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Limits of weak damping of a quantum harmonic oscillator

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In this Brief Report we analyze the limit of very weak damping of a quantum-mechanical Brownian oscillator. It is shown that the propagator for the reduced density operator of the oscillator can be written as a double path integral of the same form as that obtained in the high-temperature limit. As a direct consequence, we can write a Fokker-Planck equation for the reduced density operator of the weakly damped oscillator (at any temperature) involving only the damping and a generalized diffusion constant in momentum space.

As has already been shown in the literature,[1] the quantum dynamics of a Brownian harmonic oscillator can be obtained from the knowledge of its reduced density operator, which can be written as[1]

$$\tilde{\rho}(x,y,t)=\int\int dx'dy'J(x,y,t;x',y',0)\tilde{\rho}(x',y',0) \tag{1}$$

where

$$J(x,y,t;x',y',0)=\int_{x'}^{x}\int_{y'}^{y}\mathcal{D}x(t')\mathcal{D}y(t')\exp\frac{i}{\hbar}S_{\text{eff}}[x(t'),y(t')]\exp-\frac{1}{\hbar}\phi[x(t'),y(t')] \tag{2}$$

with

$$S_{\text{eff}}[x(t',y(t')]=\int_0^t(\tfrac{1}{2}M\dot{x}^2-\tfrac{1}{2}M\dot{y}^2-\tfrac{1}{2}M\omega_0^2x^2+\tfrac{1}{2}M\omega_0^2y^2-m\gamma x\dot{x}+M\gamma y\dot{y}-M\gamma x\dot{y}+M\gamma y\dot{x})dt' \tag{3}$$

and

$$\phi[x(t'),y(x')]=\frac{2M\gamma}{\pi}\int_0^{\Omega}\nu\coth\frac{\hbar\nu}{2kT}\int_0^t d\tau\int_0^{\tau}d\sigma[x(\tau)-y(\tau)]\cos\nu(\tau-\sigma)[x(\sigma)-y(\sigma)]\ . \tag{4}$$

Here, γ, ω_0, and Ω are the relaxation, the natural, and a cutoff frequency, respectively. T is the temperature of the environment of the oscillator. For alternative ways to express the function $J(x,y,t;x',y',0)$ the reader is referred to.[2,3]

The functional $\phi[x(t'),y(t')]$ can be simplified when we want to consider temperatures much higher than $\hbar\omega_0/k$. In this case, we can first perform the frequency integration in (4) using the approximation $\coth(\hbar\nu/2kT)\simeq 2kT/\hbar\nu$ (since the typical frequencies of motion are of the order of ω_0 and γ) and reduce (4) to

$$\phi[x(\tau),y(\tau)]=\frac{2M\gamma kT}{\hbar^2}\int_0^t[x(\tau)-y(\tau)]^2d\tau\ . \tag{5}$$

In this limit it has been shown[1] that the double convolution (1) is equivalent to a Fokker-Planck (FP) equation for the Wigner transform of $\tilde{\rho}$ (see below). Moreover, the functional $\phi[x,y]$ as written in (5) is responsible for the presence of the diffusive term in that equation.

Our main goal in this Report is to show that the kernel in (4) can also be reduced to an instantaneous contribution when we have $\gamma \ll \omega_0$ regardless of the temperature. In other words, $\phi[x,y]$ can be written as a single integral either when $kT \gg \hbar\omega_0$ (for any γ) or when $\gamma \ll \omega_0$ (for any T). Consequently we can also write a FP equation for the Wigner transform of $\tilde{\rho}$ when $\gamma \ll \omega_0$ with a generalized diffusion constant due to this new instantaneous form of (4). For regimes not described by the two limits above, a generalization of the FP equation is necessary (see Ref. 4).

Evaluating the path integral (2) within an infinitesimal time interval $[t,t+\epsilon]$ and using this result in (1) we can show that the equation of motion obeyed by $\tilde{\rho}(x,y,t)$ is (see Ref. 1)

$$\frac{\partial\tilde{\rho}}{\partial t}=-\frac{\hbar}{2Mi}\frac{\partial^2\tilde{\rho}}{\partial x^2}+\frac{\hbar}{2Mi}\frac{\partial^2\tilde{\rho}}{\partial y^2}-\gamma(x-y)\frac{\partial\tilde{\rho}}{\partial x}+\gamma(x-y)\frac{\partial\tilde{\rho}}{\partial y}+\frac{1}{2i\hbar}M\omega_0^2x^2\tilde{\rho}-\frac{1}{2i\hbar}M\omega_0^2y^2\tilde{\rho}-\frac{2M\gamma kT}{\hbar^2}(x-y)^2\tilde{\rho}\ . \tag{6}$$

Now, defining new variables $q=(x+y)/2$ and $\xi=x-y$ we can write (6) as

$$\frac{\partial\bar{\rho}}{\partial t}=-\frac{\hbar}{Mi}\frac{\partial^2\bar{\rho}}{\partial q\partial\xi}-2\gamma\xi\frac{\partial\bar{\rho}}{\partial\xi}+\frac{M\omega_0^2}{i\hbar}q\xi\bar{\rho}-\frac{2M\gamma kT}{\hbar^2}\xi^2\bar{\rho} \tag{7}$$

where $\bar{\rho}(q,\xi)\equiv\bar{\rho}(q+\xi/2,q-\xi/2)$.

