Quasienergy spectra of quantum dynamical systems

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We present a technique that yields in analytic fashion the quasienergy spectrum of bounded quantum systems in the presence of time-periodic perturbations. It also allows for the calculation of statistical averages using simple algebraic manipulations and provides tractable solutions even for systems with a large number of levels. We also report on numerical calculations for systems with few number of levels in and out of resonance, and which show the recurrences predicted by the Hogg-Huberman theorem [Phys. Rev. Lett. **48**, 711 (1982); Phys. Rev. A **28**, 22 (1983)].

I. INTRODUCTION

The dynamical behavior of quantum systems with time-periodic Hamiltonians is of fundamental interest in a number of fields. In particular, it has been the focus of much activity in the field of atomic and molecular spectroscopy, where the main concern is to predict the time behavior of such systems when subjected to laser radiation.[1–9] At a more fundamental level, the existence of classical, nonintegrable dynamical problems with chaotic solutions has led to investigations of their quantum counterparts, which surprisingly do not seem to display any of those features.

Recently, several studies of periodically driven, bounded quantum systems, showed that the energy does not display the diffusive behavior commonly associated with chaotic dynamics.[10,11] Moreover, a theorem of Hogg and Huberman[10] showed that if the quasienergy spectrum of a quantum system is discrete, then the system behaves in a recurrent, quasiperiodic fashion. Since the proof of the theorem is not constructive, the problem remains of determining the quasienergy spectrum of any periodically driven quantum dynamical system given its Hamiltonian. This paper presents a method which allows for its analytic determination in the case of time-periodic Hamiltonians. As will be shown, our technique also allows for the calculation of statistical averages using simple algebraic manipulations, and provides tractable solutions even for systems with a large number of levels.

In Sec. II we consider a many-level system in the presence of an external electromagnetic field and transform it into a time-independent problem through a suitable canonical transformation. Using an algorithm developed by Eckmann and Guenin[12] for time-independent problems we then calculate the quasienergy spectrum of the system. We show that for bounded quantum systems in the presence of a periodic perturbation, the quasienergy spectrum is indeed discrete, leading to the recurrences predicted by Hogg and Huberman. Finally in Sec. III we present numerical results using our technique and show the quasiperiodic nature of the observables.

II. QUASIENERGY SPECTRUM

A. Quasienergies and quasiperiodic states

Consider a system whose Hamiltonian oscillates in time with period T

$$H(t+T)=H(t)\ .$$

Using Floquet's theorem,[13] it can be shown[14,15] that the solutions of the Schrödinger equation can be written as

$$\Psi(t)=\sum_k \phi_k(t)e^{i\epsilon_k t/\hbar}\ ,$$

where

$$\phi_k(t)=\phi_k(t+T)\ .$$

ϵ_k is called the quasienergy and $\phi_k(t)$ the quasiperiodic state, which obeys the equation

$$[H(t)-\epsilon_k]\phi_k(t)=i\hbar\frac{\partial\phi_k}{\partial t}\ .$$

In the following subsections we develop a method for calculating the quasienergies for a system of N coupled oscillators.

B. The many-level system

Consider a system of N levels, with energies $E_1,E_2,\ldots,E_N$ in ascending order, in the presence of an external periodic field which couples only adjacent levels. The corresponding Hamiltonian in the rotating-wave approximation is given by[6,3,16]

$$H(t)=\sum_{j=1}^{N}E_j a_j^\dagger a_j+\sum_{j=1}^{N-1}\gamma_{j,j+1}a_j a_{j+1}^\dagger e^{-i\omega t}+\text{H.c.}\ , \tag{2.1}$$

where we have set $\hbar=1$ and γ_{ij}, the oscillator strength, is given by

$$\gamma_{ij}=-\vec{\mu}_{ij}\cdot\vec{\mathscr{E}}_0$$

with $\vec{\mu}_{ij}$ the dipole matrix element, $\mathscr{E}_0$ the external field, and the boson operator $a_i^\dagger$ (a_i) represents the creation (annihilation) of an atom or molecule in state i. The normalization condition is determined by

$$\sum_{i=1}^{N} \langle a_i^\dagger a_i \rangle = 1 \ . \tag{2.2}$$

In order to study the time evolution of both the level population, $\langle a_i^\dagger(t) a_i(t) \rangle$ and the energy of the system, $\langle H(t) \rangle$ we first perform a canonical transformation, $U(t)$, on Eq. (2.1) which renders it time independent. By defining $U(t)$ as

$$U(t) = \exp\left[-i\omega t \sum_{J=1}^{N} (j-1) a_j^\dagger a_j \right] \tag{2.3}$$

the new Hamiltonian becomes[17]

$$H = U^\dagger(t) H(t) U(t) - i U^\dagger(t) \frac{\partial U(t)}{\partial t} \tag{2.4}$$

or, in terms of the renormalized frequencies ω_i,

$$H = \sum_{i=1}^{N} \omega_i a_i^\dagger a_i + \sum_{j=1}^{N-1} \gamma_{j,j+1} a_j a_{j+1}^\dagger + \text{H.c.} \ , \tag{2.5}$$

