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Canonical symmetrization for the unitary bases. II. Boson and fermion bases*

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The canonical Weyl basis described in Paper I is generalized to give a boson and fermion calculus which generates the symmetric and antisymmetric bases of $U(nm)$ respectively contained in the irreducible bases of $U(n)\times U(m)$. The boson calculus may be used to find the multiplicity free Clebsch–Gordan coefficients of $U(n)$.

I. INTRODUCTION

In works by Biedenharn, Baird, Moshinsky, Louck, and Seligman,[1–5] extensive use has been made of boson operators to generate an irreducible basis of $U(n)\times U(m)$ which is simultaneously a symmetric basis of $U(nm)$. Louck has shown that this boson basis is a basis for the n-dimensional, m-particle harmonic oscillator.[6] However, symmetrization of this boson basis using the "boson calculus" has so far failed to generate all the independent bases in this space which we shall denote as $U(n)*U(m)$. The construction of a complete basis is presently a tedious task using lowering operator techniques.[7–10] Furthermore, the boson calculus itself has no justification other than a constructional validity.

In this work, we shall show the simple relationship between the boson basis and the canonical Weyl basis described in an earlier work[11] (denoted hereafter as I). This relationship leads to a new boson calculus capable of generating all the independent bases of $U(n)*U(m)$. Furthermore, we show that the canonical Weyl basis may be considered as the "special" boson basis of subspace $U(n)*S_p$ or $S_p*U(m)$ of $U(n)*U(m)$ first noted by Moshinsky.[12] This clarifies the fact that the generators of the unitary group can be used as elements of the permutation group S_p when acting on this particular basis. We also show that the canonical Weyl bases of the subspace S_p*S_p form a basis for the regular representation of S_p. As a result of this new boson calculus, it becomes a straightforward task to determine the matrix elements of the irreducible representations (IR's) of $U(n)$, and by means of the factorization lemma,[13] to calculate the multiplicity-free Clebsch–Gordan coefficients of $U(n)$.

In a similar manner, we develop a fermion calculus to generate an irreducible basis of $U(n)\times U(m)$, which is simultaneously an antisymmetric basis of $U(nm)$. We shall denote the subspace of all such irreducible bases of $U(n)\times U(m)$ as $U(n)\tilde{*}U(m)$. The fermion calculus enables us to find the antisymmetric bases of $U(nm)$ contained in the irreducible bases of $U(n)\times U(m)$. This is a very important task when dealing with fermions in atomic and nuclear physics. For example, in atomic l-shells one often needs to find the antisymmetric content of $U(4l+2)$ in symmetrized orbit–spin states of $U(2l+1)\times U(2)$. Similarly, in nuclear shells one often needs to find the antisymmetric content of $U(8l+4)$ in states of $U(2l+1)\times U(4)$ since the spin states now include isotopic spin. In the atomic case, closed form expressions already exist for the antisymmetric content in symmetrized orbit–spin states which were derived using a canonical Weyl basis.[14]

II. IRREDUCIBLE BASES FOR $U(nm)$ AND $U(n)\times U(m)$

Let $|\phi_i^l\rangle$ for $i=1,2,\ldots,n$ and $|\psi_l^j\rangle$ for $j=1,2,\ldots,m$ form bases for the lth particle of the fundamental representation of $U(n)$ and $U(m)$ respectively with generator relations:

$$e^l_{im}|\phi^k_p\rangle=\delta_{lk}\delta_{mp}|\phi^l_i\rangle,\qquad e^{jn}_l|\psi^q_k\rangle=\delta_{lk}\delta_{nq}|\psi^j_l\rangle. \tag{1}$$

The pth direct products

$$|\phi_{(i)}\rangle\equiv|\phi^1_{i_1}\phi^2_{i_2}\cdots\phi^p_{i_p}\rangle,\qquad |\psi^{(j)}\rangle\equiv|\psi^{j_1}_1\psi^{j_2}_2\cdots\psi^{j_p}_p\rangle \tag{2}$$

form a reducible bases of $U(n)$ and $U(m)$ respectively with the generators:

$$E_{im}=\sum_{l=1}^p e^l_{im},\qquad E^{jn}=\sum_{l=1}^p e^{jn}_l. \tag{3}$$

As we have seen in I, we may reduce direct products (2) using the canonical projection operators of S_p as in (4) to form a canonical Weyl basis for $U(n)$ and $U(m)$ respectively:

$$\left|\begin{matrix} & [u] & \\ \langle s\rangle & & (m)\end{matrix}\right\rangle=N^{[u]}_s\underline{P}^{[u]}_{ms}|\phi_{(i)}\rangle, \tag{4a}$$

$$\left|\begin{matrix} & [v] & \\ (n) & & \langle r\rangle\end{matrix}\right\rangle=N^{[v]}_r\bar{P}^{[v]}_{nr}|\psi^{(j)}\rangle. \tag{4b}$$

The upper bar ($^-$) denotes permutations of the superscripts and the lower bar (_) denotes permutations of the subscripts. The reason for the change in notation for the Weyl basis on the left of (4b) will become evident later.

We define the direct product basis $|\Phi^j_i(l)\rangle$ in Eq. (5):

$$|\phi^l_i\rangle\times|\psi^j_l\rangle=|\Phi^j_i(l)\rangle \tag{5}$$

Then the $|\Phi^j_i(l)\rangle$ for $i=1,2,\ldots,n$ and $j=1,2,\ldots,m$ form bases for the lth particle of the fundamental representation of $U(nm)$ with generator relations:

$$e_{im}^{jn}(l)\,|\Phi_p^q(k)\rangle = \delta_{lk}\delta_{mp}\delta_{nq}\,|\Phi_i^j(l)\rangle, \qquad (6)$$

where $e_m^{jn}(l) = e_{im}^l \times e_l^{jn}$.

