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Roton interactions in superfluid helium

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Based on a hard-sphere model for superfluid helium we derive a microscopic theory for quasiparticle interactions in this liquid. A satisfactory upper bound for the roton-roton collision frequency is obtained from this model. This theory also leads to an attractive coupling of two rotons with opposite momenta but the coupling strength is an order of magnitude larger than the phenomenological estimates based on experiments. However, as our analysis reveals, this large coupling strength is compatible with other physical considerations presented in this paper and does not contradict experimental data.

I. INTRODUCTION

The elementary excitations in superfluid helium have been the subject of many theoretical investigations during the thirty years following the pioneering work of Landau.[1] A qualitative form of the excitation spectrum was advanced by Landau on phenomenological grounds (see Fig. 1).

The first microscopic theory of the excitations in a *weakly* interacting Bose gas was developed by Bogoliubov,[2] and various attempts have been made since then to derive the excitation spectrum of a *dilute* Bose gas from first principles.[3-7] A microscopic approach for the *dense* case, based on hard-sphere interaction was put forward by Wong and Huang[8] and Meyer.[9] This type of approach,[8,9] as opposed to the method of correlated basis functions by Feenberg and co-workers[10] or a variational calculation by Feynman and Cohen,[11] seems to be very useful for exploring the interactions between excitations, such as roton-roton binding energy and roton collision frequency in superfluid helium at realistic densities.

Of the experiments which yield information on roton-roton interactions, those of Greytak[12] and Greytak *et al.*[13] on Raman scattering give the most specific information. Because of energy-momentum conservation, only pairs of excitations with equal and opposite momenta participate in the light-scattering process, a situation first appreciated by Halley.[14] Furthermore, the light-induced dipole-dipole interaction in liquid helium having a $l=2$ symmetry is responsible for the fact that in lowest order of the electric interaction only $l=2$ components of a pair of elementary excitations is observable in a Raman scattering experiment. This was pointed out first by Stephen.[15] One of the peculiarities of the observed Raman spectrum is that the two-roton peak gives rise to an energy shift of 0.37 °K below $2\Delta_0$, the minimum energy of a noninteracting roton pair. Ruvalds and Zawadowski[16] (RZ), Iwamoto,[17] and Greytak *et al.*[13] explained this shift using a model in which opposite momentum pairs form bound states in the roton region.

Other information on the roton-roton interaction is provided by roton scattering experiments. Greytak and Yan[18] measured the temperature de-

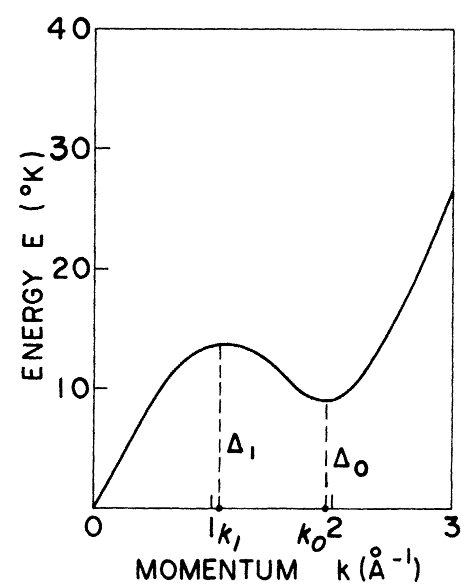

FIG. 1. Excitation spectrum of superfluid helium proposed by Landau.

pendence of the linewidth of the excitation spectrum via Raman scattering and inferred roton collision frequency. Roton collision-frequency data have also been obtained by Brewer and Edwards[19] from viscosity measurements. Both these experiments lead to a collision frequency

$$\tau^{-1}=Bn_r \ , \tag{1}$$

where n_r is the roton number density and B is about 2×10^{-10} $cm^3\,sec^{-1}$ and only very weakly temperature dependent.

Various attempts have been made to explain these effects with the help of *ad hoc* model potentials such as δ functions,[16] exponentials or step functions,[20] as well as potentials based on hydrodynamic considerations such as $3(\vec{p}_1\vec{x})(\vec{p}_2\vec{x})/x^5 - \vec{p}_1\vec{p}_2/x^3$, where $\vec{p}_1$ and $\vec{p}_2$ are the momenta of the two rotons.[20-22]

None of these approaches has succeeded in describing all experimental results consistently nor are any derived from a first-principles microscopic theory of liquid helium.

In Sec. II of this work we derive a roton-roton interaction from first principles via the hard-sphere model developed in Ref. 9, and in Sec. III we compare this interaction with previous models.[16,20,22] In Sec. IV we calculate the roton collision frequency based on our interaction. The calculation of the roton-roton binding energy is much more complicated and is not in detail discussed here.