where

$$\omega_i = E_i - (i-1)\omega \ . \tag{2.6}$$

C. The Eckmann-Guenin method and quasiperiodic states

Having transformed the original problem into a time-independent one, we now show how the eigenvectors of the Eckmann-Guenin (EG) algorithm transform into the quasiperiodic states of the Hogg-Huberman (HH) theorem on quantum recurrences. The quantity of interest is the time evolution of an operator A, i.e.,

$$A(t) = \exp(iHt) A \exp(-iHt) \ . \tag{2.7}$$

Using the HH theorem,[10,11] we can say that $A(t)$ and H are bounded, which in turn implies that $A(t)$ is equal to its power-series expansion

$$A(t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} [H,[H, \ldots, [H,A] \cdots]] \ , \tag{2.8}$$

where there is an n-fold nesting of the commutators and which we will write as

$$A(t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} (\text{ad}H)^n A \ , \tag{2.9}$$

with the operator $(\text{ad}H)^n A$ being defined by Eqs. (2.8) and (2.9). Thus, the operator A becomes associated with a vector in a 4^m-dimensional space where m is the number of degrees of freedom (here equal to the number of levels).[18] We now define a subspace of dimension less than or equal to 4^m, spanned by the vectors $(\text{ad}H)^n \underline{A}$. In this subspace the operation ad$H$ is described by multiplication to the left by a matrix $\underline{G}_{H,A}$, and such that

$$\underline{G}_{H,A} \underline{v}_A = [H, A] \equiv HA - AH \tag{2.10}$$

with

$$\underline{v}_A \equiv (\text{ad}H)^0 \underline{A} \equiv \underline{A} \tag{2.11}$$

and

$$\underline{v}_{A(t)} = \exp[it \underline{G}_{H,A}] \underline{v}_A \ . \tag{2.12}$$

The matrix $\underline{G}_{H,A}$ can be shown to transform into the matrix Hamiltonian whose eigenvalues are the quasienergies of the system, as defined by Zel'dovich[15] (see also Ref. 19) using a similarity transformation. It is easy to show that the matrix Hamiltonian is then defined by

$$H_{nn} = \omega_n \ ,$$

$$H_{nm} = \gamma_{n,n+1} \delta_{m,n+1} \ ,$$

where ω_n's are known as the detuning factors, and $\gamma_{n,n+1}$ are the Rabi oscillators.[6,19] Note that with these definitions the eigenvalues of this matrix are always real.

Since the matrix which transforms the matrix $\underline{G}_{H,A}$ into the matrix Hamiltonian is also the matrix which transforms the basis for $\underline{v}_{A(t)}$, into the basis for $\underline{A}(t)$, it follows that finding the eigenvectors of $\underline{G}_{H,A}$ is equivalent to finding the eigenvectors of H, i.e., the quasiperiodic energy states of the system.[20]

D. Algorithm for the quasienergy spectrum

In order to make the calculational procedure transparent and easy to compare with known results, we will first treat a resonant three-level system. Our aim is to find the level population as a function of time. Since we first need an expression for $\underline{a}_i^\dagger(t)$, let us proceed with the calculation of the vectors $(\text{ad}H)^n \underline{a}_i^\dagger$. This is accomplished through the following steps.

(i) Select the level whose population evolution in time we want to study, e.g., $\underline{a}_1^\dagger$.

(ii) Define: $\underline{v}_1^1 = \underline{a}_1^\dagger$.

(iii) Define the basis vector: $\underline{v}_n^1 = (\text{ad}H)^{n-1} \underline{a}_1^\dagger$. We then generate vectors until one of them, let us say $\underline{v}_{n+1}^1$, is a linear combination of the previous n vectors. For the Hamiltonian given by Eq. (2.5), $n = N$, the number of levels in the system. We thus find

$$\underline{v}_1^1 = \underline{a}_1^\dagger \ , \tag{2.13}$$

$$\underline{v}_2^1 = \omega_1 \underline{a}_1^\dagger + \gamma_{12} \underline{a}_2^\dagger \ , \tag{2.14}$$

$$\underline{v}_3^1 = (\omega_1^2 + |\gamma_{12}|^2) \underline{a}_1^\dagger + \gamma_{12}(\omega_1 + \omega_2) \underline{a}_2^\dagger + \gamma_{12}\gamma_{23} \underline{a}_3^\dagger \ , \tag{2.15}$$

$$\begin{aligned} \underline{v}_4^1 = {} & [\omega_1(\omega_1^2 + |\gamma_{12}|^2) + |\gamma_{12}|^2(\omega_1 + \omega_2)] \underline{a}_1^\dagger \\ & + \gamma_{12}(\omega_1^2 + \omega_2^2 + \omega_1\omega_2 + |\gamma_{12}|^2 + |\gamma_{23}|^2) \underline{a}_2^\dagger \\ & + \gamma_{12}\gamma_{23}(\omega_1 + \omega_2 + \omega_3) \underline{a}_3^\dagger \ . \end{aligned} \tag{2.16}$$