Multiplying (7) by $(1/2\pi\hbar)\exp(ip\xi/\hbar)$, integrating over ξ, and remembering that the Wigner transform of $\bar{\rho}(x,y)$ is given by

$$W(q,p,t)=\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\exp\frac{ip\xi}{\hbar}\bar{\rho}\left[q+\frac{\xi}{2},q-\frac{\xi}{2}\right]d\xi \tag{8}$$

we end up with

$$\frac{\partial W}{\partial t}=-\frac{\partial}{\partial q}(pW)+\frac{\partial}{\partial p}(M\omega_0^2 qW)+2\gamma\frac{\partial}{\partial p}(pW)+D\frac{\partial^2 W}{\partial p^2} \tag{9}$$

which is the well-known Fokker-Planck equation with $D=2M\gamma kT$. Since we are working in the classical limit we expect $W(q,p,t)$ to become the density of points in the phase space of the Brownian particle and, therefore, the classical limit is properly recovered.

What we are going to show from here onwards is that for a very weakly damped oscillator the double functional integral in (2) generates an equation of the same form as (9) with the diffusion constant in momentum space given by

$$D=M\gamma\omega_0\hbar\coth\frac{\hbar\omega_0}{2kT} \tag{10}$$

which clearly reduces to $D=2M\gamma kT$ when $\hbar\omega_0\ll kT$. Actually this result is in agreement with what is usually obtained in quantum optics.[5] There one treats a harmonic mode very weakly coupled to a bath of noninteracting oscillators in the Markovian approximation. The equation obtained for the reduced density operator of that mode is a FP equation with a diffusive term which only depends on T. This is exactly what is written in (9) and (10).

Equation (10) is not so easy to show as it was for $D=2M\gamma kT$. In order to achieve this result we must solve the path integral (2) which results in (see Ref. 1)

$$J(q,\xi t;q',\xi',0)=\frac{N(t)}{\pi\hbar}\exp\frac{i}{\hbar}\{[K(t)-M\gamma]q\xi+[K(t)+M\gamma]q'\xi'-L(t)q'\xi-N(t)q\xi'\}$$
$$\times\exp\left[-\frac{1}{\hbar}[A^{(1)}(t)\xi^2+A^{(2)}(t)\xi\xi'+A^{(3)}(t)\xi'^2]\right] \tag{11}$$

where $K(t)=M\omega\cot\omega t$, $L(t)=M\omega e^{-\gamma t}/\sin(\omega t)$, $N(t)=M\omega e^{\gamma t}/\sin(\omega t)$,

$$\omega=(\omega_0^2-\gamma^2)^{1/2},$$

and

$$A^{(i)}(t)=\frac{M\gamma}{\pi}\int_0^{\Omega}d\nu\,\nu\coth\frac{\hbar\nu}{2kT}A_\nu^{(i)}(t) \tag{12}$$

with

$$A_\nu^{(1)}(t)=\frac{e^{-2\gamma t}}{\sin^2\omega t}\int_0^t\int_0^t\sin\omega\tau\cos\nu(\tau-\sigma)\times\sin\omega\sigma e^{\gamma(\tau+\sigma)}d\tau\,d\sigma, \tag{13}$$

$$A_\nu^{(2)}(t)=\frac{2e^{-\gamma t}}{\sin^2\omega t}\int_0^t\int_0^t\sin\omega\tau\cos\nu(\tau-\sigma)\times\sin\omega(t-\sigma)e^{\gamma(\tau+\sigma)}d\tau\,d\sigma, \tag{14}$$

$$A_\nu^{(3)}(t)=\frac{1}{\sin^2\omega t}\int_0^t\int_0^t\sin\omega(t-\tau)\cos\nu(\tau-\sigma)\times\sin\omega(t-\sigma)e^{\gamma(\tau+\sigma)}d\tau\,d\sigma. \tag{15}$$

Now, evaluating one of the double integrals in (13)–(15) and taking the limit $\gamma\to 0$ we can show that all $A_\nu^{(i)}(t)$'s are proportional to $\delta(\nu-\omega_0)$ with multiplicative factors given by

$$C^{(1)}(t)=\frac{\pi}{\sin^2\omega_0 t}\int_0^t\sin^2\omega_0 t\,d\tau, \tag{16}$$

$$C^{(2)}(t)=\frac{2\pi}{\sin^2\omega_0 t}\int_0^t\sin\omega_0\tau\sin\omega_0(t-\tau)d\tau, \tag{17}$$

$$C^{(3)}(t)=\frac{\pi}{\sin^2\omega_0 t}\int_0^t\sin^2\omega_0(t-\tau)d\tau, \tag{18}$$

and therefore

$$A^{(i)}(t)=M\gamma\omega_0\coth\frac{\hbar\omega_0}{2kT}C^{(i)}(t). \tag{19}$$

With this form for $A^{(i)}(t)$ the expression (11) is exactly the same we could have obtained had we started from the path integral (2) with $\phi[x(t'),y(t')]$ replaced by

$$\phi[x(t'),y(t')]=\frac{1}{\hbar^2}M\gamma\hbar\omega_0\coth\frac{\hbar\omega_0}{2kT}\times\int_0^t[x(t')-y(t')]^2dt' \tag{20}$$

which leads us [see (5)–(9)] to a FP equation with $D = M\gamma\hbar\omega_0 \coth(\hbar\omega_0/2kT)$. This result was conjectured in a previous paper[1] but it was not properly shown in that occasion.

In conclusion, we have shown that in the two regimes, $kT \gg \hbar\omega_0$ (for any γ) or $\gamma \ll \omega_0$ (for any T), one can alternatively describe the quantum dynamics of a Brownian particle by a FP equation for the Wigner transform of $\tilde{\rho}$. The diffusion constant in momentum space is given either by $D = 2M\gamma kT$ (first case) or by $D = M\gamma\hbar\omega_0 \coth(\hbar\omega_0/2kT)$ (second case).

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