II. ROTON-ROTON INTERACTION

The hard-sphere interaction in liquid helium in the two-body potential approximation is given by[9]

$$H=\sum_{\vec{k}}\frac{\hbar^2k^2}{2m}a^\dagger_{\vec{k}}a_{\vec{k}}+\frac{1}{2V}\sum_{\vec{p},\vec{q},\vec{k}}\frac{4\pi a\hbar}{m}\times V(\vec{k},\vec{p})\,a^\dagger_{\vec{p}}\,a^\dagger_{\vec{q}}\,a_{\vec{p}+\vec{k}}\,a_{\vec{q}-\vec{k}} \ , \tag{2}$$

$$V(\vec{k},\vec{p})=\cos(ka)_+-a[\vec{p}(\vec{p}+\vec{k})/k]\,j_1(ka)_- \ , \tag{3}$$

where the $\pm$ subscripts mean that we actually should write $a\pm\epsilon$, with $\epsilon\to0^+$ in order to insure that the derivatives on the wave function are outside the hard-sphere region. Mass and diameter of the helium atom are taken to be $m=6.7\times10^{-24}$ g and $a=2.1$ Å. The $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$ are the boson plane-wave operators.

In order to keep track of the zero momentum condensate we replace the zero momentum operator $\hat{N}_0$ by

$$\hat{N}_0=N-\sum_{\vec{k}}{}'\,a^\dagger_{\vec{k}}a_{\vec{k}} \ , \tag{4}$$

and perform a Bolsterli transformation[23]

$$\begin{aligned} b_{\vec{k}}&=a^\dagger_0(\hat{N}_0+1)^{-1/2}a_{\vec{k}} \ ,\\ b^\dagger_{\vec{k}}&=a^\dagger_{\vec{k}}(\hat{N}_0+1)^{-1/2}a_0 \ ,\\ \hat{N}_0&=N-\sum_{\vec{k}}{}'\,b^\dagger_{\vec{k}}b_{\vec{k}} \end{aligned} \tag{5}$$

($k\neq0$). Here N is the total number of particles and $\hat{N}_0$ is the zero momentum number operator.

H is minimized with respect to a normalized set of variational ground states $\{|\psi_0'\rangle\}$

$$|\psi_0'\rangle=\prod_{\vec{k}>0}^{\infty}(1-U_k^2)^{1/2}e^{-U_kb^\dagger_{\vec{k}}b^\dagger_{\vec{k}}}|\psi_0\rangle \ , \tag{6}$$

where $\vec{k}>0$ means that the product is taken over an open semisphere in k space, such that $\vec{k}$ is inside, $-\vec{k}$ is outside. $\vec{k}=0$ is excluded too. The ground states $\{|\psi_0'\rangle\}$ are induced by the Bogoliubov transformation

$$\begin{aligned} \beta_{\vec{k}}&=(1-U_k^2)^{-1/2}(b_{\vec{k}}+U_kb^\dagger_{-\vec{k}}) \ ,\\ \beta^\dagger_{\vec{k}}&=(1-U_k^2)^{-1/2}(b^\dagger_{\vec{k}}+U_kb_{-\vec{k}}) \ , \end{aligned} \tag{7}$$

such that

$$\beta_{\vec{k}}|\psi_0'\rangle=0;\quad b_{\vec{k}}|\psi_0\rangle=0;\quad |\psi_0\rangle=(N!)^{1/2}(a_0^\dagger)^N|0\rangle \tag{8}$$

for every $\vec{k}\neq0$.

The Bogoliubov functional U_k is then determined by the integral equation resulting from the variational problem.[9] The lowest-order terms in H that have not been diagonalized in this procedure represent our quasiparticle interaction:

$$H_{p,\mathrm{int}}=\frac{g}{2V}\sum_{\vec{k},\vec{p}}{}'\,V(\vec{k}-\vec{p},\vec{p})[f^{(1)}(|\vec{k}|,|\vec{p}|)\beta^\dagger_{\vec{k}}\beta_{\vec{k}}\beta^\dagger_{\vec{p}}\beta_{\vec{p}}+f^{(2)}(|\vec{k}|,|\vec{p}|)\beta^\dagger_{\vec{k}}\beta^\dagger_{-\vec{k}}\beta_{\vec{p}}\beta_{-\vec{p}}] \ , \tag{9}$$