The direct product basis

$$|\phi_{\langle i\rangle}\rangle \times |\psi^{\langle j\rangle}\rangle = |\Phi_{i_1}^{j_1}(1)\Phi_{i_2}^{j_2}(2)\cdots\Phi_{i_p}^{j_p}(p)\rangle \equiv |\Phi_{\langle i\rangle}^{\langle j\rangle}\rangle. \qquad (7)$$

is a reducible basis of $U(n)\times U(m)$ with generators $E_{im}\times E^{jn}$. We may reduce this basis using (4) and find the irreducible bases $|{}^{[u]}_{\langle s\rangle}\rangle \times |{}^{[v]}_{\langle r\rangle}\rangle$ of $U(n)\times U(m)$ are simply

$$\left|\begin{matrix}[u] \\ \langle s\rangle \quad (m)\end{matrix}\right\rangle \times \left|\begin{matrix}[v] \\ (n) \quad \langle r\rangle\end{matrix}\right\rangle = N_r^{[v]} N_s^{[u]} \bar{P}_{nr}^{[v]} \underline{P}_{ms}^{[u]}\,|\Phi_{\langle i\rangle}^{\langle j\rangle}\rangle. \qquad (8)$$

The pth direct product basis $|\Phi_{\langle i\rangle}^{\langle j\rangle}\rangle$ also forms a reducible bases of $U(nm)$ with generators:

$$E_{im}^{jn} = \sum_{l=1}^{p} e_{im}^{jn}(l). \qquad (9)$$

Again, we may reduce the bases of $U(nm)$ using the canonical projection operators of S_p to form a canonical irreducible Weyl basis $|{}^{[\lambda]}_{\langle t\rangle\,(0)}\rangle$ as in (10):

$$\left|\begin{matrix}[\lambda] \\ \langle t\rangle \quad (o)\end{matrix}\right\rangle = N_t^{[\lambda]} P_{ot}^{[\lambda]}\,|\Phi_{\langle i\rangle}^{\langle j\rangle}\rangle. \qquad (10)$$

III. BOSON BASES

A. Boson calculus

From (10) we find the symmetric states of $U(nm)$ are

$$|[p0\cdots 0]\rangle = \frac{1}{\sqrt{p!}} \sum_{q\in S_p} [q]\,|\Phi_{\langle i\rangle}^{\langle j\rangle}\rangle. \qquad (11)$$

We may construct linear combinations of these symmetric states of $U(nm)$ from the irreducible bases of $U(n)\times U(m)$ in (8) using the Clebsch–Gordan coefficients of the canonical bases of S_p since the bases in (8) are also irreducible bases $|{}^{[u]}_{(m)}\rangle \times |{}^{[v]}_{(n)}\rangle$ of $S_p\times S_p$. We find that

$$\left|\begin{matrix}[u] \\ \langle s\rangle * \langle r\rangle\end{matrix}\right\rangle = \frac{1}{(l[u])^{1/2}} \sum_n \left|\begin{matrix}[u] \\ \langle s\rangle \quad (n)\end{matrix}\right\rangle \times \left|\begin{matrix}[u] \\ (n) \quad \langle r\rangle\end{matrix}\right\rangle,$$
$$= \frac{N_r^{[u]} N_s^{[u]}}{(l[u])^{1/2}} \sum_n \bar{P}_{nr}^{[u]} \underline{P}_{ns}^{[u]}\,|\Phi_{\langle i\rangle}^{\langle j\rangle}\rangle \qquad (12)$$

is symmetric under all permutations $[q]$ of S_p as can easily be verified directly:

$$[q]\sum_n \bar{P}_{nr}^{[u]} \underline{P}_{ns}^{[u]} = \sum_n [\bar{q}]\bar{P}_{nr}^{[u]}[\underline{q}]\underline{P}_{ns}^{[u]}$$
$$= \sum_{n,t,t'} D_{tn}^{[u]}[q] D_{t'n}^{[u]}[q] \bar{P}_{tr}^{[u]} \underline{P}_{t's}^{[u]}$$
$$= \sum_t \bar{P}_{tr}^{[u]} \underline{P}_{ts}^{[u]}.$$

Hence, $|{}^{[u]}_{\langle s\rangle*\langle r\rangle}\rangle$ must be some linear combination of symmetric bases (11) of $U(nm)$. We shall denote the subspace of all bases $|{}^{[u]}_{\langle s\rangle*\langle r\rangle}\rangle$ of the direct product space $|\Phi_{\langle i\rangle}^{\langle j\rangle}\rangle$ as $U(n)*U(m)$. We wish to find the symmetric content of $U(nm)$ in our bases $|{}^{[u]}_{\langle s\rangle*\langle r\rangle}\rangle$ of $U(n)*U(m)$. This is equivalent to determining the symmetric canonical irreducible bases of $U(nm)$ contained in the irreducible basis $|{}^{[u]}_{\langle s\rangle}\rangle \times |{}^{[u]}_{\langle r\rangle}\rangle$ of $U(n)\times U(m)$.

We shall accomplish this task by using boson operators to generate our bases. We may write our symmetric basis in terms of boson operators as in (13) below:

$$\sum_{q\in S_p} [q]\,|\Phi_{\langle i\rangle}^{\langle j\rangle}\rangle = \frac{1}{\sqrt{p!}} \sum_{q\in S_p} [q]\, a_{\langle i\rangle}^{\langle j\rangle}\,|0\rangle, \qquad (13)$$

where $a_{\langle i\rangle}^{\langle j\rangle}|0\rangle = a_{i_1}^{j_1} a_{i_2}^{j_2}\cdots a_{i_p}^{j_p}|0\rangle$, and $a_i^{j\dagger} = \bar{a}_i^j$. The boson operators obey the following commutation relations:

$$[a_i^j, a_{i'}^{j'}] = [\bar{a}_i^j, \bar{a}_{i'}^{j'}] = 0, \qquad (14a)$$

$$[\bar{a}_i^j, a_{i'}^{j'}] = \delta_{jj'}\delta_{ii'}. \qquad (14b)$$

We may now expand the generators of $U(n)$, $U(m)$, and $U(nm)$ in terms of the boson operators as follows[15]:

$$E_{im} = \sum_{l=1}^{p} a_i^l \bar{a}_m^l, \qquad (15a)$$

$$E^{jn} = \sum_{l=1}^{p} a_l^j \bar{a}_l^n, \qquad (15b)$$

$$E_{im}^{jn} = a_i^j \bar{a}_m^n. \qquad (15c)$$

From (13) we see that $|{}^{[u]}_{\langle s\rangle*\langle r\rangle}\rangle$ may be put in terms of boson operators as in (16):

$$\left|\begin{matrix}[u] \\ \langle s\rangle * \langle r\rangle\end{matrix}\right\rangle = \frac{N_r^{[u]} N_s^{[u]}}{(l^{[u]}p!)^{1/2}} \sum_n \bar{P}_{nr}^{[u]} \underline{P}_{ns}^{[u]} a_{\langle i\rangle}^{\langle j\rangle}\,|0\rangle. \qquad (16)$$