At this stage we notice that the fourth vector can be expressed as a linear combination of the other three, i.e.,

$$\underline{v}_4^1 = \alpha \underline{v}_1^1 + \beta \underline{v}_2^1 + \gamma \underline{v}_3^1 \ , \tag{2.17}$$

with

$$\begin{aligned} \alpha &= \omega_1\omega_2\omega_3 - \omega_3 |\gamma_{12}|^2 - \omega_1 |\gamma_{23}|^2 \ , \\ \beta &= |\gamma_{12}|^2 + |\gamma_{23}|^2 - \omega_1\omega_2 - \omega_2\omega_3 - \omega_1\omega_3 \ , \\ \gamma &= \omega_1 + \omega_2 + \omega_3 \ . \end{aligned} \tag{2.18}$$

(iv) Construct the matrix $\underline{G}_{H,a_1^\dagger}$. It is an $n\times n$ matrix, n being the number of linearly independent $\underline{v}_i^1$, with the ith column given by the vector $\underline{v}_{i+1}^1$ in the $\{\underline{v}_i^1\}$ representation, i.e.,

$$\underline{G}_{H,a_1^\dagger}=\begin{bmatrix}0 & 0 & \alpha\\ 1 & 0 & \beta\\ 0 & 1 & \gamma\end{bmatrix}.$$

As we discussed it in Sec. II C, $\underline{G}_{H,a_1^\dagger}$ can be transformed into a time-independent Hamiltonian matrix by a similarity transformation. The eigenvalues are therefore real, and they cannot depend on the particular operator whose dynamics we are calculating.

(v) For the resonant case: $E_3-E_1=2\omega$ (also known as "no detuning"[3,5]), or $\omega_3-\omega_1=0$. The quasienergy spectrum is then given by the eigenvalues of $\underline{G}$, i.e.,

$$\begin{aligned}\lambda_0&=\omega_1\,,\\ \lambda_1&=(\omega_1+\omega_2)/2+\sqrt{[(\omega_1-\omega_2)/2]^2+\gamma^2}\,,\\ \lambda_2&=(\omega_1+\omega_2)/2-\sqrt{[(\omega_1-\omega_2)/2]^2+\gamma^2}\,,\end{aligned} \quad (2.19)$$

with $\gamma^2=|\gamma_{12}|^2+|\gamma_{23}|^2$. The eigenvectors are in turn expressed as

$$\underline{x}_1=\begin{bmatrix}\omega_1\omega_2-\gamma^2\\ -(\omega_1+\omega_2)\\ 1\end{bmatrix}, \quad (2.20a)$$

$$\underline{x}_2=\begin{bmatrix}1\\ -(\omega_2+2\omega_1-\lambda_1)/(\omega_1\lambda_2)\\ 1/(\omega_1\lambda_2)\end{bmatrix}, \quad (2.20b)$$

$$\underline{x}_3=\begin{bmatrix}1\\ -(\omega_2+2\omega_1-\lambda_2)/(\omega_1\lambda_1)\\ 1/(\omega_1\lambda_1)\end{bmatrix}. \quad (2.20c)$$

Notice that the eigenvalues of $\underline{G}_{H,a_1^\dagger}$ coincide with those of Białynicka-Birula *et al.*,[6] Eq. (17), for the case where $\omega_1=0$ and all γ's are equal to one.

(vi) In order to express the time evolution of the operator $a_1^\dagger(t)$, we now need to compute[12]

$$\underline{a}_1^\dagger(t)=\sum_{i=1}^{3}e^{i\lambda_i t}\mu_i\underline{x}_i\,, \quad (2.21)$$

where μ_i are the components of the vector $\underline{v}_1^1\equiv(1,0,0)$ in the basis $\{\underline{x}_i\}$. After some simple algebra we obtain

