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H_2 AND H_∞ FILTERING DESIGN SUBJECT TO IMPLEMENTATION UNCERTAINTY*

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Abstract. This paper presents new filtering design procedures for discrete-time linear systems. It provides a solution to the problem of linear filtering design, assuming that the filter is subject to parametric uncertainty. The problem is relevant, since the proposed filter design incorporates real world implementation constraints that are always present in practice. The transfer function and the state space realization of the filter are simultaneously computed. The design procedure can also handle plant parametric uncertainty. In this case, the plant parameters are assumed not to be exactly known but belonging to a given convex and closed polyhedron. Robust performance is measured by the H_2 and H_∞ norms of the transfer function from the noisy input to the filtering error. The results are based on the determination of an upper bound on the performance objectives. All optimization problems are linear with constraint sets given in the form of LMI (linear matrix inequalities). Global optimal solutions to these problems can be readily computed. Numerical examples illustrate the theory.

Key words. robust filtering, implementation uncertainty, fragility, linear matrix inequalities

AMS subject classifications. 93E11, 93E25, 60G35

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1. Introduction.

In the 1980s, a great deal of effort was dedicated to the study of implementation issues of filters and controllers [1, 2, 3, 4]. The motivation was to devise design techniques that would lead to filters and controllers that could perform well when implemented on a digital computer. The main objectives were (a) to minimize the degradation of performance caused by computation of signals in a finite precision computational architecture, and (b) to minimize the impact of truncation and rounding on the coefficients of the filter or controller. These objectives were addressed using many different techniques (see [1] for details). Among these techniques, a popular approach to dealing with degradation of the signals was to model rounding and truncation as noise [5], whereas rounding and truncation of the filter or controller coefficients was addressed by studying the *sensitivity* of these parameters to variations [1]. The great development of the computer industry in the 1990s brought to the signal processing and control practitioner processors with more and more bits of precision at very low cost, which somewhat dimmed the importance of the topic. The fact that every few years the computer industry provides processors with longer wordlength is used by some to justify the design of filters and controllers with little or no regard to finite precision perturbation effects. In fact, for many simple systems, this increase in wordlength means that the quantization effects can be practically ignored. However, faster and more precise computers also provide the opportunity to increase the complexity of the systems, in terms of both more sophisticated algorithms

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and number of devices. As observed in [6], this increased complexity will eventually face limitations in bandwidth, that is, the speed at which the devices communicate, reducing the sampling rates (relative to the available processor wordlength). In this scenario, a careful analysis of perturbation effects on filters and controllers certainly will be required. Also, in many consumer electronics products, inexpensive processors (say, fixed point digital signal processors (DSPs)) are usually preferred. These processors often impose nontrivial wordlength limitations, and thus better design algorithms are needed to deal with them. Some recent efforts along this line are reported in [7].

In fact, the importance of robustness to filter and control parametric perturbations seems to have been *rediscovered* by the end of the 1990s with the paper [8]. This work, despite the controversy it raised [9], showed that many robust control design methods, which were targeted to deal with plant uncertainty, could be particularly sensitive to parameter uncertainty on the controller. The authors use a series of numerical examples to illustrate that a very small perturbation on the coefficients of controllers could lead to a loss of stability of the closed-loop system [8]. Since then, many authors have addressed the problem of robustness to parametric control or filter perturbation under the label of *fragility* [10, 11, 12, 13, 14].

While many works on filter *sensitivity* are more concerned with the problem of choosing an appropriate realization for a given filter transfer function [2, 3], many works on *fragility* seem to focus more on the robustness of the filter transfer function rather than its realization [12]. The approach developed in this paper blends these two issues by simultaneously designing the optimal filter transfer function *and* its realization. The strategy is to modify the filtering procedure introduced in [15] to take into account robustness with respect to filter parametric variations. Variations of the filter parameters are allowed inside a region specified by a quadratic matrix inequality. The maximum allowed norm of the filter uncertainty is specified as a percentage of the norm of the nominal filter parameters. The ability to specify the uncertainty in the filter parameters relative to the size of the nominal filter is especially important when the transfer function and the state space realization of the filter are to be designed simultaneously. This model is also very appropriate to model perturbations on the parameters coming from truncation on a floating-point computational architecture, where rounding and truncation introduce errors relative to the size of the original numbers.

In this paper, guaranteed cost functions are developed to provide upper bounds on the maximum value of the H_2 or H_∞ norm of the uncertain transfer function from an exogenous noise input to the filtering error on the filter uncertainty region. This paper introduces and completely solves these H_2 and H_∞ guaranteed cost filtering design problems. The design conditions are expressed as linear matrix inequalities (LMIs), and hence numerical solutions can be readily computed [16]. In contrast to [15, 17], the results specify not only the transfer function of the filter but also its realization. Illustrative examples show the effectiveness of the proposed approach. An interesting feature observed in the examples is that the filters designed by the proposed technique have less round-off gain than the standard Kalman filter [2, 5], although such a performance measure is not directly addressed in the optimization process. The design procedures introduced in this paper admit straightforward extensions to simultaneously handle plant parameter uncertainty specified in terms of convex bounded polyhedrons. These extensions can be derived to contemplate both the quadratic stability [17] and the extended stability [18] approaches. In the former, a single quadratic Lyapunov function is used to evaluate the performance on the uncertainty region, while in the latter a parameter dependent Lyapunov function [19] is built.

The notation is standard. Lowercase letters denote vectors while capital letters represent matrices. The symbol $(^T)$ is used to indicate the transpose of vectors and matrices. If a symmetric matrix X is positive definite, this is indicated by $X > 0$.

2. Preliminary results on filtering. Consider the linear discrete-time time-invariant system

$$x(k+1) = Ax(k) + Bw(k), \tag{1}$$

$$z(k) = C_z x(k) + D_z w(k), \tag{2}$$

$$y(k) = C_y x(k) + D_y w(k), \tag{3}$$

where all matrices and vectors are assumed to have appropriate dimensions. The *optimal filtering* problem consists of designing a linear filter

$$x_f(k+1) = A_f x_f(k) + B_f y(k), \tag{4}$$

$$z_f(k) = C_f x_f(k) + D_f y(k), \tag{5}$$

which makes use of the plant output $y(k)$ to produce the filtered output $z_f(k)$, with the objective of minimizing a norm of the transfer function from the noise input $w(k)$ to the filtering error $e(k) := z(k) - z_f(k)$. Collecting the filter parameters in the matrix

$$\mathcal{F} := \begin{bmatrix} D_f & C_f \\ B_f & A_f \end{bmatrix}, \tag{6}$$

we can state the optimal filtering problem as the optimization problem

$$\min_{\mathcal{F}} \|H_{we}(z;\mathcal{F})\|_p. \tag{7}$$

The values of $p = \{2, \infty\}$ are the choices usually found in the literature. The next lemmas revisit the solutions of the optimal filtering problems given in [17]. The solution is given as LMI conditions formulated in terms of the transformed set of filter parameters

$$\mathcal{K} := \begin{bmatrix} R & L \\ F & Q \end{bmatrix}, \tag{8}$$

defined with respect to the above partitioning.