It follows from (14) that

$$[\bar{q}]\,a_{\langle i\rangle}^{\langle j\rangle}|0\rangle = [\underline{q}^{-1}]\,a_{\langle i\rangle}^{\langle j\rangle}|0\rangle \qquad (17)$$

and, hence,

$$\bar{P}_{nr}^{[u]} a_{\langle i\rangle}^{\langle j\rangle}|0\rangle = \underline{P}_{rn}^{[u]} a_{\langle i\rangle}^{\langle j\rangle}|0\rangle. \qquad (18)$$

Using (18), we may simplify the expression for $|{}^{[u]}_{\langle s\rangle*\langle r\rangle}\rangle$ in (16) to find

$$\left|\begin{matrix}[u] \\ \langle s\rangle * \langle r\rangle\end{matrix}\right\rangle = N_r^{[u]} N_s^{[u]} (l^{[u]}/p!)^{1/2} \underline{P}_{rs}^{[u]} a_{\langle i\rangle}^{\langle j\rangle}\,|0\rangle. \qquad (19)$$

If we let

$$M_{rs}^{[u]} = N_r^{[u]} N_s^{[u]} (l^{[u]}/p!)^{1/2}, \qquad (20)$$

then we have the following relation between our basis of $U(n)*U(m)$ and the boson operators:

$$\left|\begin{matrix}[u] \\ \langle s\rangle * \langle r\rangle\end{matrix}\right\rangle = M_{rs}^{[u]} \underline{P}_{rs}^{[u]} a_{\langle i\rangle}^{\langle j\rangle}\,|0\rangle \qquad (21a)$$

or, similarly,

$$\left|\begin{matrix}[u] \\ \langle r\rangle * \langle s\rangle\end{matrix}\right\rangle = M_{rs}^{[u]} \bar{P}_{rs}^{[u]} a_{\langle j\rangle}^{\langle i\rangle}\,|0\rangle. \qquad (21b)$$

Equations (21) illustrate the reciprocity between upper and lower projection operators for bases of $U(m)*U(n)$

Since $P_{rs}^{[u]} = C_{rs}^{[u]} O_{rs}^{[u]}$, where $C_{rs}^{[u]}$ is a positive constant, and since the seminormal canonical projection operators $O_{rs}^{[u]}$ may be easily generated, we now have a convenient and straightforward method of finding the symmetric content of $U(nm)$ in the irreducible bases of $U(n)\times U(m)$ which we call the boson calculus. As an example of the use of the boson calculus, we find the highest weight bases $|{}^{11}_{2}\rangle \times |{}^{11}_{2}\rangle$ of $U(3)\times U(3)$ in terms of boson operators. Using Eq. (21a) with seminormal projection operators, we have

$$\underline{O}^{[210]}_{\substack{12\\3}\,\substack{12\\3}} a_1^1 a_1^1 a_2^2\,|0\rangle = 2\underline{S}_{12}\underline{A}_{13}\underline{S}_{12} a_1^1 a_1^1 a_2^2\,|0\rangle,$$
$$= 8(a_1^1 a_1^1 a_2^2 - a_1^1 a_2^1 a_1^2)\,|0\rangle.$$

Normalizing, we find $|{}^{11}_{2}\rangle \times |{}^{11}_{2}\rangle$ in terms of the symmetric Weyl bases of $U(6)$ with three particles

$$\left|\begin{matrix}11\\2\end{matrix}\right\rangle\times\left|\begin{matrix}11\\2\end{matrix}\right\rangle=\left(\sqrt{2}\left|\begin{matrix}1&1&2\\1&1&2\end{matrix}\right\rangle-\left|\begin{matrix}1&1&2\\1&2&1\end{matrix}\right\rangle\right)\Big/\sqrt{3}. \tag{22}$$

Similarly,

$$\underset{\sim}{O}^{[210]}_{\substack{12\ 12\\3\ \ 3}}a_1^1a_2^2a_3^2|0\rangle=2\underline{S}_{12}\underline{A}_{13}\underline{S}_{12}a_1^1a_2^2a_3^2|0\rangle$$
$$=2(2a_1^1a_2^2a_3^2+2a_2^1a_1^2a_3^2-a_1^1a_3^2a_2^2-a_2^1a_3^2a_1^2$$
$$-a_3^1a_1^2a_2^2-a_3^1a_2^2a_1^2)|0\rangle, \tag{23a}$$

$$\underset{\sim}{O}^{[210]}_{\substack{12\ 13\\3\ \ 2}}a_1^1a_2^2a_3^2|0\rangle=4\underline{S}_{12}\underline{[23]}\underline{A}_{12}a_1^1a_2^2a_3^2|0\rangle$$
$$=4(a_1^1a_3^2a_2^2+a_3^1a_1^2a_2^2-a_2^1a_3^2a_1^2-a_3^1a_2^2a_1^2|0\rangle, \tag{23b}$$

$$\underset{\sim}{O}^{[100]}a_1^1a_2^2a_3^2|0\rangle=4\underline{S}_{123}a_1^1a_2^2a_3^2|0\rangle$$
$$=(a_1^1a_2^2a_3^2+a_1^1a_3^2a_2^2+a_3^1a_1^2a_2^2+a_3^1a_2^2a_1^2$$
$$+a_2^1a_3^2a_1^2+a_2^1a_1^2a_3^2)|0\rangle. \tag{23c}$$