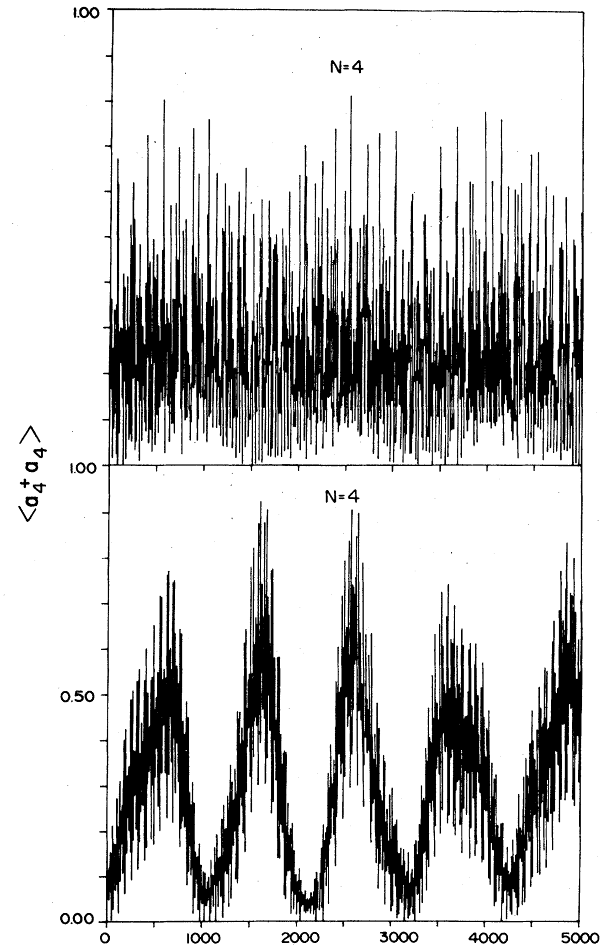

FIG. 1. The expectation value of the fourth-level population, as a function of the number of "maximum period" ($=2\pi$/minimum of the absolute value of the quasienergies). The different figures show the variation in the response for different energy levels. For the lower figure $E_1=1.0$, $E_2=1.1$, $E_3=2.5$, $E_4=4.0$, with one pair of resonant levels, and for the upper figure $E_1=1.0$, $E_2=1.1$, $E_3=3.0$, $E_4=4.0$ with three pairs of resonant levels. Equal-Rabi case.

$$\begin{aligned}\mu_1&=-1/\gamma^2\,,\\ \mu_2&=[(\lambda_2-\omega_1)\omega_1\lambda_2]/[(\lambda_2-\lambda_1)\gamma^2]\,,\\ \mu_3&=[\omega_1\lambda_1(\omega_1-\lambda_1)]/[(\lambda_2-\lambda_1)\gamma^2]\,.\end{aligned} \quad (2.22)$$

It is now clear from Eq. (2.21) that the eigenvalues of $\underline{G}$ determine the periodicity or quasiperiodicity of the system. Substituting Eq. (2.22) into Eq. (2.21), we find

$$\begin{aligned}\underline{a}_1^\dagger(t)=&-\frac{1}{\gamma^2}[\lambda_1\lambda_2\underline{v}_1^1-(\omega_1+\omega_2)\underline{v}_2^1+\underline{v}_3^1]e^{i\lambda_0 t}+\left[\frac{(\lambda_2-\omega_1)\omega_1\lambda_2}{(\lambda_2-\lambda_1)\gamma^2}\right]\left[\underline{v}_1^1-\left[\frac{\omega_2+2\omega_1-\lambda_1}{\omega_1\lambda_2}\right]\underline{v}_2^1+\frac{1}{\omega_1\lambda_2}\underline{v}_3^1\right]e^{i\lambda_1 t}\\ &+\left[\frac{\omega_1\lambda_1(\omega_1-\lambda_1)}{(\lambda_2-\lambda_1)\gamma^2}\right]\left[\underline{v}_1^1-\left[\frac{\omega_2+2\omega_1-\lambda_2}{\omega_1\lambda_1}\right]\underline{v}_2^1+\frac{1}{\omega_1\lambda_1}\underline{v}_3^1\right]e^{i\lambda_2 t}\,,\end{aligned} \quad (2.23)$$

where $\underline{v}\,^1_1$, $\underline{v}\,^1_2$, and $\underline{v}\,^1_3$ are given by Eqs. (2.13), (2.14), and (2.15), respectively. After that substitution into Eq. (2.23), we find

$$\underline{a}\,^\dagger_1(t)=\left\{\frac{|\gamma_{23}|^2}{\gamma^2}e^{i\omega_1 t}+\frac{|\gamma_{12}|^2}{\gamma^2}\left[\left[\frac{\lambda_2-\omega_1}{\lambda_2-\lambda_1}\right]e^{i\lambda_1 t}+\left[\frac{\omega_1-\lambda_1}{\lambda_2-\lambda_1}\right]e^{i\lambda_2 t}\right]\right\}\underline{a}\,^\dagger_1$$
$$+\gamma_{12}\left[\frac{\lambda_2-\omega_1}{\lambda_2-\lambda_1}\right]\left[\frac{\lambda_1-\omega_1}{\gamma^2}\right](e^{i\lambda_1 t}-e^{i\lambda_2 t})\underline{a}\,^\dagger_2+\frac{\gamma_{12}\gamma_{23}}{\gamma^2}\left[-e^{i\omega_1 t}+\left[\frac{\lambda_2-\omega_1}{\lambda_2-\lambda_1}\right]e^{i\lambda_1 t}+\left[\frac{\omega_1-\lambda_1}{\lambda_2-\lambda_1}\right]e^{i\lambda_2 t}\right]\underline{a}\,^\dagger_3 \tag{2.24}$$

which determines the time-dependent population of level 1 through multiplication by its complex conjugate and by taking statistical averages with appropriate initial conditions.