LEMMA 1 (H_2 filtering). *There exist a matrix $\mathcal{K}$, partitioned as in* (8), *and symmetric matrices Y, Z, W such that the LMI*

$$\begin{bmatrix} Z & \bullet & \bullet & \bullet & \bullet \\ Z & Y & \bullet & \bullet & \bullet \\ A^T Z & A^T Y + C_y^T F^T + Q^T & Z & \bullet & \bullet \\ A^T Z & A^T Y + C_y^T F^T & Z & Y & \bullet \\ B^T Z & B^T Y + D_y^T F^T & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} > 0, \tag{9}$$

$$\begin{bmatrix} W & \bullet & \bullet & \bullet \\ C_z^T - C_y^T R^T - L^T & Z & \bullet & \bullet \\ C_z^T - C_y^T R^T & Z & Y & \bullet \\ D_z^T - D_y^T R^T & \mathbf{0} & \mathbf{0} & I \end{bmatrix} > 0, \tag{10}$$

$$\operatorname{trace}(W) < \mu \tag{11}$$

have a feasible solution if and only if the filter

$$\mathcal{F} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & V^{-1} \end{bmatrix} \mathcal{K} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & Z^{-1}U^{-1} \end{bmatrix}, \tag{12}$$

where U and V are nonsingular otherwise arbitrary matrices chosen to satisfy $Y + VUZ = Z$, is such that

$$\|H_{we}(z;\mathcal{F})\|_2^2 < \mu. \tag{13}$$

LEMMA 2 (H_∞ filtering). *There exist a matrix $\mathcal{K}$, partitioned as in* (8), *and symmetric matrices Y, Z such that the LMI*

$$\begin{bmatrix} Z & \bullet & \bullet & \bullet & \bullet & \bullet \\ Z & Y & \bullet & \bullet & \bullet & \bullet \\ A^T Z & A^T Y + C_y^T F^T + Q^T & Z & \bullet & \bullet & \bullet \\ A^T Z & A^T Y + C_y^T F^T & Z & Y & \bullet & \bullet \\ B^T Z & B^T Y + D_y^T F^T & \mathbf{0} & \mathbf{0} & \mu \mathbf{I} & \bullet \\ \mathbf{0} & \mathbf{0} & C_z - RC_y - L & C_z - RC_y & D_z - RD_y & \mu \mathbf{I} \end{bmatrix} > 0 \tag{14}$$

have a feasible solution if and and only if the filter $\mathcal{F}$ given in (12) *is such that*

$$\|H_{we}(z;\mathcal{F})\|_\infty < \mu. \tag{15}$$

The above lemmas are generalizations of the results obtained in [17]. Here, the assumptions that the filter(4)–(5) is strictly proper and that the matrix D_z is null have been removed. There is virtually no change from the proofs presented in [15, 17] to the ones required to prove Lemmas 1 and 2. These proofs are omitted for brevity and the interested reader is referred to [15, 17] for more details. The constraints stated in Lemmas 1 and 2 are all LMI, and hence solutions to the optimization problem (7) can be obtained by minimizing the scalar μ subject to the given inequalities. The resulting problems are convex and their global optimal solutions can be obtained via convex programming techniques [16].

Once a solution to the inequalities stated in Lemmas 1 or 2 has been found, the user is asked to pick an arbitrary nonsingular matrix U and then solve for V to satisfy $Y + VUZ = Z$ (or choose V and solve for U). This will produce the optimal filter parameters $\mathcal{F}$ through (12). Notice that this is done a posteriori, and that this arbitrary choice does not affect the optimality of the solution. In fact, it is possible to show that the transfer function of the filter associated with the parameters (12) is not affected by the choice of U and V (see [17] for details). The main role of these matrices is to parameterize a particular state space realization of the filter, a fact that will be explored in the next sections.

3. Problem statement. The main purpose of this paper is to derive conditions for the design of filters subject to parametric perturbations. More specifically, it is assumed that the parameters of the filter(4)–(5) are subject to an additive perturbation of the form

$$\mathcal{F} = \mathcal{F}_0 + \Delta_{\mathcal{F}}. \tag{16}$$

The symbol $\mathcal{F}_0$ denotes *nominal* filter parameters, and the unknown perturbation $\Delta_{\mathcal{F}}$ is assumed to be in the set

$$\mathbb{F}_{\mathcal{R}}(\mathcal{F}_0) := \left\{ \Delta_{\mathcal{F}} \,:\, \Delta_{\mathcal{F}}^T \mathcal{R}^{-1} \Delta_{\mathcal{F}} \leq \gamma^2 \mathcal{F}_0^T \mathcal{R}^{-1} \mathcal{F}_0 \right\}. \tag{17}$$

In contrast to the conventional norm bounded uncertainty model, where the right-hand side of the inequality given in (17) is usually constant, the uncertainty set $\mathbb{F}_{\mathcal{R}}(\mathcal{F}_0)$ relates the size of the parametric perturbation $\Delta_{\mathcal{F}}$ to the size of the nominal filter parameters $\mathcal{F}_0$. These factors are weighted by the inverse of an arbitrary positive definite matrix $\mathcal{R}$. In this way, by setting the scalar $0 \le \gamma \le 1$, the size of the perturbation $\Delta_{\mathcal{F}}$ can be specified *relative* to the size of the nominal filter parameters $\mathcal{F}_0$, which are yet to be determined. The inequality (17) can also be translated as a more standard norm bound relation of the kind

$$\Delta_{\mathcal{F}} \in \mathbb{F}_{\mathcal{R}}(\mathcal{F}_0) \quad \Rightarrow \quad \|\Delta_{\mathcal{F}}\|_{\mathcal{R}^{-1}} \le \gamma \|\mathcal{F}_0\|_{\mathcal{R}^{-1}}, \tag{18}$$

where $\|\cdot\|_{\mathcal{R}}$ denotes a weighted Frobenius or two norm. This inequality is evidence that the norm of $\Delta_{\mathcal{F}} \in \mathbb{F}_{\mathcal{R}}(\mathcal{F}_0)$ is limited to being a fraction of the norm of the nominal filter $\mathcal{F}_0$. Another interpretation is obtained in terms of a norm bound on the amplitude of the noise signal

$$w_{\Delta_{\mathcal{F}}}(k) = \begin{pmatrix} w_y(k) \\ w_{x_f}(k) \end{pmatrix} := \Delta_{\mathcal{F}} \begin{pmatrix} y(k) \\ x_f(k) \end{pmatrix}, \tag{19}$$

for which

$$\Delta_{\mathcal{F}} \in \mathbb{F}_{\mathcal{R}}(\mathcal{F}_0) \quad \Rightarrow \quad \|w_{\Delta_{\mathcal{F}}}(k)\|_{\mathcal{R}^{-1}} \le \gamma \left\| \mathcal{F}_0 \begin{pmatrix} y(k) \\ x_f(k) \end{pmatrix} \right\|_{\mathcal{R}^{-1}}. \tag{20}$$

The above interpretation relates the uncertainty set $\mathbb{F}_{\mathcal{R}}(\mathcal{F}_0)$ to the uncertainty models considered in the recent work [20].