Collecting terms and normalizing, we find

$$\begin{matrix}\left|\begin{matrix}12\\3\end{matrix}\right\rangle\times\left|\begin{matrix}12\\2\end{matrix}\right\rangle\\ \left|\begin{matrix}13\\2\end{matrix}\right\rangle\times\left|\begin{matrix}12\\2\end{matrix}\right\rangle\\ |123\rangle\times|122\rangle\end{matrix}=\begin{pmatrix}-2/\sqrt{6}&1/\sqrt{6}&1/\sqrt{6}\\0&-1/\sqrt{2}&1/\sqrt{2}\\1/\sqrt{3}&1/\sqrt{3}&1/\sqrt{3}\end{pmatrix}\begin{matrix}\left|\begin{matrix}2&2&1\\1&2&3\end{matrix}\right\rangle\\ \left|\begin{matrix}2&1&2\\1&2&3\end{matrix}\right\rangle\\ \left|\begin{matrix}1&2&2\\1&2&3\end{matrix}\right\rangle\end{matrix} \tag{24}$$

The bases generated by our boson calculus must be identical to those generated by Baird and Biedenharn[16] by antisymmetrizing the columns of the "boson tableau". However, column antisymmetrization can only be applied to derive certain bases, and merely represents a simplification of the permutational content of the canonical projection operators when acting on such bases. Column antisymmetrization can be used for all bases $|{}^{[u]}_{\langle s\rangle}\rangle\times|{}^{[u]}_{\langle r\rangle}\rangle$ where both $|{}^{[u]}_{\langle s\rangle}\rangle$ and $|{}^{[u]}_{\langle r\rangle}\rangle$ have nondegenerate weights. For example, using column symmetrization to generate the basis in Eq. (22), we have

$$\left|\begin{matrix}11\\2\end{matrix}\right\rangle\times\left|\begin{matrix}11\\2\end{matrix}\right\rangle=\frac{1}{\sqrt{3}}\begin{bmatrix}a_1^1&a_1^1\\a_2^2&\end{bmatrix}|0\rangle$$
$$=\frac{1}{\sqrt{3}}(a_1^1a_2^2-a_2^1a_1^2)|a_1^1|0\rangle$$
$$=\frac{1}{\sqrt{3}}(a_1^1a_1^1a_2^2-a_1^1a_2^1a_1^2)|0\rangle. \tag{25}$$

We have antisymmetrized with respect to the subscripts of the columns in the "boson tableau." Because of the reciprocity in Eqs. (21), we could have equally well antisymmetrized with respect to the superscripts.

In the case where one of the bases $|{}^{[u]}_{\langle s\rangle}\rangle$ or $|{}^{[u]}_{\langle r\rangle}\rangle$ has a semimaximum weight [a highest weight for $U(n-1)$ or $U(m-1)$ respectively] and the other has a nondegenerate weight, we may again use column antisymmetrization to derive a boson basis. Thus, corresponding to our previous result, we have

$$\left|\begin{matrix}13\\2\end{matrix}\right\rangle\times\left|\begin{matrix}12\\2\end{matrix}\right\rangle=\frac{1}{\sqrt{2}}\begin{bmatrix}a_1^1&a_3^2\\a_2^2&\end{bmatrix}|0\rangle$$
$$=\frac{1}{\sqrt{2}}(a_1^1a_2^2-a_2^1a_1^2)a_3^2|0\rangle$$
$$=\frac{1}{\sqrt{2}}(a_1^1a_2^2a_3^2-a_1^2a_2^1a_3^2)|0\rangle. \tag{26}$$

However, in most cases where $|{}^{[u]}_{\langle s\rangle}\rangle$ or $|{}^{[u]}_{\langle r\rangle}\rangle$ have degenerate weights, column antisymmetrization fails to generate an orthonormal basis, and lowering operator techniques must be employed. Thus, to find $|{}^{12}_{3}\rangle\times|{}^{12}_{2}\rangle$, we must lower the basis $|{}^{11}_{3}\rangle\times|{}^{12}_{2}\rangle$ as shown below:

$$\left|\begin{matrix}12\\3\end{matrix}\right\rangle\times\left|\begin{matrix}12\\2\end{matrix}\right\rangle=\frac{1}{\sqrt{2}}E_{21}\left|\begin{matrix}11\\3\end{matrix}\right\rangle\times\left|\begin{matrix}12\\2\end{matrix}\right\rangle,$$
$$=\frac{E_{21}}{\sqrt{6}}\begin{bmatrix}a_1^1&a_1^2\\a_3^2&\end{bmatrix}|0\rangle,$$
$$=\frac{1}{\sqrt{6}}(-2a_1^2a_2^2a_3^1+a_1^2a_2^1a_3^2+a_1^1a_2^2a_3^2)|0\rangle. \tag{27}$$

In general, this lowering technique is very tedious and our boson calculus represents a considerable simplification for deriving the boson bases.

B. Weyl bases

Let $m=p$ and $|{}_{\langle s\rangle*\langle r\rangle}^{[u]}\rangle$ be a basis of $U(n)*U(p)$, where $\langle r\rangle$ has a weight with maximum degeneracy; that is, let $\psi^{(j)}=\psi_1^1\psi_2^2\cdots\psi_p^p$ so that the standard tableau of $U(p)$, $T^{[u]}_{\langle r\rangle}$, is the same as the standard tableau of S_p, $T^{[u]}_{(r)}$. Then from Eq. (21a) we have

$$\left|\begin{matrix}[u]\\\langle s\rangle*\langle r\rangle\end{matrix}\right\rangle=M^{[u]}_{rs}\underline{P}^{[u]}_{rs}a^1_{i_1}a^2_{i_2}\cdots a^p_{i_p}|0\rangle. \tag{28}$$

Comparing this with Eq. (4a), we see there is a one-to-one correspondence between the boson bases $|{}_{\langle s\rangle*\langle r\rangle}^{[u]}\rangle$ and Weyl bases $|{}_{\langle s\rangle(r)}^{[u]}\rangle$. Since $|{}_{\langle s\rangle*\langle r\rangle}^{[u]}\rangle$ transforms like a Weyl basis under permutations $[\underline{q}]$ and generators E_{ij}, we have

$$\left|\begin{matrix}[u]\\\langle s\rangle*\langle r\rangle\end{matrix}\right\rangle=\left|\begin{matrix}[u]\\\langle s\rangle\ \ (r)\end{matrix}\right\rangle \tag{29a}$$

when $\quad T^{[u]}_{\langle r\rangle}=T^{[u]}_{(r)}$.