In order to calculate the time evolution of any other operator, we have to go back to step one and define a new basis $\{\underline{v}\,^j_i\}$, with $\underline{v}\,^n_j=(\mathrm{ad}H)^{n-1}\underline{a}\,^\dagger_j$. This basis, although different from the one we first computed, will still generate the same matrix $\underline{G}_{H,a_1^\dagger}$. Notice however, that to find the time evolution of the operator $a_j^\dagger$, it is the vector $\underline{v}\,^1_j=(1,0,0)$, which now enters into Eq. (2.10). Therefore, Eq. (2.23) can be generalized to read as

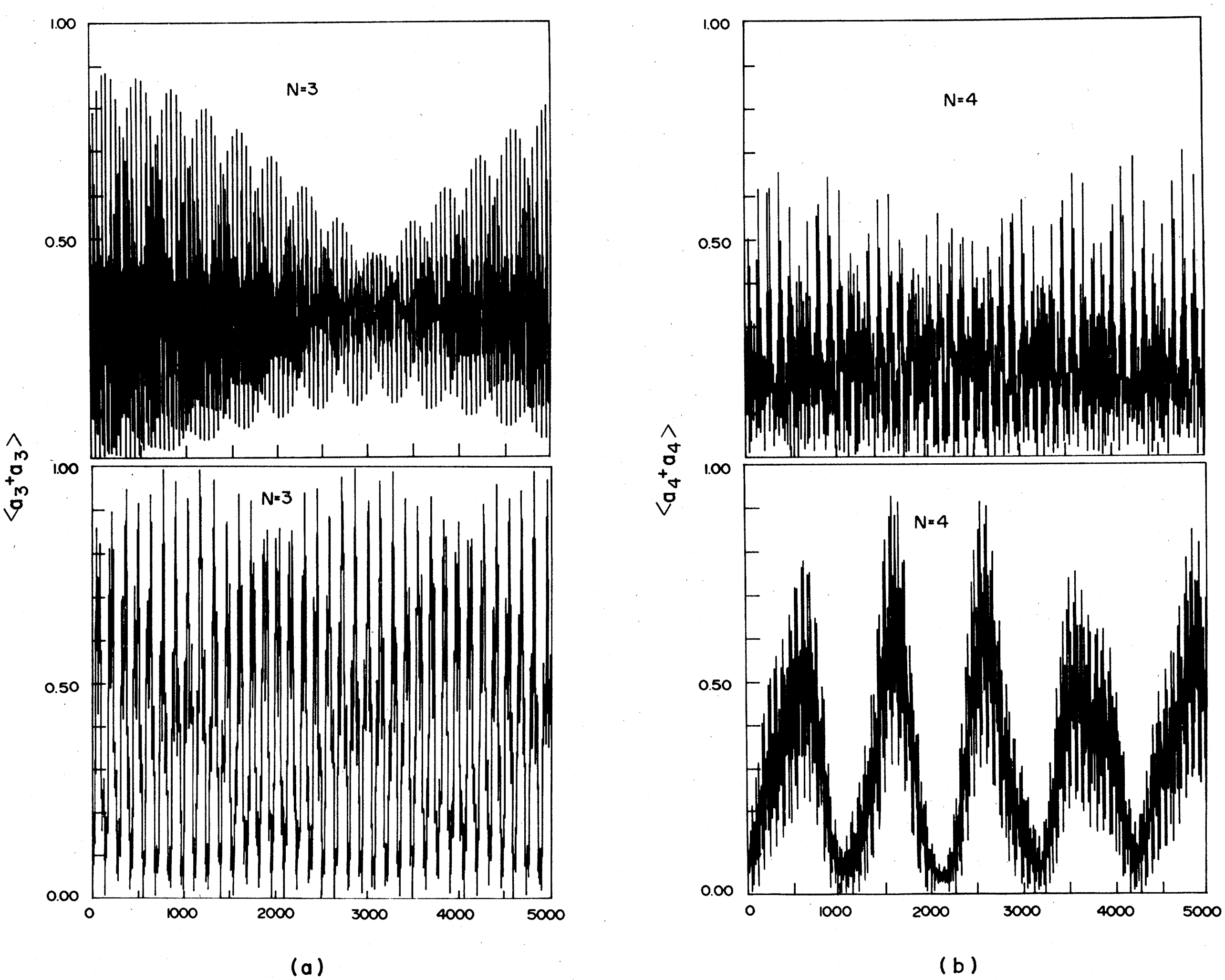

FIG. 2. The expectation value of the Nth level population, as a function of the number of maximum periods. The upper figures have $\gamma_{m,m+1}=\sqrt{m}\,\gamma_{12}$ with $\gamma_{12}=1$, and the lower figures show the equal-Rabi case, for (a) $N=3$, (b) $N=4$.

