory does not put on equal footing the electric charge and the monopole, since the former is introduced through the electric current and the latter is a hole moving in space-time.[3,4] Notice that the topology of space-time becomes even more exotic when generalized monopoles are present.[5]

A way out has been looked for by many authors[2,6] via the introduction of a second potential B_μ. They did not completely succeed in dispensing with an exotic space-time whenever they wanted to stick to ordinary vector–tensor algebra. However, just on the basis of both a vector potential $A \in \sec \Lambda^1 \tau M \subset \sec \mathscr{C}(\tau M,g)$ [where $\mathscr{C}(\tau M,g)$ is the Clifford bundle constructed in the tangent bundle τM of the Lorentz manifold M equipped with the Lorentz metric g, and sec means a section of the bundle] and a pseudovector potential $\gamma_5 B \in \sec \mathscr{C}(\tau M,g)$, we recently constructed[7] a rather satisfactory formalism for magnetic monopoles without strings (i.e., living in the ordinary Minkowski space-time, $\mathbb{R}^{1,3}$), by making recourse to the Clifford algebra $\mathbb{R}_{1,3}$ or more precisely to the Clifford bundle $\mathscr{C}(\tau M,g)$ [where $(T_x M,g) = \mathbb{R}^{1,3}$]. Here, $\mathbb{R}_{1,3}$ is an algebra sufficiently powerful to allow adding together tensors of different ranks (grades). In Ref. 8, for example, both the electric and the magnetic current are vectorial, while in our approach they are represented by a vectorial and a pseudovectorial current, respectively (and nevertheless we can add them together[7]). Our formalism can be considered satisfactory for the reasons we shall see below. See also Ref. 9. Some analogous, but nonequivalent, results have been obtained in Refs. 10,11.

II. FROM CLIFFORD TO KÄHLER

In this paper we want, first of all, to pass from the $\mathscr{C}(\tau M,g)$ language, used in Ref. 7, to the $\mathscr{K}(\tau^* M,\hat{g})$ language, i.e., to the language of the differential forms in $\tau^* M$, the cotangent bundle with metric $\hat{g}$ (equipped with the Kähler algebra).[12–14] This paves the way, incidentally, for a generalization of our "monopoles without string" to non-Abelian gauge groups.

The new language will allow us to approach the question of a suitable formalism for interacting charges and mono-

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poles without string from a *geometrical* point of view in the space-time manifold.[15]

We recall that $\mathscr{K}(T_x^*M,\hat{g}) = \mathscr{C}(T_xM,g) = \mathbb{R}_{1,3}$, the so-called space-time algebra.[16] Now $\mathscr{K}(T_x^*M,\hat{g})$, as a linear space over the real field, can be written

$$\Lambda^0(T_x^*M) + \Lambda^1(T_x^*M) + \Lambda^2(T_x^*M) + \Lambda^3(T_x^*M) + \Lambda^4(T_x^*M), \tag{1}$$

where $\Lambda^k(T_x^*M)$ is the $\binom{4}{k}$-dimensional space of the k-forms. Here, $\Lambda(T_x^*M) = \Sigma\Lambda^k(T_x^*M)$ is called the Cartan algebra, and the pair $[\Lambda(T_x^*M),\hat{g}_x]$ is called the Hodge algebra. An analogous terminology exists for the vector bundles associated with these algebras.[9]

In $\mathscr{K}(\tau^*M,\hat{g})$ there is a particular differential operator ∂ odd in the $\mathbb{Z}_2$-gradation of the algebra.[17] To introduce ∂, consider first, for any $t^*\in \sec \tau^*M \subset \sec \mathscr{K}(\tau^*M,\hat{g})$ and any $t\in \sec \tau M$, the bilinear tensorial map of type $(1,1)$ given by

$$\Psi \to t^*\nabla_t\Psi, \tag{2}$$

where Ψ is any element of $\sec \mathscr{K}(\tau^*M,\hat{g})$ and ∇_t is the covariant derivative of Ψ (considered as an element of the tensor bundle). Then ∂ is defined as the tensorial trace of the map:

$$\partial = \mathrm{Tr}(t^*\nabla_t). \tag{3}$$

In terms of a local basis $\{\gamma^\mu\}$ of one-form fields and its dual basis $\{e_\mu\}$ of vector fields, we can write

$$\partial = \gamma^\mu \nabla_{e_\mu}. \tag{3$'$}$$

In particular, taking any local neighborhood $U\subset M$ with a local basis $\{dx^\mu\}$, so that $\partial = \gamma^\mu\nabla_\mu$, we can show[9,13] that for any $\Psi\in \sec(\Lambda\tau^*M,\hat{g}) \subset \sec \mathscr{K}(\tau^*M,\hat{g})$:

$$\partial\Psi = dx^\mu \wedge (\nabla_\mu \Psi) + \partial_\mu \lrcorner (\nabla_\mu\Psi), \tag{4}$$

where $\lrcorner$ is the usual contraction operator of the theory of differential forms. We have

$$dx^\mu \wedge (\nabla_\mu\Psi) = d\Psi, \tag{5}$$

$$\partial_\mu \lrcorner (\nabla_\mu \Psi) = -\delta\Psi, \tag{6}$$

where d is the usual differential, and δ is the Hodge coderivative operator, here defined as

$$\delta\Psi_k = (-1)^k *^{-1} d * \Psi_k \tag{7}$$

where $*$ is the Hodge star operator and $\Psi_k\in \sec \mathscr{K}(\tau^*M,\hat{g})$. The power of the Kähler bundle formalism appears clearly, once we add to the fundamental formula

$$\partial\Psi = (d-\delta)\Psi \tag{8}$$

the result[9,13,18,19]

$$\gamma^5\Psi_k = (-1)^t * \Psi_k, \tag{9}$$

where $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$ is the volume element,[20] and where $t=1$ for $k=1,2,3$ and $t=2$ for $k=0,4$ in the particular case of the space time algebra $\mathbb{R}_{1,3}$ and with our conventions. We also have that $\partial^2 = (d-\delta)^2$ is the D'Alambertian operator.

III. GENERALIZED POTENTIAL AND FIELD: A SATISFACTORY FORMALISM

Before going on, observe that the "completed" Maxwell equations, $\delta F = -J_e$, $dF = -*J_m$, where $F \in \sec(\Lambda^2\tau^*M,\hat{g}) \subset \sec \mathscr{K}(\tau^*M,\hat{g})$ is the electromagnetic field and J_e, $J_m \in \sec(\Lambda^1\tau^*M,\hat{g}) \subset \sec \mathscr{K}(\tau^*M,\hat{g})$ are, respectively, the electric and magnetic currents, can be written as[21]

$$\partial F = J_e - *J_m = J_e + \gamma^5 J_m \equiv \overline{J}. \tag{10}$$

With the introduction of the generalized potential[21] $\overline{A} \equiv A + \gamma^5 B$, where $A,B\in \sec(\Lambda^1\tau^*M,\hat{g}) \subset \sec \mathscr{K}(\tau^*M,\hat{g})$, we get $F = \partial\overline{A} = \partial\wedge A + \partial\cdot(\gamma^5 B)$, once we impose the Lorentz gauge $\partial \circ A = 0$.[22] Then we can write Eq. (10) as:

$$\partial^2 A = J_e, \quad \partial^2 B = J_m. \tag{11}$$

In our previous work[7] we wrote Eqs. (10) and (11) in $\mathscr{C}(\tau M,g)$ instead of $\mathscr{K}(\tau^*M,\hat{g})$. There we succeeded in introducing a noncoventional Lagrangian that yields the correct field equations when varied with respect to the generalized potential. Our approach, however, cannot overcome the "no-go theorems" by Rosenbaum *et al.*[8]; for instance Rohrlich[8] showed that a single Lagrangian can yield both the field equations and the charge and pole motion equations *only* in the trivial case when $J_m = kJ_e$, where k is a constant. *Nevertheless* in our approach we need to apply the variational principle just once, since our Lagrangian[7] *implies* even the correct coupling of the currents to the field. In fact, as shown in detail in Refs. 9 and 23, the "completed" Maxwell equations [Eq. (10)] *imply*, if $S^\mu \equiv -\frac{1}{2}F\gamma_\mu F$, that

$$\partial_\mu S^\mu = F\cdot J_e + (\gamma^5 F)\cdot J_m, \tag{12}$$

where $S^\mu\gamma^\nu = E^{\mu\nu}$ is the symmetric energy-momentum of the electromagnetic field. Calling $K_e = F\cdot J_0$ and $K_m = -(\gamma^5 F)\cdot J_m$, and by projecting on the Pauli algebra $\mathbb{R}_{3,0}$, one does *consequently* find the expected expressions for the forces (in particular the *Lorentz forces*) acting on a charge and a monopole:

$$\mathbf{K}_e = \rho_e \mathbf{E} + \mathbf{J}_e \times \mathbf{H}, \tag{13a}$$

$$\mathbf{K}_m = -\rho_m \mathbf{E} + \mathbf{J}_m \times \mathbf{E}. \tag{13b}$$