Also,

$$M^{[u]}_{rs}=N^{[u]}_{s} \tag{29b}$$

when $\quad T^{[u]}_{\langle r\rangle}=T^{[u]}_{(r)}$.

The commutivity of boson operators in Eq. (28) illustrates the fact that a reordering of the notation for single particle states leaves the Weyl basis unchanged. It is evident that the Weyl bases $|{}_{\langle s\rangle*\langle r\rangle}^{[u]}\rangle$ form a subspace $U(n)*S_p\subset U(n)*U(p)$.

Now let $n=p$ and $|{}_{\langle s\rangle*\langle r\rangle}^{[u]}\rangle$ be a basis of $U(p)*U(m)$, where $\langle s\rangle$ has a weight with maximum degeneracy; that is, let $\phi_{(i)}=\phi_1^1\phi_2^2\cdots\phi_p^p$ so that $T^{[u]}_{\langle s\rangle}=T^{[u]}_{(s)}$. Then from Eq. (21b) we have

$$\left|\begin{matrix}[u]\\(s)*\langle r\rangle\end{matrix}\right\rangle=M^{[u]}_{rs}\overline{P}^{[u]}_{sr}a_1^{j_1}a_2^{j_2}\cdots a_p^{j_p}|0\rangle. \tag{30}$$

Comparing this with Eq. (4b), we see there is a one-to-one corespondence between the boson bases $|{}_{(s)*\langle r\rangle}^{[u]}\rangle$ and the Weyl bases $|{}_{(s)\langle r\rangle}^{[u]}\rangle$. Since $|{}_{(s)*\langle r\rangle}^{[u]}\rangle$ transforms like a Weyl basis under permutations $[\overline{q}]$ and generators E^{ij}, we have

$$\left|\begin{matrix}[u]\\\langle s\rangle*\langle r\rangle\end{matrix}\right\rangle=\left|\begin{matrix}[u]\\(s)\ \ \langle r\rangle\end{matrix}\right\rangle \tag{31a}$$

when $T^{[u]}_{\langle s\rangle}=T^{[u]}_{(s)}$.

Also,

$$M^{[u]}_{rs} = N^{[u]}_r \tag{31b}$$

when $T^{[u]}_{\langle s\rangle} = T^{[u]}_{(s)}$.

It is evident that the Weyl bases $|{}^{[u]}_{(s)*\langle r\rangle}\rangle$ form a subspace $S_p * \mathrm{U}(m) \subset U(p) * \mathrm{U}(m)$. The reason for our choice of notation in (4b) is now clear.

Moshinsky[17] has shown that for the "special" Gel'fand bases $|{}_{\langle s\rangle}{}^{[u]}{}_{\langle r\rangle}\rangle$ of $\mathrm{U}(n) * S_p$

$$E^{nm}E^{mn} = (\bar{1}) + (\overline{nm}), \tag{32a}$$

and similarly for the "special" Gel'fand bases $|{}_{\langle r\rangle}{}^{[u]}{}_{\langle s\rangle}\rangle$ of $S_p * \mathrm{U}(m)$:

$$E_{nm}E_{mn} = (\underline{1}) + (\underline{nm}). \tag{32b}$$

Equation (32a) is easily verified since both generators E^{mn} of $\mathrm{U}(n)$ and *state* permutations $(\bar{q})$ of S_p commute with the *particle* operators $\underline{P}^{[u]}_{rs}$. Thus, we have

$$\begin{aligned}
&E^{nm}E^{mn}\underline{P}^{[u]}_{rs}|\phi_{(i)}\rangle \\
&= \underline{P}^{[u]}_{rs}E^{nm}E^{mn}|\phi^{1\,2\cdots n\cdots m\cdots p}_{i_1 i_2\cdots i_n\cdots i_m\cdots i_p}\rangle \\
&= \underline{P}^{[u]}_{rs}[|\phi^{1\,2\cdots n\cdots m\cdots p}_{i_1 i_2\cdots i_n\cdots i_m\cdots i_p}\rangle + |\phi^{1\,2\cdots n\cdots m\cdots p}_{i_1 i_2\cdots i_n\cdots i_m\cdots i_p}\rangle] \\
&= \underline{P}^{[u]}_{rs}[(\bar{1}) + (\overline{nm})]|\phi^{1\,2\cdots n\cdots m\cdots p}_{i_1 i_2\cdots i_n\cdots i_m\cdots i_p}\rangle \\
&= [(\bar{1}) + (\overline{nm})]\underline{P}^{[u]}_{rs}|\phi_{(i)}\rangle
\end{aligned}$$

Equation (32b) may be verified in a similar manner. Other more complicated expressions may also be derived for the r-cycles of S_p in terms of the generators of $\mathrm{U}(n)$ or $\mathrm{U}(m)$ when operating on these "special" Gel'fand bases. However, it is more important to note that the upper and lower Gel'fand invariant operators

$$\bar{I}^m_k = \sum^m_{i_1, i_2, \ldots, i_k} E^{i_1 i_2}E^{i_2 i_3}\cdots E^{i_k i_1}, \tag{33a}$$

$$\underline{I}^n_k = \sum^n_{i_1, i_2, \ldots, i_k} E_{i_1 i_2}E_{i_2 i_3}\cdots E_{i_k i_1} \tag{33b}$$