$$\underline{a}_n^\dagger(t) = -\frac{1}{\gamma^2}[\lambda_1\lambda_2 \underline{v}_1^n - (\omega_1+\omega_2)\underline{v}_2^n + \underline{v}_3^n] e^{i\lambda_0 t} + \frac{(\lambda_2-\omega_1)\omega_1\lambda_2}{(\lambda_2-\lambda_1)\gamma^2}\left[\underline{v}_1^n - \frac{(\omega_2+2\omega_1-\lambda_1)}{\omega_1\lambda_2}\underline{v}_2^n + \frac{1}{\omega_1\lambda_2}\underline{v}_3^n\right]e^{i\lambda_1 t}$$
$$+ \frac{\omega_1\lambda_1(\omega_1-\lambda_1)}{(\lambda_2-\lambda_1)\gamma^2}\left[\underline{v}_1^n - \frac{(\omega_2+2\omega_1-\lambda_2)}{\omega_1\lambda_1}\underline{v}_2^n+\frac{1}{\omega_1\lambda_1}\underline{v}_3^n\right]e^{i\lambda_2 t} . \quad (2.25)$$

In order to obtain results which can be compared with existing ones, we impose the initial conditions $\langle \underline{a}_1^\dagger \underline{a}_1 \rangle = 1$, $\langle \underline{a}_2^\dagger \underline{a}_2 \rangle = \langle \underline{a}_3^\dagger \underline{a}_3 \rangle = 0$. Multiplying Eq. (2.25) by its complex conjugate and taking statistical averages, we find

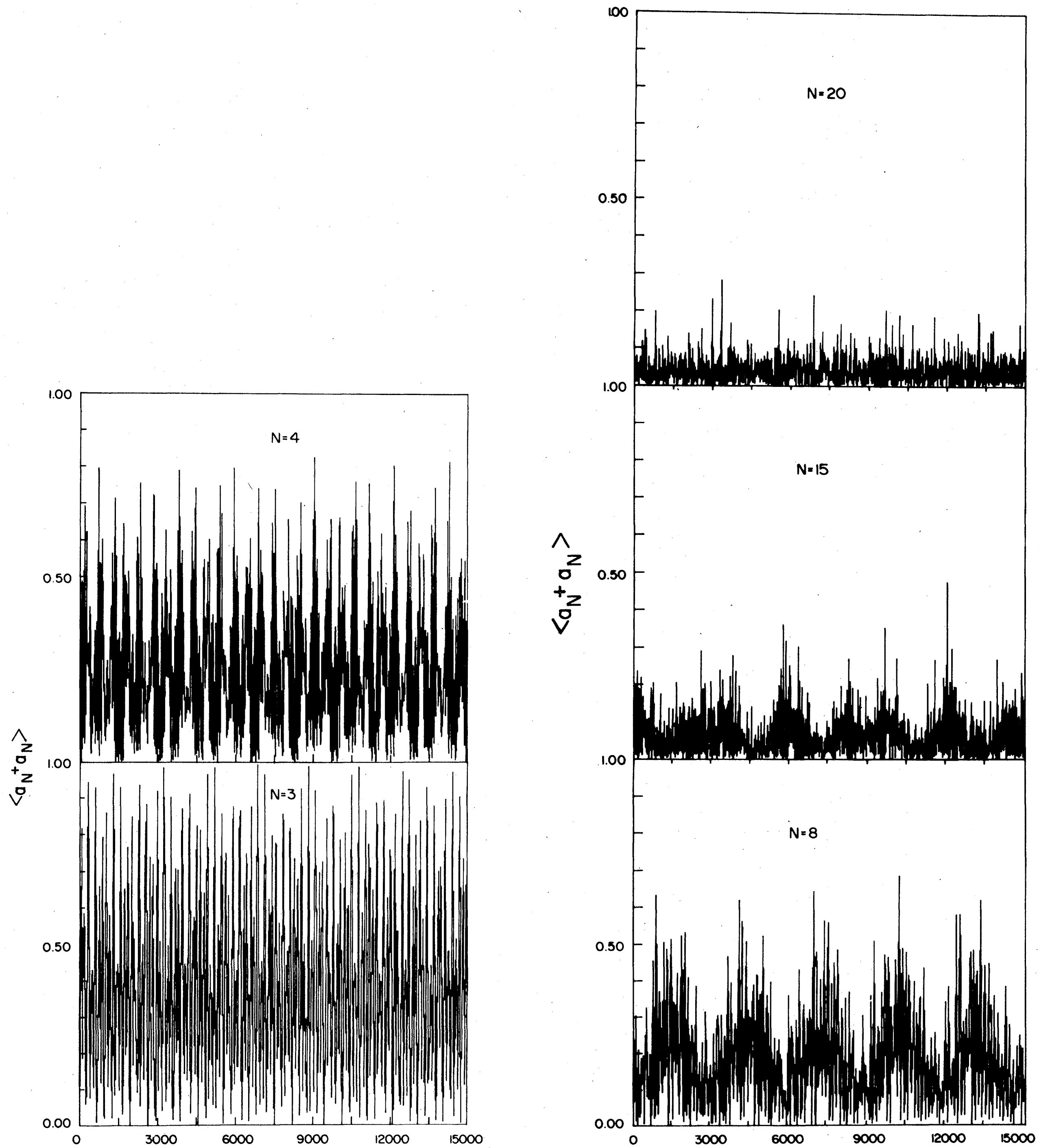

FIG. 3. The expectation value of the Nth level population, as a function of the number of maximum periods, for different levels systems, in the equal-Rabi case.