IV. GENERALIZED CONNECTION AND CURVATURE

As is well known, in a gauge theory[24] the potentials are pullbacks of *connections* in the PFB $\pi: P\to M$ with group G, and the associated field is the pullback of the connection *curvature*. In the case of standard electromagnetism, the field $F\in \sec(\Lambda^2\tau^*M,\hat{g})$ is derived from a potential $A\in \sec(\Lambda^1\tau^*M,\hat{g})$, i.e.,

$$F = dA. \tag{14}$$

However the Hodge decomposition theorem[25] (valid for compact spaces) assures us that more generally, if $F\in \sec(\Lambda^2\tau^*M,\hat{g})$, then there always exist $A\in \sec(\Lambda^1\tau^*M,\hat{g})$, $*B\in \sec(\Lambda^3\tau^*M,\hat{g})$ and $C\in \sec(\Lambda^2\tau^*M,\hat{g})$, with $dC = \delta C = 0$, such that F can be *uniquely decomposed* into

$$F = dA + \delta * B + C. \tag{15}$$

The Hodge decomposition naturally suggests naming *generalized connection* the quantity

$$\overline{A} = A - *B \in \sec(\Lambda^1\tau^*M,\hat{g}) + \sec(\Lambda^3\tau^*M,\hat{g}) \quad (16)$$

and *generalized curvature*[26] the quantity

$$F = \partial\overline{A} = (d-\delta)\overline{A} = dA + \delta{*}B - d{*}B - \delta A. \quad (17)$$

Then

$$F \in \sec(\Lambda^0\tau^*M,\hat{g}) + \sec(\Lambda^2\tau^*M,\hat{g}) + \sec(\Lambda^4\tau^*M,\hat{g}).$$

If we want F to be still a two-form, then the last two addenda in Eq. (17) have to vanish, and we automatically end up with the Lorentz gauge condition

$$d{*}B = \delta A = 0, \quad (18)$$

and are left with

$$F = dA + \delta{*}B. \quad (17')$$

The field equations are obtained by evaluating ∂F, with $\partial \equiv d - \delta$:

$$(d-\delta)(dA + \delta{*}B - d{*}B - \delta A) = \partial^2 A - \partial^2{*}B, \quad (19)$$

which writes

$$\partial F = J_e - {*}J_m \quad (20)$$

once we identify $\partial^2 A \equiv J_e$; $\partial^2 B \equiv J_m$. Equations (19) are of course the "completed" Maxwell equations, now deduced within a geometrical context via a natural generalization of the definitions of connection and curvature: a generalization inspired by the "correspondences" $\partial = d - \delta$ and $* = (-1)^r\gamma^5$, and by the Hodge decomposition theorem.

V. FURTHER REMARKS

(i) A rather interesting consequence of the geometrical interpretation just presented is that Eq. (17) can be assumed as a *new definition* of F, without imposing any longer the Lorentz gauge, since even in this case we get the right "completed" Maxwell equations [as it is clear from Eqs. (18) and (19)].

(ii) The introduction of our "monopoles without string" for the more general case of non-Abelian groups is discussed in Refs. 27 and 28. Here we want to emphasize once more that, for our aims, the ordinary tensorial language is too poor, since—among the others—it does not satisfactorily distinguish between scalar and pseudoscalar quantities, as on the contrary it is strictly required by physics. For instance, it is an essential character of the Lagrangian density of Ref. 7 to be the sum of a scalar and a pseudoscalar part.[7,29]

(iii) At last, let us take advantage of the present opportunity for pointing out some misprints that appeared in the previous paper,[7] that might make it difficult for the interested reader to rederive those results of ours: (1) at page 234, column 2, line 18: the two expressions $\partial\cdot\overline{J}$ ought rather to read $\partial\circ\overline{J}$; (2) at page 235, Eqs. (14) and (15): all three expressions should be written $\tilde{\overline{J}}\circ\overline{A}$; (3) at page 235: the last-term in the rhs of Eq. (17) ought to be eliminated; (4) at page 236, column 1, line 22: "pseudoscalars" should be corrected into "pseudovectors." Let us stress that the "ball product" $\circ$ is *not* a new fundamental product since in terms of the Clifford product we have, for A, $B \in \sec\mathscr{C}(\tau M,g)$, that $A\circ B \equiv \frac{1}{2}(A\tilde{B} + B\tilde{A})$.