of $\mathrm{U}(m)$ and $\mathrm{U}(n)$ may be expanded in terms of the upper and lower *state* r-cycle class operators of S_p, $\bar{K}^{pm}_r$ for $r = 1, 2, \ldots, k$ and $\underline{K}^{pn}_r$ for $r = 1, 2, \ldots, k$ respectively as has been shown in I. The boson bases $|{}^{[u]}_{\langle s\rangle *\langle r\rangle}\rangle$ in (21a) are eigenvectors of these upper and lower class operators since these bases transform like irreducible bases $|{}^{[u]}_{\langle s\rangle}\rangle$ of S_p under lower permutations $(\underline{q})$, and like irreducible bases $|{}^{[u]}_{\langle r\rangle}\rangle$ of S_p under upper permutations $(\bar{q})$. It is for this reason that the projected bases $|{}^{[u]}_{\langle s\rangle *\langle r\rangle}\rangle$ forms a Gel'fand bases $|{}^{[u]}_{\langle s\rangle}\rangle \times |{}^{[u]}_{\langle r\rangle}\rangle$ of $\mathrm{U}(n) \times \mathrm{U}(m)$ for different standard tableaus $T^{[u]}_{\langle s\rangle} = T^{[u]}_{(s)}\phi_{(i)}$ of $\mathrm{U}(n)$ and different standard tableaus $T^{[u]}_{\langle r\rangle} = T^{[u]}_{(r)}\psi^{(j)}$ of $\mathrm{U}(m)$.

Finally, let $m = p$, $n = p$, and $T^{[u]}_{\langle r\rangle} = T^{[u]}_{(r)}$, $T^{[u]}_{\langle s\rangle} = T^{[u]}_{(s)}$. Then the boson basis

$$\left|\begin{matrix}[u]\\ (s) * (r)\end{matrix}\right\rangle = M^{[u]}_{rs}\underline{P}^{[u]}_{rs}a^1_1 a^2_2\cdots a^p_p|0\rangle$$

is a canonical irreducible basis of S_p under permutations $[\underline{q}]$ and $[\bar{q}]$. The bases $|{}^{[u]}_{(s)*(r)}\rangle$ form a subspace $S_p * S_p \subset \mathrm{U}(p) * \mathrm{U}(p)$ and are a bases for the regular representation of S_p.

From Eqs. (29b) and (20) we have the relation

$$(l^{[u]}/p!)^{1/2}N^{[u]}_r N^{[u]}_s = N^{[u]}_s$$

when $T^{[u]}_{\langle r\rangle} = T^{[u]}_{(r)}$. Therefore,

$$N^{[u]}_r = (p!/l^{[u]})^{1/2} = \sqrt{H([u])}. \tag{34}$$

when $T^{[u]}_{(r)} = T^{[u]}_{\langle r\rangle}$. This also follows directly from evaluating (3.25) of I.

C. Factorization lemma[18]

One of the most important aspects of the boson calculus is that we may use it to determine the matrix elements of the IR's of the unitary group. Then, by means of the factorization lemma, we can generate the multiplicity-free Clebsch–Gordan coefficients of the unitary group.

Let $D^{[1]}(U)$ be the fundamental or self-representation of $\mathrm{U}(n)$ given by

$$D^{[1]}(U) = \begin{pmatrix} u^1_1 & u^2_1 & \cdots & u^n_1 \\ u^1_2 & u^2_2 & \cdots & u^n_2 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ u^1_n & u^2_n & \cdots & u^n_n \end{pmatrix}, \tag{35}$$

and let $n = m$ so that $|{}^{[u]}_{\langle s\rangle *\langle r\rangle}\rangle$ is a basis of $\mathrm{U}(n) * \mathrm{U}(n)$. We multiply the boson bases $|{}^{[u]}_{\langle s\rangle *\langle r\rangle}\rangle$ by the constant $L([u])$ such that

$$L([u])\left|\begin{matrix}[u]\\ \langle s\rangle * \langle r\rangle\end{matrix}\right\rangle = L([u])M^{[u]}_{rs}\underline{P}^{[u]}_{rs}a^{(j)}_{(i)}|0\rangle$$

contains the term $a^{(j)}_{(i)}|0\rangle$ just once when (i) and (j) have highest weight in $\mathrm{U}(n)$. Then Louck has proven that[19]

$$D^{[u]}_{\langle s\rangle\langle r\rangle}(U) = L([u])M^{[u]}_{rs}\underline{P}^{[u]}_{rs}u^{(j)}_{(i)}, \tag{36}$$

where $u^{(j)}_{(i)} = u^{j_1}_{i_1}u^{j_2}_{i_2}\cdots u^{j_p}_{i_p}$. For example, from Eq. (25) we have that $L([210]) = \sqrt{3}$. From (24) it follows that

$$\begin{matrix} D^{[210]}_{\substack{12\\3}\,\substack{12\\2}}(U) \\ D^{[210]}_{\substack{13\\2}\,\substack{12\\2}}(U) \\ D^{[300]}_{123\;122}(U) \end{matrix} = \begin{pmatrix} -\sqrt{2} & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & -\sqrt{3}/\sqrt{2} & \sqrt{3}/\sqrt{2} \\ 1 & 1 & 1 \end{pmatrix} \begin{matrix} u^2_1 u^2_2 u^1_3 \\ u^2_1 u^1_2 u^2_3 \\ u^1_1 u^2_2 u^2_3 \end{matrix}. \tag{37}$$

Ciftan and Biedenharn have shown that[20]

$$L([u]) = \sqrt{H([u])}, \tag{38}$$

where $H([u])$ is the product of hook-lengths described in I.

We now have an explicit means of calculating the canonical Clebsch–Gordan coefficients of $\mathrm{U}(n)$ by using the factorization lemma. Let

$$D^{[v]}_{\langle q\rangle\langle p\rangle}(a) = \sqrt{H([v])}\,M^{[v]}_{pq}\underline{P}^{[v]}_{pq}a^{(j)}_{(i)}.$$

The factorization lemma can then be written as

$$\left\langle\begin{matrix}[u]\\ \langle s\rangle * \langle r\rangle\end{matrix}\right|D^{[v]}_{\langle q\rangle\langle p\rangle}(a)\left|\begin{matrix}[\lambda]\\ \langle n\rangle * \langle m\rangle\end{matrix}\right\rangle = \left(\frac{H([u])}{H([\lambda])}\right)^{1/2}\sum_{\delta} C^{[v][\lambda][u]^{\delta}}_{\langle q\rangle\langle n\rangle\langle s\rangle}C^{[v][\lambda][u]^{\delta}}_{\langle p\rangle\langle m\rangle\langle r\rangle} \tag{39}$$

where $[u]^{\delta}$ is the δth IR $[u]$ contained in the direct product $[v]\times[\lambda]$. Since the left-hand side of (39) can be calculated explicitly using the seminormal canonical projection operators, we can directly evaluate the product of Clebsch–Gordan coefficients of the unitary group on the right of (39). Because of the sum on the right on (39), only the multiplicity-free coefficients can be uniquely determined.