$$\langle \underline{a}_2^\dagger(t)\underline{a}_2(t)\rangle=\tfrac{1}{2}(\gamma^2/M^2)(1+\alpha)\sin^2(Mt) \tag{2.26}$$

$$\langle \underline{a}_3^\dagger(t)\underline{a}_3(t)\rangle=\tfrac{1}{4}(1-\alpha^2)\{1+\cos^2(Mt)+(\Delta/2M)^2\sin^2(Mt)-2\cos[(\Delta/2)t]\cos(Mt)-(\Delta/M)\sin[(\Delta/2)t]\sin(Mt)\}\ , \tag{2.27}$$

where

$$M=(\lambda_1-\lambda_2)/2=\{[(\omega_1-\omega_2)/2]^2+\gamma^2\}^{1/2} \tag{2.28}$$

$$\Delta/2-M=\omega_1-\lambda_1\ , \tag{2.29}$$

$$\Delta/2+M=\omega_1-\lambda_2\ , \tag{2.30}$$

$$\alpha=(|\gamma_{12}|^2-|\gamma_{23}|^2)/\gamma^2\ . \tag{2.31}$$

These results are identical to those obtained by Senitzky using completely different methods.[3]

E. Multifrequency fields

The techniques which we have developed in the previous sections can be simply generalized to the case of n external frequencies impinging on a quantum system. In this case the Hamiltonian can be written as[6]

$$H(t)=\sum_{i=1}^{N}E_i a_i^\dagger a_i+\sum_{j=1}^{N-1}\gamma_j a_{j+1}^\dagger a_j e^{i\omega_j t}+\text{H.c.}\ , \tag{2.32}$$

where ω_j is the frequency of the jth laser beam, $\gamma_j=-\vec{\mu}_{ij}\cdot\vec{\mathscr{E}}_j$, and $\mathscr{E}_j$ measures the amplitude, phase, and polarization of the plane wave with frequency ω_j.

To apply the EG method once again we need a time-independent Hamiltonian, which is obtained by performing the unitary transformation

$$U(t)=\exp\left[-it\sum_{k=1}^{N}\omega_k\alpha_k a_k^\dagger a_k\right]\ . \tag{2.33}$$