The weighting factor $\mathcal{R}$ plays an interesting role in the definition of $\mathbb{F}_{\mathcal{R}}(\mathcal{F}_0)$ and can have a major impact on the reduction of the conservatism of the design conditions to be derived. Roughly speaking, the matrix $\mathcal{R}$ can play the same role as a *scaling matrix*[1] in robust H_∞ analysis [21]. Using the LMI conditions to be derived in the next section, one can simultaneously perform the design of both the filter parameters $\mathcal{F}$ and the scaling matrix $\mathcal{R}$. If desired, one can also set $\mathcal{R}$ to a constant value without destroying the linearity of the design conditions. However, notice that, if the objective of fixing $\mathcal{R}$ is to establish a certain fixed weight on (17–18) and (20), say, $\mathcal{R} = \bar{\mathcal{R}}$, then one can still use a scaling matrix $\mathcal{R} = \lambda \bar{\mathcal{R}}$, where λ is a positive scalar to be determined. Leaving the scalar λ as a variable can be of much help in reducing conservatism (see the numerical example in section 6).

Throughout the rest of this paper, the norm minimization problem defined in (7) is replaced with

$$\min_{\mathcal{F}_0} \rho_p\left(\mathcal{F}_0\right), \tag{21}$$

where the function ρ_p is a *guaranteed cost* function, that is, it satisfies the inequality

$$\|H_{we}(z; \mathcal{F}_0 + \Delta_{\mathcal{F}})\|_p \le \rho_p(\mathcal{F}_0) \quad \forall \Delta_{\mathcal{F}} \in \mathbb{F}_{\mathcal{R}}(\mathcal{F}_0). \tag{22}$$

In other words, the function ρ_p is an upper bound to the H_p norm of the uncertain transfer function $H_{we}(z; \mathcal{F}_0 + \Delta_{\mathcal{F}})$ that holds for all $\Delta_{\mathcal{F}} \in \mathbb{F}_{\mathcal{R}}(\mathcal{F}_0)$.

[1]Notice that when $\mathcal{R}$ is a scalar it can be canceled on both sides of (17) and (18).

4. Main result. We are not aware of any available design technique that can effectively solve the filter design problems stated in the previous section, where the filter parameter is subject to uncertainties $\Delta_{\mathcal{F}} \in \mathbb{F}_{\mathcal{R}}(\mathcal{F}_0)$. In the following paragraphs we will show that a much simpler design problem, that is, one that can be stated as a set of LMI, can be obtained if uncertainties are introduced in the transformed set of parameters $\mathcal{K}$ defined in (8). That is, we will consider the filter design problem, where the transformed set of parameters $\mathcal{K}$ is perturbed as

$$\mathcal{K} = \mathcal{K}_0 + \Delta_{\mathcal{K}}, \qquad \Delta_{\mathcal{K}} \in \mathbb{F}_{\mathcal{W}}(\mathcal{K}_0), \tag{23}$$

where the scaling $\mathcal{W}$ will be chosen to maintain equivalence between $\mathbb{F}_{\mathcal{R}}(\mathcal{F}_0)$ and $\mathbb{F}_{\mathcal{W}}(\mathcal{K}_0)$. More specifically, $\mathcal{W}$ will be chosen to ensure that we can find the optimal solution to problem (21)–(22) by solving an equivalent but simpler problem, where the perturbations act on the transformed set of filter variables. This is made possible due to the result in the following lemma.

LEMMA 3. *Let $\mathcal{S}$ and $\mathcal{T}$ be any square and nonsingular matrices of appropriate dimensions. Then $\Delta_{\mathcal{F}} \in \mathbb{F}_{\mathcal{R}}(\mathcal{F}_0)$ if and only if $\mathcal{S}\Delta_{\mathcal{F}}\mathcal{T} \in \mathbb{F}_{\mathcal{S}\mathcal{R}\mathcal{S}^T}(\mathcal{S}\mathcal{F}_0\mathcal{T})$.*

Proof. The proof follows immediately from using the assumption that matrices $\mathcal{S}$ and $\mathcal{T}$ are nonsingular and properly factorizing the variables and matrices appearing in the definition of $\mathbb{F}_{\mathcal{R}}(\mathcal{F}_0)$. ∎

Lemma 3 deserves two remarks. The first is that it makes explicit how the scaling matrix $\mathcal{W}$ must be chosen to cope with the one to one change of variables in the form $\mathcal{K} = \mathcal{S}\mathcal{F}\mathcal{T}$ that will be used to parameterize the transformed set of filter parameters. Notice that the corresponding "transformed" scaling $\mathcal{W} = \mathcal{S}\mathcal{R}\mathcal{S}^T$ depends exclusively on $\mathcal{S}$. Second, equivalence between $\mathbb{F}_{\mathcal{R}}(\mathcal{F}_0)$ and $\mathbb{F}_{\mathcal{W}}(\mathcal{K}_0)$ is achieved when the change of variables is performed simultaneously on the nominal filter $\mathcal{F}_0$ and on the parametric uncertainty $\Delta_{\mathcal{F}}$. These properties enables us to determine a solution to problem (21) by equivalently rewriting the inequality that defines the guaranteed cost function (22) in the form

$$\|H_{we}(z; \mathcal{K}_0 + \Delta_{\mathcal{K}})\|_p \le \rho_p(\mathcal{K}_0) \quad \forall \Delta_{\mathcal{K}} \in \mathbb{F}_{\mathcal{S}\mathcal{R}\mathcal{S}^T}(\mathcal{K}_0), \tag{24}$$

which is expressed entirely in terms of the transformed variables $(\mathcal{K}_0, \Delta_{\mathcal{K}}) = (\mathcal{S}\mathcal{F}_0\mathcal{T}, \mathcal{S}\Delta_{\mathcal{F}}\mathcal{T})$. Notice that the assumption that $\mathcal{S}$ and $\mathcal{T}$ are nonsingular and square matrices is naturally satisfied whenever the order of the filter is the same as the order of the plant.

In the following lemma we develop an inequality associated with a perturbation on the transformed set of parameters $\mathcal{K}$. This inequality will be used to derive the main result of this paper.