IV. FERMION BASES

From (10) we find the antisymmetric states of U(nm) are

$$|[11\cdots 1]\rangle = \frac{1}{\sqrt{p!}} \sum_{q\in S_p} \epsilon_q[q]\,|\Phi^{(j)}_{(i)}\rangle. \qquad (40)$$

We may construct linear combinations of these antisymmetric states of U(nm) from the irreducible bases of U(n)×U(m) using the Clebsch–Gordan coefficients of the canonical bases of S_p. We find that

$$\left|\begin{matrix}[u]\\ \langle s\rangle\,\tilde{*}\,\langle\tilde{r}\rangle\end{matrix}\right\rangle = \frac{1}{(l^{[u]})^{1/2}} \sum_n \epsilon_{\sigma_{nm}} \left|\begin{matrix}[u]\\ \langle s\rangle\ (n)\end{matrix}\right\rangle \times \left|\begin{matrix}[\tilde{u}]\\ (\tilde{n})\ \langle\tilde{r}\rangle\end{matrix}\right\rangle,$$

$$= \frac{N^{[\tilde{u}]}_{\tilde{r}} N^{[u]}_{s}}{(l^{[u]})^{1/2}} \sum_n \epsilon_{\sigma_{nm}} \overline{P}^{[\tilde{u}]}_{\tilde{n}\tilde{r}} \underline{P}^{[u]}_{ns} |\Phi^{(j)}_{(i)}\rangle \qquad (41)$$

is antisymmetric under all permutations $[q]$ of S_p for any standard tableau $T^{[u]}_{(m)}$. This can be shown, using (1.23) of I, since

$$[q]\sum_n \epsilon_{\sigma_{nm}} \overline{P}^{[\tilde{u}]}_{\tilde{n}\tilde{r}} \underline{P}^{[u]}_{ns} = \sum_n \epsilon_{\sigma_{nm}} [\overline{q}] \overline{P}^{[\tilde{u}]}_{\tilde{n}\tilde{r}} [\underline{q}] \underline{P}^{[u]}_{ns}$$

$$= \sum_{n,t,t'} \epsilon_{\sigma_{nm}} \epsilon_{\sigma_{tn}} \epsilon_q D^{[u]}_{tn}[q] D^{[u]}_{t'n}[q] \overline{P}^{[\tilde{u}]}_{\tilde{t}\tilde{r}} \underline{P}^{[u]}_{t's}$$

$$= \epsilon_q \sum_t \epsilon_{\sigma_{tm}} \overline{P}^{[\tilde{u}]}_{\tilde{t}\tilde{r}} \underline{P}^{[u]}_{ts}.$$

Hence, $|{}^{[u]}_{\langle s\rangle\tilde{*}\langle\tilde{r}\rangle}\rangle$ must be some linear combination of antisymmetric bases (40) of U(nm). From (8) we see the basis $|{}^{[u]}_{\langle s\rangle\tilde{*}\langle\tilde{r}\rangle}\rangle$ is an irreducible basis $|{}^{[u]}_{\langle s\rangle}\rangle \times |{}^{[\tilde{u}]}_{\langle\tilde{r}\rangle}\rangle$ of U(n)×U(m). We shall denote the subspace of all bases $|{}^{[u]}_{\langle s\rangle\tilde{*}\langle\tilde{r}\rangle}\rangle$ of the direct product space $|\Phi^{(j)}_{(i)}\rangle$ as U(n) $\tilde{*}$ U(m).

It is important to determine the antisymmetric content of U(nm) in our bases $|{}^{[u]}_{\langle s\rangle\tilde{*}\langle\tilde{r}\rangle}\rangle$ of U(n) $\tilde{*}$ U(m), for this will be equivalent to determining the antisymmetric irreducible bases of U(nm) contained in the irreducible basis $|{}^{[u]}_{\langle s\rangle}\rangle \times |{}^{[\tilde{u}]}_{\langle\tilde{r}\rangle}\rangle$ of U(n)×U(m). In what follows we shall show a simple and straightforward means of finding this antisymmetric content.

For this purpose it is convenient to use fermion operators to generate our bases. We may write our antisymmetric basis in terms of fermion operators as in (42) below:

$$\sum_{q\in S_p} \epsilon_q[q]\,|\Phi^{(j)}_{(i)}\rangle = \frac{1}{\sqrt{p!}} \sum_{q\in S_p} \epsilon_q[q]\, a^{(j)}_{(i)}|0\rangle, \qquad (42)$$

where $a^{(j)}_{(i)}|0\rangle = a^{j_1}_{i_1} a^{j_2}_{i_2}\cdots a^{j_p}_{i_p}|0\rangle$, and $a^{j\dagger}_{i} = \overline{a}^{j}_{i}$. The fermion operators obey the following anticommutation relations:

$$[a^j_i, a^{j'}_{i'}]_+ = [\overline{a}^j_i, \overline{a}^{j'}_{i'}]_+ = 0, \qquad (43a)$$

$$[\overline{a}^j_i, a^{j'}_{i'}]_+ = \delta_{jj'}\delta_{ii'}. \qquad (43b)$$

We may also expand the generators of U(n), U(m), and U(nm) in terms of the fermion operators as in (15).