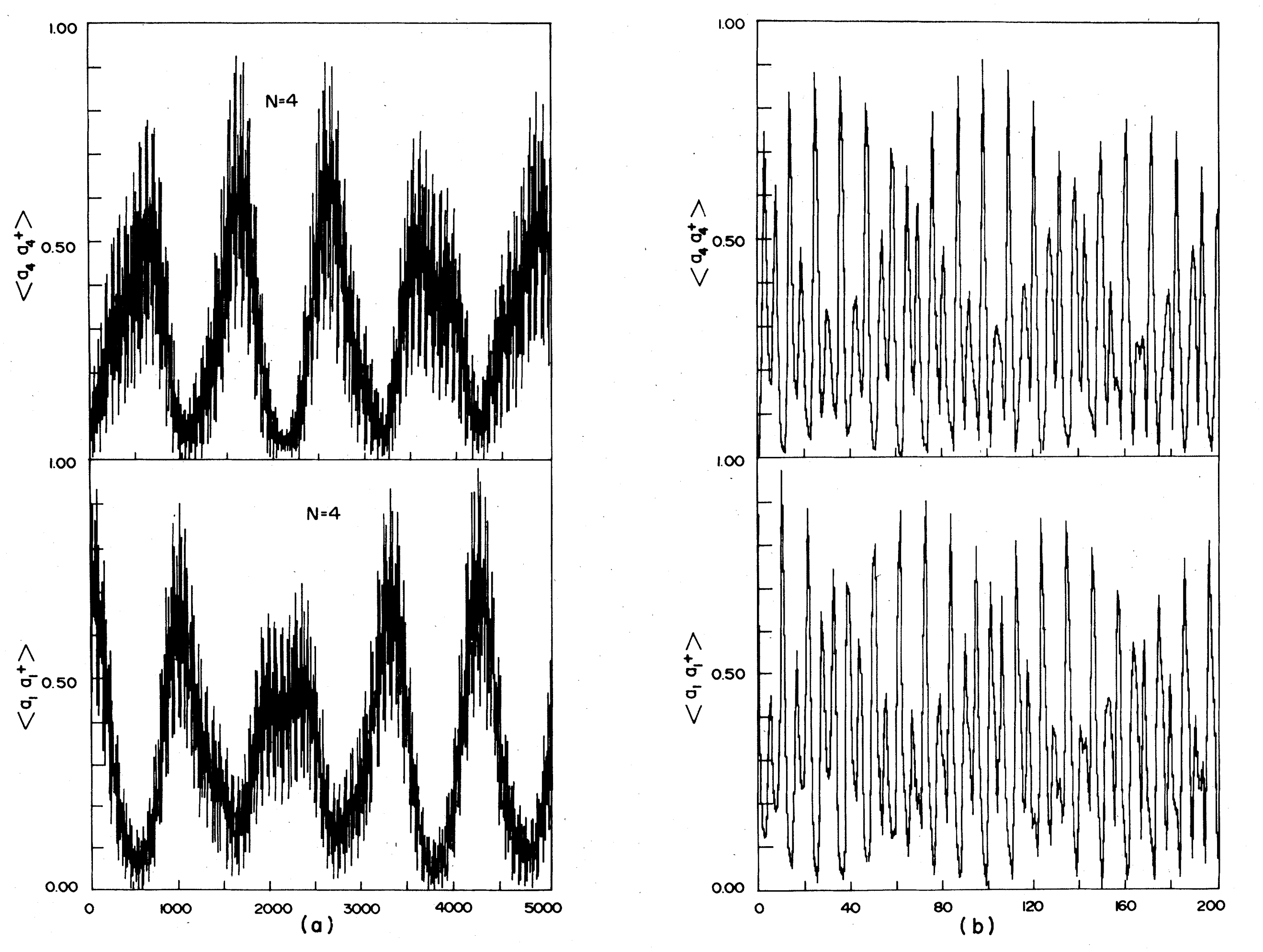

FIG. 4. The statistical average of the first (lower figure) and fourth (upper figure) level, for a four-level system, with $\omega_4-\omega_1\sim 0$ (resonant), as a function of the number of maximum periods (n): (a) $n=5000$; (b) $n=200$ to show the two-level behavior in detail. The oscillator strengths are all equal to one.

The transformed Hamiltonian becomes

$$H=\sum_{i=1}^{N}\epsilon_i a_i^\dagger a_i+\sum_{j=1}^{N-1}(\gamma_j a_j a_{j+1}^\dagger+\gamma_j^* a_{j+1}a_j^\dagger) \quad (2.34)$$

with

$$\epsilon_i=E_i-\alpha_i\omega_i \quad (2.35)$$

and the α_i's are recursively related by

$$\alpha_{j+1}\omega_{j+1}-\alpha_j\omega_j=-\omega_j \ . \quad (2.36)$$

III. NUMERICAL RESULTS

We now present some numerical results which illustrate the points made in the previous sections. As it will become clear for all conditions studied we have recurrence in the system. In general all the curves are more complicated than those shown in Ref. 5. This can be easily understood, by looking at the form of the quasienergies given by Eq. (2.19). It is clear that the quasienergies will be incommensurate even for the case where all the Rabi oscillator strengths are equal. This in turn implies quasiperiodicity in the response of the system to the external field.

In all the figures shown, we have used the boundary condition that at $t=0$ the system is in its ground state. The horizontal axis measures time in units of $2\pi/\lambda_{\min}$, where $\lambda_{\min}$ is the minimum of the absolute values of the quasienergies, and the vertical axis depicts the level population.

We have considered only two cases, (i) all γ_{ij}'s equal to one (equal Rabi case), and (ii) the harmonic case, $\gamma_{m,m+1}=\sqrt{m}\,\gamma_{1,2}$ and $\gamma_{1,2}=1$. In all cases, the external frequency was set to $\omega=0.9999$.

Figure 1 shows the time variation of the fourth-level population for a four-level system. For the upper figure, $E_1=1.0$, $E_2=1.1$, $E_3=3.0$, $E_4=4.0$, with three almost resonant sets of levels ($\omega_i-\omega_j=0$): first and third, first and fourth, and third and fourth. For the lower figure, we have changed E_3 to 2.5. For both figures the oscillator strengths are all equal. Figure 2 shows behavior of the Nth level population in time for $N=3$ [Fig. 2(a)] and $N=4$ [Fig. 2(b)] as we change the oscillator strength. Figure 3 shows how the level population varies, and always recurs, increasing the number of levels, all in the equal-Rabi case. Finally, Fig. 4 shows the behavior of two-resonant levels. Figure 4(b) has an expanded axis so that one can compare the two-level behavior with that obtained by Shore.[8] We see in all figures that the quantum system recurs in time as predicted by the Hogg-Huberman theorem.

IV. SUMMARY

In this paper, we have presented a general technique which allows for the calculation of the quasienergy spectra of bounded quantum systems in the presence of time periodic perturbations. We showed that it is possible to transform the problem into a tractable algorithm first proposed by Eckmann and Guenin for time-independent problems, and which can then be used to study the analytic nature of the quasienergies. At a more fundamental level, this work allows for *a priori* criteria for deciding whether the conditions of the Hogg-Huberman theorem for quantum recurrences are satisfied.

These techniques ought to be of use in problems dealing with the interaction between strong electromagnetic radiation and matter, as we illustrated through a few examples. Finally, we should also point out that a different numerical technique has recently been proposed by Nauts and Wyatt,[21] and which could become complementary to our methods.

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