LEMMA 4. *If there exists a symmetric and positive definite matrix $\mathcal{W}$ such that*

$$\begin{bmatrix} \mathcal{Q} + \mathcal{B}\mathcal{K}_0\mathcal{C} + \mathcal{C}^T\mathcal{K}_0^T\mathcal{B}^T - \mathcal{B}\mathcal{W}\mathcal{B}^T & \gamma\mathcal{C}^T\mathcal{K}_0^T \\ \gamma\mathcal{K}_0\mathcal{C} & \mathcal{W} \end{bmatrix} > 0, \tag{25}$$

then

$$\mathcal{Q} + \mathcal{B}\left(\mathcal{K}_0 + \Delta_{\mathcal{K}}\right)\mathcal{C} + \mathcal{C}^T\left(\mathcal{K}_0 + \Delta_{\mathcal{K}}\right)^T\mathcal{B}^T > 0 \quad \forall\, \Delta_{\mathcal{K}} \in \mathbb{F}_{\mathcal{W}}(\mathcal{K}_0). \tag{26}$$

Proof. Applying the Schur complement on (25), one obtains that for all $\Delta_{\mathcal{K}} \in \mathbb{F}_{\mathcal{W}}(\mathcal{K}_0)$,

$$\begin{aligned} \mathcal{Q} + \mathcal{B}\mathcal{K}_0\mathcal{C} + \mathcal{C}^T\mathcal{K}_0^T\mathcal{B}^T &> \mathcal{B}\mathcal{W}\mathcal{B}^T + \gamma^2\mathcal{C}^T\mathcal{K}_0^T\mathcal{W}^{-1}\mathcal{K}_0\mathcal{C} \\ &> \mathcal{B}\mathcal{W}\mathcal{B}^T + \mathcal{C}^T\Delta_{\mathcal{K}}^T\mathcal{W}^{-1}\Delta_{\mathcal{K}}\mathcal{C} \\ &> -\mathcal{B}\Delta_{\mathcal{K}}\mathcal{C} - \mathcal{C}^T\Delta_{\mathcal{K}}^T\mathcal{B}^T, \end{aligned}$$

which recovers (26). ∎

The condition stated in the above lemma is only sufficient. Yet it has been extensively used in the filtering and control to characterize computable robustness conditions as, for instance, in [12, 21]. However, notice that the scaling matrix $\mathcal{W}$ enters the above condition linearly so that is can be freely optimized. This will help reduce the conservatism of this condition.

The above two lemmas will be combined to show that the optimal solution to the problem (21) subject to the transformed guaranteed cost function (24), for $p = \{2, \infty\}$, can be formulated and solved in terms of LMI conditions. We first consider the case when the multiplier $\mathcal{R}$ is a free optimization variable.

THEOREM 1 (H_2 filtering). *If there exist matrices G and $\mathcal{K}_0$, partitioned as in* (8)*, and symmetric matrices Y, Z, W, E, H such that the LMI*

$$\begin{bmatrix} Z & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ Z & Y - H & \bullet & \bullet & \bullet & \bullet & \bullet \\ A^T Z & A^T Y + C_y^T F^T + Q^T & Z & \bullet & \bullet & \bullet & \bullet \\ A^T Z & A^T Y + C_y^T F^T & Z & Y & \bullet & \bullet & \bullet \\ B^T Z & B^T Y + D_y^T F^T & \mathbf{0} & \mathbf{0} & \mathbf{I} & \bullet & \bullet \\ \mathbf{0} & \mathbf{0} & \gamma R C_y + \gamma L & \gamma R C_y & \gamma R D_y & E & \bullet \\ \mathbf{0} & \mathbf{0} & \gamma F C_y + \gamma Q & \gamma F C_y & \gamma F D_y & G^T & H \end{bmatrix} > 0, \tag{27}$$

$$\begin{bmatrix} W - E & \bullet & \bullet & \bullet & \bullet & \bullet \\ C_z^T - C_y^T R^T - L^T & Z & \bullet & \bullet & \bullet & \bullet \\ C_z^T - C_y^T R^T & Z & Y & \bullet & \bullet & \bullet \\ D_z^T - D_y^T R^T & \mathbf{0} & \mathbf{0} & \mathbf{I} & \bullet & \bullet \\ \mathbf{0} & \gamma R C_y + \gamma L & \gamma R C_y & \gamma R D_y & E & \bullet \\ \mathbf{0} & \gamma F C_y + \gamma Q & \gamma F C_y & \gamma F D_y & G^T & H \end{bmatrix} > 0, \tag{28}$$

$$\operatorname{trace}(W) < \mu \tag{29}$$

have a feasible solution, then the nominal filter

$$\mathcal{F}_0 = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & V^{-1} \end{bmatrix} \mathcal{K}_0 \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & Z^{-1} U^{-1} \end{bmatrix}, \tag{30}$$

where U and V are nonsingular otherwise arbitrary matrices chosen to satisfy $Y + VUZ = Z$, is such that

$$\|H_{we}(z; \mathcal{F}_0 + \Delta_{\mathcal{F}})\|_2^2 \le \rho_2(\mathcal{F}_0) := \mu \quad \forall \Delta_{\mathcal{F}} \in \mathbb{F}_{\mathcal{R}}(\mathcal{F}_0), \tag{31}$$

where $\mathbb{F}_{\mathcal{R}}(\mathcal{F}_0)$ is as defined in (17) *with the scaling matrix*

$$\mathcal{R} := \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & V^{-1} \end{bmatrix} \begin{bmatrix} E & G \\ G^T & H \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & V^{-T} \end{bmatrix}. \tag{32}$$

Proof. Defining

$$\Delta_{\mathcal{K}} := \begin{bmatrix} \Delta_R & \Delta_L \\ \Delta_F & \Delta_Q \end{bmatrix}, \qquad \mathcal{S} := \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & V \end{bmatrix}, \qquad \mathcal{T} := \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & UZ \end{bmatrix},$$

the nominal filter parameters (30) and their parametric perturbations can be recovered from ($\mathcal{K}_0$, $\Delta_{\mathcal{K}}$), for U, V, and Z nonsingular, by the formulas

$$\mathcal{F}_0 = \mathcal{S}^{-1} \mathcal{K}_0 \mathcal{T}^{-1}, \quad \Delta_{\mathcal{F}} = \mathcal{S}^{-1} \Delta_{\mathcal{K}} \mathcal{T}^{-1}.$$

Hence, it is possible to conclude from the result of Lemma 3 that the constraint $\Delta_{\mathcal{F}} \in \mathbb{F}_{\mathcal{R}}(\mathcal{F}_0)$ can be replaced without loss of generality by $\Delta_{\mathcal{K}} \in \mathbb{F}_{\mathcal{S}\mathcal{R}\mathcal{S}^T}(\mathcal{K}_0)$, as indicated in (24).