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[14]Let us consider that the metric tensor $g \in \sec(\tau^*M \times \tau^*M)$ induces the "dual metric" $\hat{g}$ in the spaces $\Lambda^k(\tau^*M)$ (Ref. 3):

$$\hat{g}(\varphi_1,\varphi_2)\gamma^5 = \varphi_1 \wedge {*}\varphi_2,$$

where $\varphi_1,\varphi_2 \in \sec(\Lambda^k\tau^*M,\hat{g})$ is the so-called Hodge bundle. For future reference, note that, in the particular case in which $\varphi_1 = \varphi_2 = \varphi \in \sec(\Lambda^2\tau^*M,\hat{g})$, it holds:

$$\hat{g}(\varphi,\varphi) = -\hat{g}({*}\varphi,{*}\varphi).$$

[15]For a *completely* geometric formulation (and generalization to arbitrary gauge groups) of the theory, we ought however to make recourse to a spliced bundle $\pi:P\circ P \to M$ with group $G \times G$ where M is an arbitrary space-time with nonzero, Lorentzian curvature: cf. Ref. 27.

[16]By adopting Hestenes' notations (cf. the second one of Ref. 13), we call *space-time algebra* the Clifford algebra $\mathbb{R}_{1,3}$ that we called "Dirac algebra" in Ref. 7. More correctly we shall reserve the name Dirac algebra for $\mathbb{R}_{4,1} \simeq \mathbb{C}(4)$. Notice, incidentally, that the *Majorana algebra* $\mathbb{R}_{3,1}$ is quite different from $\mathbb{R}_{1,3}$, so that *two* algebras [$\mathbb{R}_{1,3} \simeq \mathbb{H}(2)$, and $\mathbb{R}_{3,1} \simeq \mathbb{R}(4)$] can be naturally associated with Minkowski space-time; and this can have a bearing on physics (even for the mathematical problems with tachyons, for instance). At last, the *Pauli algebra* is $\mathbb{R}_{3,0} \simeq \mathbb{C}(2)$.

[17]Recall that we denote the Clifford product in $\mathscr{C}(\tau M,g)$, as well as in $\mathscr{K}(\tau^*M,\hat{g})$, by the mere *juxtaposition* of symbols.

[18]See, e.g., D. Hestenes and G. Sobczyk, *Clifford Algebra to Geometrical Calculus* (Reidel, Dordrecht, 1984); D. Hestenes, "A Unified Language for Mathematics and Physics" in *Clifford Algebras and their Applications in Mathematical Physics*, edited by J. S. R. Chisholm and A. K. Common (Reidel, Dordrecht, 1986); E. Tonti, Rend. Sem. Mat. Fis. Milano **46**, 163 (1976); P. Lounesto, Ann. Inst. Henri Poincaré A **33**, 53 (1980); I. Porteous, *Topological Geometry* (Cambridge U.P., Cambridge, 1981); W. A. Rodrigues, Jr., and V. L. Figueiredo, in *Proceedings of the 8th Italian Conference on General Relativity and Gravitational Physics*, edited by M. Cerdonio, R. Cianci, M. Francaviglia, and M. Toller (World Scientific, Singapore, 1989), pp. 467–471.

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[20]Recall that, whereas γ^5 is the volume element in $\mathscr{K}(\tau^*M,\hat{g})$, in Ref. 7 $\gamma_5 = e_0e_1e_2e_3 \in \mathscr{C}(\tau M,g)$ and $\{e_\mu\}$ is an orthonormal basis of $\mathbf{R}^{1,3}$.

[21]Cf. R. Mignani and E. Recami, Nuovo Cimento A **30**, 533 (1975) and references cited therein.

[22]Note that the scalar product between $\Psi_r \in \Lambda^r(T_x^*M)$ and $\Psi_k \in \Lambda^k(T_x^*M)$ is defined by $\Psi_r \cdot \Psi_k = \langle \Psi_r \Psi_k \rangle_{|r-s|}$; i.e., it is the component in $\Lambda^{|r-s|}(T_x^*M)$ of the Clifford product of Ψ_r and Ψ_k. Sometimes we make recourse also the *ball product* ($\circ$) which, in terms of the Clifford product, is defined as follows: $A \circ B = \frac{1}{2}(A\widetilde{B} + B\widetilde{A})$. The tilde operation, in its turn, is defined as follows: $D = d_1d_2\cdots d_r$; $\widetilde{D} = d_r\cdots d_2d_1$, where the $d_i \in \mathbf{R}^{1,3}$, $i = 1,2,\ldots,r$.

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[26]Such a terminology is, of course, acceptable only when working in the base manifold. Despite this fact, the theory of electromagnetism with monopoles without string, containing the potentials A and B, can be formulated as a PFB $\pi: P \circ P \to M$ with group $U(1) \times U(1)$, where A and B are "parts" of a genuine connection in the sense of a PFB theory.[9,27]

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