From (42) we see that $|{}^{[u]}_{\langle s\rangle\tilde{*}\langle\tilde{r}\rangle}\rangle$ may be put in terms of fermion operators as in (44):

$$\left|\begin{matrix}[u]\\ \langle s\rangle\,\tilde{*}\,\langle\tilde{r}\rangle\end{matrix}\right\rangle = \frac{N^{[\tilde{u}]}_{\tilde{r}} N^{[u]}_{s}}{(l^{[u]}p!)^{1/2}} \sum_n \epsilon_{\sigma_{nm}} \overline{P}^{[\tilde{u}]}_{\tilde{n}\tilde{r}} \underline{P}^{[u]}_{ns} a^{(j)}_{(i)}|0\rangle. \qquad (44)$$

By using the anticommutation relations (43), it follows that

$$[\overline{q}]\, a^{(j)}_{(i)}|0\rangle = \epsilon_q [\underline{q}^{-1}]\, a^{(j)}_{(i)}|0\rangle. \qquad (45)$$

From the above relation and (1.23) of I, we have

$$\overline{P}^{[\tilde{u}]}_{\tilde{n}\tilde{r}} a^{(j)}_{(i)}|0\rangle = \epsilon_{\sigma_{nr}} \underline{P}^{[u]}_{rn} a^{(j)}_{(i)}|0\rangle. \qquad (46)$$

Using (46), we may simplify the expression for $|{}^{[u]}_{\langle s\rangle\tilde{*}\langle\tilde{r}\rangle}\rangle$ in (44) to find

$$\left|\begin{matrix}[u]\\ \langle s\rangle\,\tilde{*}\,\langle\tilde{r}\rangle\end{matrix}\right\rangle = N^{[\tilde{u}]}_{\tilde{r}} N^{[u]}_{s} (l^{[u]}/p!)^{1/2} \epsilon_{\sigma_{rm}} \underline{P}^{[u]}_{rs} a^{(j)}_{(i)}|0\rangle.$$

If we let

$$M^{[u]}_{\tilde{r}s} = N^{[\tilde{u}]}_{\tilde{r}} N^{[u]}_{s} (l^{[u]}/p!)^{1/2},$$

then we have the following simple relation between our basis of U(n) $\tilde{*}$ U(m) and the fermion operators:

$$\left|\begin{matrix}[u]\\ \langle s\rangle\,\tilde{*}\,\langle\tilde{r}\rangle\end{matrix}\right\rangle = \epsilon_{\sigma_{rm}} M^{[u]}_{\tilde{r}s} \underline{P}^{[u]}_{rs} a^{(j)}_{(i)}|0\rangle, \qquad (47a)$$

or similarly,

$$\left|\begin{matrix}[u]\\ \langle\tilde{r}\rangle\,\tilde{*}\,\langle s\rangle\end{matrix}\right\rangle = \epsilon_{\sigma_{rm}} M^{[u]}_{\tilde{r}s} \overline{P}^{[u]}_{rs} a^{(i)}_{(j)}|0\rangle. \qquad (47b)$$

Equations (47) illustrate the reciprocity between upper and lower projection operators for bases of U(m) $\tilde{*}$ U(n) and U(n) $\tilde{*}$ U(m).

We now have a convenient and straightforward method of finding the antisymmetric content of U(nm) in the irreducible bases of U(n)×U(m) which we call the fermion calculus. From the expression $\underline{P}^{[u]}_{rs} a^{(j)}_{(i)}|0\rangle$, we may find the antisymmetric content of U(nm) in the bases $|{}^{[u]}_{\langle s\rangle}\rangle \times |{}^{[\tilde{u}]}_{\langle\tilde{r}\rangle}\rangle$ when the $a^{(j)}_{(i)}$ are fermion operators, or we may find the symmetric content of U(nm) in the bases $|{}^{[u]}_{\langle s\rangle}\rangle \times |{}^{[u]}_{\langle r\rangle}\rangle$ when the $a^{(j)}_{(i)}$ are boson operators. Hence, the fermion calculus can be generated by the seminormal projection operators $O^{[u]}_{rs}$ acting on fermion operators, and the boson calculus can be generated by the same seminormal projection operators acting on boson operators.

To illustrate this point, we use the results of Eqs. (23a) and (23b) to find the antisymmetric content of $|{}^{12}_{3}\rangle \times |{}^{12}_{2}\rangle$ and $|{}^{13}_{2}\rangle \times |{}^{12}_{2}\rangle$ respectively by letting the $a^{(j)}_{(i)}$ be fermion operators. Our case is somewhat special since $\langle r\rangle = \langle\tilde{r}\rangle$. The fermion basis analogous to Eq. (23c) is shown below.

$$\underline{O}^{[100]} a^1_1 a^2_2 a^2_3|0\rangle = 4\underline{A}_{123} a^1_1 a^2_2 a^2_3|0\rangle,$$

$$= 4(a^1_1a^2_2a^2_3 - a^1_1a^2_3a^2_2 + a^1_3a^2_1a^2_2 - a^1_3a^2_2a^2_1 + a^1_2a^2_3a^2_1 - a^1_2a^2_1a^2_3)|0\rangle.$$

For convenience we let $\psi^1 = \psi^+$, $\psi^2 = \psi^-$, $\Phi^1_1 = \Phi_{1^+}$, etc. Collecting terms and normalizing the above bases, we find

$$\begin{matrix} \left|\begin{matrix}12\\3\end{matrix}\right\rangle \times \left|\begin{matrix}+\\-\end{matrix}\right\rangle^{-} \\[2ex] \left|\begin{matrix}13\\2\end{matrix}\right\rangle \times \left|\begin{matrix}+\\-\end{matrix}\right\rangle^{-} \\[2ex] \left|\begin{matrix}1\\2\\3\end{matrix}\right\rangle \times |+--\rangle \end{matrix} = \begin{pmatrix} 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 2/\sqrt{6} & -1/\sqrt{6} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix} \begin{matrix} \left|\begin{matrix}1^-\\2^-\\3^+\end{matrix}\right\rangle \\ \left|\begin{matrix}1^-\\2^+\\3^-\end{matrix}\right\rangle \\ \left|\begin{matrix}1^+\\2^-\\3^-\end{matrix}\right\rangle \end{matrix}. \qquad (48)$$

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