With that in mind, it suffices to show that (27)–(29) guarantee robustness with respect to all $\Delta_{\mathcal{K}} \in \mathbb{F}_{\mathcal{S}\mathcal{R}\mathcal{S}^T}(\mathcal{K}_0)$. This can be done with the help of Lemma 4. Notice that a perturbed version of (9), where $\mathcal{K}$ is replaced with $\mathcal{K}_0 + \Delta_{\mathcal{K}}$, can be written as (26) with

$$\mathcal{Q} := \begin{bmatrix} Z & Z & ZA & ZA & ZB \\ Z & Y & YA & YA & YB \\ A^T Z^T & A^T Y^T & Z & Z & \mathbf{0} \\ A^T Z^T & A^T Y^T & Z & Y & \mathbf{0} \\ B^T Z & B^T Y & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}, \quad \mathcal{B} := \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathcal{C}^T := \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ C_y^T & \mathbf{I} \\ C_y^T & \mathbf{0} \\ D_y^T & \mathbf{0} \end{bmatrix},$$

while a perturbed inequality (10) is in the form (26) with

$$\mathcal{Q} := \begin{bmatrix} W & C_z & C_z & D_z \\ C_z^T & Z & Z & \mathbf{0} \\ C_z^T & Z & Y & \mathbf{0} \\ D_z^T & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}, \qquad \mathcal{B} := \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \qquad \mathcal{C}^T := - \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ C_y^T & \mathbf{I} \\ C_y^T & \mathbf{0} \\ D_y^T & \mathbf{0} \end{bmatrix}.$$

Therefore we can define the variables

$$\begin{bmatrix} E & G \\ G^T & H \end{bmatrix} := \mathcal{S}\mathcal{R}\mathcal{S}^T = \mathcal{W}$$

to obtain both inequalities (27) and (28) directly from Lemma 4. □

THEOREM 2 (H_∞ filtering). *If there exist matrices G and $\mathcal{K}_0$, partitioned as in* (8)*, and symmetric matrices Y, Z, E, H such that the LMI*

$$\begin{bmatrix} Z & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ Z & Y - H & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ A^T Z & A^T Y + C_y^T F^T + Q^T & Z & \bullet & \bullet & \bullet & \bullet & \bullet \\ A^T Z & A^T Y + C_y^T F^T & Z & Y & \bullet & \bullet & \bullet & \bullet \\ B^T Z & B^T Y + D_y^T F^T & \mathbf{0} & \mathbf{0} & \mu \mathbf{I} & \bullet & \bullet & \bullet \\ \mathbf{0} & G & C_z - RC_y - L & C_z - RC_y & D_z - RD_y & \mu \mathbf{I} - E & \bullet & \bullet \\ \mathbf{0} & \mathbf{0} & \gamma RC_y + \gamma L & \gamma RC_y & \gamma RD_y & \mathbf{0} & E & \bullet \\ \mathbf{0} & \mathbf{0} & \gamma FC_y + \gamma Q & \gamma FC_y & \gamma FD_y & \mathbf{0} & G^T & H \end{bmatrix} > 0, \tag{33}$$

has a feasible solution, then the nominal filter $\mathcal{F}_0$ given in (30) *is such that*

$$\|H_{we}(z; \mathcal{F}_0 + \Delta_{\mathcal{F}})\|_\infty \le \rho_\infty(\mathcal{F}_0) := \mu \quad \forall \Delta_{\mathcal{F}} \in \mathbb{F}_{\mathcal{R}}(\mathcal{F}_0), \tag{34}$$

where $\mathbb{F}_{\mathcal{R}}(\mathcal{F}_0)$ is defined with the scaling matrix $\mathcal{R}$ given by (32)*.*

Proof. This proof follows the same pattern as the proof of Theorem 1 and is thus omitted. □

The constraints stated in Theorems 1 and 2 are all LMI. The scalar μ can be used to define the guaranteed cost function (22). The global optimal solution to the guaranteed cost problem (21) can be obtained by minimizing the scalar μ subject to the given LMI.

It is interesting to observe that under the assumption that the scaling matrix $\mathcal{R}$ is a free variable, the filter provided by Theorem 1 shares with the one proposed in [17] the property that its state space realization is irrelevant as far as the upper bound of the estimation error is concerned. As in Lemmas 1 and 2, the state space parameterization of the optimal filter obtained in Theorems 1 and 2 can be arbitrarily chosen by changing the matrices U and V. However, notice that, from (32), the choice of V does affect the multiplier $\mathcal{R}$. For instance, for the particular choice $V = \mathbf{I}$, the sets $\mathbb{F}_{\mathcal{R}}(\mathcal{F}_0) \equiv \mathbb{F}_{\mathcal{S}\mathcal{R}\mathcal{S}^T}(\mathcal{K}_0)$. Due to the coupling condition (32), this property does not remain valid when the scaling matrix is fixed. This special case is treated in detail in the following paragraphs.

When $\mathcal{R}$ is a given constant matrix, the variable V, which is associated with the choice of filter realization, becomes part of the optimization variables by the relation (32). In general, the introduction of (32) in the form of a constraint in the optimization design problem destroys the desired convexity properties. However, in the important case when $\mathcal{R}$ is a given matrix with the block diagonal structure

$$\bar{\mathcal{R}} = \begin{bmatrix} \bar{\mathcal{R}}_1 & \mathbf{0} \\ \mathbf{0} & \bar{\mathcal{R}}_2 \end{bmatrix}, \tag{35}$$

one can show that convexity is preserved, still leading to an LMI design problem. In this case, which is possibly the most meaningful for modeling implementation uncertainty, the following corollaries to Theorems 1 and 2 apply.

COROLLARY 3. *Let $\bar{\mathcal{R}}$ be partitioned as in* (35)*. If there exist a positive scalar λ, matrix $\mathcal{K}_0$, partitioned as in* (8)*, and symmetric matrices Y, Z, W, E, H such that the LMI* (27)–(29) *with the additional linear constraints*

$$E = \lambda \bar{\mathcal{R}}_1, \quad G = \mathbf{0}, \tag{36}$$

have a feasible solution, then the nominal filter $\mathcal{F}_0$ given in (30) *with*

$$V = \lambda^{-1/2} H^{1/2} \bar{\mathcal{R}}_2^{-1/2}$$

is such that $\|H_{we}(z; \mathcal{F}_0 + \Delta_{\mathcal{F}})\|_2^2 \leq \rho_2(\mathcal{F}_0) := \mu$ for all $\Delta_{\mathcal{F}} \in \mathbb{F}_{\bar{\mathcal{R}}}(\mathcal{F}_0)$.

COROLLARY 4. *Let $\bar{\mathcal{R}}$ be partitioned as in* (35)*. If there exist a positive scalar λ, matrix $\mathcal{K}_0$, partitioned as in* (8)*, and symmetric matrices Y, Z, E, H such that the LMI* (33) *with the additional linear constraints* (36) *has a feasible solution, then the nominal filter $\mathcal{F}_0$ given in* (30) *with $V = \lambda^{-1/2} H^{1/2} \bar{\mathcal{R}}_2^{-1/2}$ is such that $\|H_{we}(z; \mathcal{F}_0 + \Delta_{\mathcal{F}})\|_\infty \leq \rho_\infty(\mathcal{F}_0) := \mu$ for all $\Delta_{\mathcal{F}} \in \mathbb{F}_{\bar{\mathcal{R}}}(\mathcal{F}_0)$.*

Proof. Corollaries 3 and 4 can be proved in the same way. If $\mathcal{R} = \lambda \bar{\mathcal{R}}$, given in (35), then from (32) and (36) we have that

$$V^{-1} H V^{-T} = \lambda \bar{\mathcal{R}}_2,$$

which is satisfied by the choice of $V = \lambda^{-1/2} H^{1/2} \bar{\mathcal{R}}_2^{-1/2}$. Also notice that $\mathbb{F}_{\bar{\mathcal{R}}}(\mathcal{F}_0) = \mathbb{F}_{\lambda \bar{\mathcal{R}}}(\mathcal{F}_0)$. □

In Corollaries 3 and 4, the matrix V (and, consequently, matrix U) is automatically chosen by the optimization problem and cannot be picked by the designer, as in Theorems 1 and 2. This implies that the state space realization of the optimal filter is obtained as a result of the optimization procedure. In this sense, Theorems 1 and 2 simultaneously design the optimal filter transfer function *and* its realization. This result is in accordance with the well-known fact that some realizations of the same filter transfer function can be better than others for implementation [1, 2].

Also notice that, as in [17, 15], all of the above results can be shown to reduce to the standard Kalman filter and to the central H_∞ filter when $\gamma = 0$. In fact, with

$\gamma = 0$ the scaling matrices E, G, and H can be set arbitrarily close to zero, reducing these inequalities to the ones given in [17].

An interesting comment on the technical device used to prove Theorems 1 and 2 is that, to the authors' knowledge, it is the first time that a filtering or control robustness property has been derived directly from the transformed inequalities given in Lemmas 1 and 2. The robustness analysis was performed with respect to the transformed set of filter parameters $\mathcal{K}$ instead of the actual filter parameters $\mathcal{F}$. Working with the transformed parameters $\mathcal{K}$ instead of $\mathcal{F}$ was the key that permitted us to both incorporate and keep the scaling matrix $\mathcal{R}$ as an extra variable in the obtained design inequalities.

5. Extension to plant parameter uncertainty. In this section the assumption that the plant parameters are exactly known is relaxed. Following [17], the plant parameters, collected in the matrix

$$\mathcal{M} := \begin{bmatrix} A & B \\ C_z & D_z \\ C_y & D_y \end{bmatrix}, \tag{37}$$

are allowed to be unknown but to belong to the convex hull of N given extreme matrices (see [22]). That is,

$$\mathcal{M} \in \mathbb{M} := \mathrm{co}\left\{ \mathcal{M}_i := \begin{bmatrix} A_i & (B)_i \\ (C_z)_i & (D_z)_i \\ (C_y)_i & (D_y)_i \end{bmatrix}, \quad i = 1, \dots, N \right\}. \tag{38}$$

The goal is to derive design procedures that enable one to take into account the filter parameter uncertainty as well as the plant parameter uncertainty. This can be done by defining guaranteed cost functions that satisfy the general inequality

$$\|H_{we}(z; \mathcal{F}_0 + \Delta_{\mathcal{F}}, \mathcal{M})\|_p \le \rho_p(\mathcal{F}_0) \qquad \forall\ \Delta_{\mathcal{F}} \in \mathbb{F}_{\mathcal{R}}(\mathcal{F}_0) \quad \forall\ \mathcal{M} \in \mathbb{M}. \tag{39}$$

In the case of plant parametric uncertainty, the uncertain transfer function $H_{we}(z; \mathcal{F}_0 + \Delta_{\mathcal{F}}, \mathcal{M})$ depends on both the filter perturbation $\Delta_{\mathcal{F}}$ and the uncertain plant parameters $\mathcal{M}$. The guaranteed cost ρ_p provides an upper bound to the H_p norm of $H_{we}(z; \mathcal{F}_0 + \Delta_{\mathcal{F}}, \mathcal{M})$, which holds for all $\Delta_{\mathcal{F}} \in \mathbb{F}_{\mathcal{R}}(\mathcal{F}_0)$ *and* all $\mathcal{M} \in \mathbb{M}$. Following [17, 22], a guaranteed cost function ρ_2 can be built by generating N copies of the LMI (27)–(29) whose plant parameters correspond to those of $\mathcal{M}_i$, $i = 1, \dots, N$. The same procedure can be applied to generate ρ_∞ from appropriate versions of the inequalities given in Theorem 2.

The rationale behind this procedure is that the LMI (27)–(29) and (33) are all affine on the parameters of the uncertain matrix $\mathcal{M}$. Therefore, a convex combination of feasible inequalities (27)–(29) and (33) can be used to generate appropriate feasible inequalities for each $M \in \mathbb{M}$ (see [17, 22]). It is also straightforward to generate robust filtering conditions, which use a parameter dependent Lyapunov function to test stability following the methods of [18, 19]. The derivation of these extensions and the corresponding LMI conditions are left to the interested reader.

6. Numerical example. Consider the system in the form (1)–(3) with matrices

$$\left[\begin{array}{c|c} A & B \\ \hline C_z & D_z \\ \hline C_y & D_y \end{array}\right] = \left[\begin{array}{cc|ccc} 0.8 & 0.9 & 1 & 0 & 0 \\ 0.3 & -0.5 & 0 & 1 & 0 \\ \hline 1 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 \end{array}\right].$$

TABLE 1
H_2 filter transfer functions.

γ	0.01	0.05	0.1
Design I	$\frac{0.63z(z+0.91)}{(z-0.33)(z+0.53)}$	$\frac{0.86z(z+0.74)}{(z-0.09)(z+0.60)}$	$\frac{0.62z(z+0.70)}{(z-0.02)(z+0.64)}$
Design II	$\frac{0.73z(z+0.85)}{(z-0.25)(z+0.55)}$	$\frac{1.01z(z+0.68)}{z(z+0.68)}$	$\frac{0.65z(z+0.68)}{z(z+0.68)}$

TABLE 2
H_2 filtering performance.

	Nominal cost			Guaranteed cost			Round-off gain		
γ	0.01	0.05	0.1	0.01	0.05	0.1	0.01	0.05	0.1
Kalman	1.36	1.36	1.36		—		13.74	13.74	13.74
Design I	1.36	1.75	3.26	1.56	3.61	5.63	10.88	1.14	0.07
Design II	1.39	1.94	3.29	1.68	3.67	5.65	7.00	0.00	0.00

In the next sections we will design filters $\mathcal{F}_0$ to minimize an upper bound to the H_p norm, $p = \{2, \infty\}$, of $H_{we}(z; \mathcal{F}_0 + \Delta_{\mathcal{F}})$, where $\Delta_{\mathcal{F}} \in \mathbb{F}_{\mathcal{R}}(\mathcal{F}_0)$ for the values of $\gamma = \{0.01, 0.05, 0.1\}$.

6.1. *H_2* filtering. A standard stationary Kalman filter has been designed to serve as a template for the H_2 filtering design. The transfer function $\mathcal{F}_{\mathrm{K}}(z)$ of the Kalman filter is given by

$$\mathcal{F}_{\mathrm{K}}(z) = \frac{0.58z(z+0.96)}{(z-0.38)(z+0.52)}.$$

The following two H_2 filter design methods have been tried:

Design I: Minimize μ subject to the LMI (27)–(29) with a full variable scaling $\mathcal{R}$ (Theorem 1).

Design II: Minimize μ subject to the LMI (27)–(29) and the linear constraint (36) with a fixed scaling $\bar{\mathcal{R}} = \mathbf{I}$ (Corollary 3).

The transfer functions of Designs I and II are given in Table 1. These filters are associated with the performance measures given in Table 2. In this table the "Nominal cost" is the H_2 norm of $H_{we}(z; \mathcal{F}_0)$, and the "Guaranteed cost" is the square root of the value of μ obtained by solving the problems in Theorem 1 and Corollary 3. The "Round-off gain" shown in the third column of Table 2 is a measure that has not been directly optimized by solving the design problems of this paper. It was computed after determining the minimal round-off gain realizations for the designed filters according to [2, 5].

It is important to notice that the solution of the problems in Theorem 1 and Corollary 3 implies the simultaneous design of a filter realization. Moreover, if one is to use these results to compare performance with a given filter realization $\mathcal{F}$, it is necessary to impose an additional constraint relating $\mathcal{F}$ and $\mathcal{K}$. As noted before, such a relationship is nonlinear and destroys the convexity of the problem. For these reasons, and to be able to compare our results with other techniques, we have arbitrarily chosen $V = \mathbf{I}$, in which case the relationship between $\mathcal{F}$ and $\mathcal{K}$ becomes linear. This is, in a certain sense, equivalent to fixing the admissible filter realizations. Additionally, it also implies $\mathbb{F}_{\mathcal{R}}(\mathcal{F}_0) \equiv \mathbb{F}_{\mathcal{S}\mathcal{R}\mathcal{S}^T}(\mathcal{K}_0)$, which seems to be an appropriate choice for comparing a given realization to one obtained by the methods proposed in this paper.

Table 3
H_∞ filter transfer functions.

γ	0.01	0.05	0.1
Design III	$\frac{0.72(z+1.09)}{z+0.24}$	$\frac{1.16(z+0.57)}{z+0.51}$	$\frac{1.20(z+0.13)}{z+0.13}$
Design IV	$\frac{0.85(z+0.93)}{z+0.36}$	$\frac{1.11(z+0.73)}{z+0.60}$	$\frac{1.17(z+0.70)}{z+0.65}$

Using this idea, we have computed guaranteed cost for the standard Kalman filter. However, no results are shown in the table since the LMI in Theorem 1 and Corollary 3 become infeasible for $\gamma = 3.3 \times 10^{-4}$ and $\gamma = 1.2 \times 10^{-4}$, respectively. This is evidence of the importance of allowing the optimization to freely tune the filter realization.

Also notice that the optimal round-off gain realizations of the filters produced by Designs I and II have lower round-off gains than the optimal coordinates of the Kalman filter, although this measure of performance has not been directly optimized. This asserts the effectiveness of the uncertainty domain $\mathbb{F}_{\mathcal{R}}(\mathcal{F}_0)$ in producing *non-fragile* filters.

It is interesting to try to interpret the effect of the parameter perturbation on the performance measures and in the filter transfer functions. From Table 1, one can notice that, as the parameter uncertainty increases, the filter transfer function tends to a constant, with no dynamics. This trend can help explain why the round-off gains have decreased accordingly in Table 2. This effect is even accentuated in Design II, where the filter optimization has fewer parameters with which to play. It seems interesting that, to maximize the performance in the presence of an increasing implementation uncertainty, the designed filter has been made simpler by the design procedure.

6.2. H_∞ filtering. This time we have designed a standard H_∞ filter to serve as a template for the H_∞ filtering design. The transfer function $\mathcal{F}_{\mathrm{H}}(z)$ of the standard H_∞ filter is given by

$$\mathcal{F}_{\mathrm{H}}(z) = \frac{0.76(z+1.02)}{(z+0.29)}.$$

It is interesting to notice that the standard optimal H_∞ filter design already presents a pole-zero cancellation. In fact, this feature will be present in all designed filters. As before, two H_∞ filter design methods have been tried as follows:

Design III: Minimize μ subject to the LMI (33) with a full variable scaling $\mathcal{R}$ (Theorem 2).

Design IV: Minimize μ subject to the LMI (33) and the linear constraint (36) with a fixed scaling $\bar{\mathcal{R}} = \mathbf{I}$ (Corollary 4).

The transfer functions of Designs III and IV are given in Table 3 and their performance measures in Table 4. The guaranteed costs for the standard H_∞ filter have been computed by setting $V = \mathbf{I}$ and solving the design LMI for the given realization. The costs in the first line corresponds to the case when the scaling $\mathcal{R}$ has been optimized, whereas the costs in the second line have been obtained with $\bar{\mathcal{R}} = \mathbf{I}$.

The same trends observed in the H_2 filter design appear in the H_∞ design. Notice especially the tendency to simplify the filter by reducing it to a constant scaling. Interestingly enough, this tendency now appears more accentuated in Design III,

TABLE 4
H_∞ filtering performance.

	Nominal cost			Guaranteed cost			Round-off gain		
γ	0.01	0.05	0.1	0.01	0.05	0.1	0.01	0.05	0.1
H_∞	1.44	1.44	1.44	1.96 13.67	5.19 178.8	11.64 593.5	6.98	6.98	6.98
Design III	1.56	3.61	5.63	1.66	3.93	7.28	10.88	1.14	0.07
Design IV	1.68	3.67	5.65	1.75	4.89	7.82	7.00	0.00	0.00

TABLE 5
Actual H_2 filtering performance for the example in section 6.1.

	$\bar{\sigma}$		
γ	0.01	0.05	0.1
Kalman	1.36	1.55	2.13
Design II	1.42	2.05	3.59

where the scaling $\mathcal{R}$ has been allowed to be optimized. Notice again a significant reduction in the round-off gain.

6.3. Estimating conservativeness. In the previous section, the performances of the designed filters have been evaluated with respect to guaranteed cost functions, which are upper bounds to the norm of the filtering error system. In this section we attempt to access the filter performance by directly evaluating an estimate of the actual error system norms. The idea is to estimate the conservativeness of the method and to evaluate its practical usefulness. We restrict our attention to the case of H_2 filtering design with $\mathcal{R} = \mathbf{I}$.

For each filter design $\mathcal{F}_0$, we randomly generate a number of perturbation matrices $\Delta_{\mathcal{F}_j}$, $j = 1, \ldots, M$, in $\mathbb{F}_\mathcal{R}(\mathcal{F}_0)$. The following procedure was used in this generation:

1. Generate a square matrix Δ_j, with the same number of rows as in $\mathcal{F}_0$, where all entries are normally distributed random real numbers with zero mean and unitary variance.
2. Compute $\Delta_{\mathcal{F}j} = \frac{\gamma}{\|\Delta_j\|}\Delta_j \mathcal{F}_0$ and $\mathcal{F}_j = \mathcal{F}_0 + \Delta_{\mathcal{F}j}$.
3. If $\mathcal{F}_j$ is asymptotically stable, set $\sigma_j = \|H_{we}(z; \mathcal{F}_j)\|_2$; otherwise set $\sigma_j = \infty$.

All filter perturbations generated by the above procedure are guaranteed to be in the boundary of the set $\mathbb{F}_\mathcal{R}(\mathcal{F}_0)$, and an estimate of the error system norm can be computed as

$$\bar{\sigma} := \max_{j=1,\ldots,M} \sigma_j \approx \sup_{\Delta_\mathcal{F} \in \mathbb{F}_\mathcal{R}(\mathcal{F}_0)} \|H_{we}(z; \mathcal{F}_0 + \Delta_\mathcal{F})\|_2.$$

Strictly speaking, $\bar{\sigma}$ provides a lower bound to the error system norm, which serves as a good approximation for the worst case norm as M becomes large. In our experiments we have set $M = 1000$.

We start by evaluating the problem described in section 6.1. The results of the above numerical experiment applied to the filters previously labelled Kalman and Design II are shown in Table 5 for several values of γ. Note that the performance of Design II is, as expected, always below the designed guaranteed cost but, surprisingly, above the performance of the nominal Kalman filter design. We credit this apparently surprising behavior to a relative insensitivity of this particular example to variations on the filter parameters, rather than to an overconservativeness of our approach. We try to support this claim in the following paragraphs.

TABLE 6
Actual H_2 filtering performance for the second example in section 6.3.

	$\bar{\sigma}$		
γ	0.01	0.05	0.1
Kalman	1.30	∞	∞
Corollary 3	1.28	1.35	1.43

The previous example might leave the impression that the proposed design procedure produces robust filters at the expense of sacrificing performance; this impression is possibly due to the implicit conservativeness in the design inequalities. In order to show that the proposed procedure can indeed lead to efficient robust designs, we consider another simple example with

$$\left[\begin{array}{c|c} A & B \\ \hline C_z & D_z \\ \hline C_y & D_y \end{array}\right] = \left[\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -0.5 & 0.5 & 0.1 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ \hline -0.5 & 0.25 & 0.5 & 0 & 1 \end{array}\right].$$

For the above example, the Kalman filter has a nominal performance of $\|H_{we}(z;\mathcal{F}_0)\|_2 = 1.28$. The results of the above numerical experiment applied to the Kalman filter for several values of γ are shown in the first row of Table 6. These results show that the Kalman filter is extremely sensitive to parameter variations: a relative perturbation of size $\gamma = 0.01$ already implies some loss of performance but, more important, for higher values of γ, the Kalman filter becomes unstable (indicated as an infinite cost). This highly sensitive system seems to provide a better benchmark for our design methodology. After computing the robust filters using Corollary 3, we run the numerical experiment and obtain the performance estimate shown in the second row of Table 6. In this example, the design procedure not only produced a filter, which performs as efficiently as the nominal Kalman filter for $\gamma = 0.01$, but also produced robust filters for $\gamma = 0.05$ and $\gamma = 0.1$, which were able to withstand large parameter perturbations without becoming unstable and without sacrificing too much performance.

In the above example, an aspect that might have contributed to the sensitivity of the Kalman filter to parameter variations is the increased order of the filter. Generally speaking, it seems natural to expect that state space realizations of filters become more sensitive to parameter variations as the order of the filter (and the associated matrix dimensions) increases. In order to verify this trend we modify the system used in section 6.1 to augment its order. More specifically, we introduce a delay on the measurement signal $y(k)$. This produces the third order system

$$\left[\begin{array}{c|c} A & B \\ \hline C_z & D_z \\ \hline C_y & D_y \end{array}\right] = \left[\begin{array}{ccc|ccc} 0.8 & 0.9 & 0 & 1 & 0 & 0 \\ 0.3 & -0.5 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right].$$

Due to the introduction of the delay on the measurement, the Kalman filter now has a nominal performance of $\|H_{we}(z;\mathcal{F}_0)\|_2 = 1.72$.

Table 7 shows the results of the numerical experiment performed on both the nominal Kalman filter design and the filters produced by Corollary 3. Note that now

TABLE 7
Actual H_2 filtering performance for the example in section 6.1 *with a measurement delay.*

	$\bar{\sigma}$		
γ	0.01	0.05	0.1
Kalman	1.74	2.16	3.70
Corollary 3	1.75	2.15	3.62

the performances of the Kalman filter and the robust filter designed for $\gamma = 0.01$ are practically the same, whereas Corollary 3 produces filters that perform better than the Kalman filter for values of γ greater than 0.05. These results agree with the statement that we should expect the Kalman filter to become more sensitive to parameter variations as the order of the filter increases, in which case the procedure we have proposed provides an effective way to design robust filters.

Finally, note that the main source of conservatism in this design comes from the fact that the guaranteed cost functions we have used evaluate performance with respect to parameter perturbations $\Delta_{\mathcal{F}} \in \mathbb{F}_{\mathcal{R}}(\mathcal{F}_0)$, which are allowed to vary with time. Indeed, this explains the gap between the (time-varying) guaranteed cost values in Table 2 and the (time-invariant) values of $\bar{\sigma}$ in Table 5.

7. Conclusions. A new procedure has been proposed for designing filters which are robust in the presence of perturbations on the filter parameters. The filters are obtained by minimizing guaranteed H_2 and H_∞ cost functions developed by confining the filter parametric uncertainty in a region defined by a quadratic inequality. The size of this uncertainty region depends on the size of the filter parameters, and the maximum allowed parametric perturbation is specified as a percentage of the size of the filter gains. Both the transfer function and the realization of the robust filter are simultaneously designed. The optimization problems to be solved have constraints specified in terms of LMI, whose global optimal solutions can be determined using convex programming. The numerical examples suggest that the proposed technique may produce filters with reduced round-off noise gain, although this performance measure is not directly optimized in the design process.

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