$$H_{r,\mathrm{int}}=\frac{g}{2V}\sum_{\substack{\vec{p},\vec{k},\vec{q}:\,\vec{q}\neq\vec{k},\vec{p}\neq-\vec{q}\\ \vec{p}\neq-\vec{k},\vec{p}+\vec{k}\neq\vec{q}}}{}'\,V(\vec{k},\vec{p})h(|\vec{p}|,|\vec{q}|,|\vec{p}+\vec{k}|,|\vec{q}-\vec{k}|)\,\beta^\dagger_{\vec{p}}\beta^\dagger_{\vec{q}}\beta_{\vec{p}+\vec{k}}\beta_{\vec{q}-\vec{k}} \ , \tag{10}$$

where

$$f^{(l)}(|\vec{k}|, |\vec{p}|) = (1 - U_k U_p)^l/(1 - U_k^2)(1 - U_p^2), \tag{11}$$

$$h(|\vec{p}|, |\vec{q}|, |\vec{p}+\vec{k}|, |\vec{q}-\vec{k}|) = [(1 - U_p^2)(1 - U_q^2)(1 - U_{\vec{p}+\vec{k}}^2)(1 - U_{\vec{q}-\vec{k}}^2)]^{-1}(1+ \text{terms in } U^2 \text{ and } U^4), \tag{12}$$

and

$$g = 4\pi a \hbar^2/m. \tag{13}$$

III. COMPARISON OF VARIOUS THEORIES

Let us compare the most important previous models for roton interactions[16,20] with Eqs. (9) and (10).[17] RZ initially proposed[16] a δ-function interaction in configuration space and generalized it later[16] to

$$V(\hat{k}_0, \hat{q}_0) = \sum_{l,m} g_4^{(l)} 4\pi Y_{lm}^*(\hat{k}_0) Y_{lm}(\hat{q}_0) = \sum_l g_4^{(l)}(2l+1) P_l(\cos\theta_{\hat{k}_0\hat{q}_0}) \tag{14}$$

in order to investigate the different angular momentum channels separately. Here the Y_{lm} are spherical harmonics, P_l Legendre polynomials, and θ the angle between the incoming roton $\vec{q}_0$ and the outgoing roton $\vec{k}_0$. The quantities $\hat{q}_0$ and $\hat{k}_0$ are the corresponding unit vectors. Since Raman induced excitations have $l = 2$ symmetry the effective interaction strength in (14) is $g_4^{(2)}$. Using Green's-function techniques RZ could fit the experimental binding energy of 0.37 °K for two rotons of opposite momenta by choosing the effective interaction strength $g_4^{(2)} = -0.12 \times 10^{-38}$ erg cm^3. The quantity investigated by RZ is the reduced quasiparticle Green's function

$$G_2^{lm}(|\vec{k}_1|, \Omega) = \lim_{\vec{k}=0} \frac{4\pi}{(2\pi)^8} \int G_2(k_1 K - k_1; k_3) \times Y_{lm}(\hat{k}_1) Y_{lm}^*(\hat{k}_3)\, dk_3^4\, d\omega_{k_1}\, d\Omega_{\vec{k}_1}, \tag{15}$$

where $k_1 \equiv [\omega_{k_1}, \vec{k}_1]$, $k_3 \equiv [\omega_{k_3}, \vec{k}_3]$, $K \equiv [\Omega, \vec{K}]$ represent the 4-momentum of the excitations and the center-of-mass motion, respectively. The two-particle Green's function satisfies

$$\begin{aligned} G_2(k_1 K - k_1; k_3) &= i(2\pi)^4[\delta^4(k_1 - k_3) + \delta^4(K - k_1 - k_3)] \\ &\quad \times G^{(0)}(k_1) G^{(0)}(K - k_1) + i(2\pi)^{-4} G^{(0)}(k_1) G^{(0)}(K - k_1) \\ &\quad \times \int d q_3^4\, V(k_1 K - k_1; q_3 K - q_3) G_2(q_3 K - q_3; k_3), \end{aligned} \tag{16}$$

where the $G^{(0)}(k)$'s are the free-one-quasiparticle Green's functions and V is the bare vertex function for an instantaneous interaction thus (14)–(16) immediately lead to

$$G_2^{lm}(|\vec{k}_1|, \Omega) = G_2^{(0)lm}(|\vec{k}_1|, \Omega) + G_2^{(0)lm}(|\vec{k}_1|, \Omega) g^{(l)} F^{lm}(\Omega) \times [1 - g^{(l)} F^{lm}(\Omega)]^{-1}, \tag{17}$$

where

$$F^{lm}(\Omega) = \frac{4\pi}{(2\pi)^3} \int_0^\infty \frac{k^2 Z_1^2(|\vec{k}|)\, dk}{\Omega - 2\epsilon_{\vec{k}} + i\Gamma}. \tag{18}$$

Setting $\Gamma = 0$ and chosing $\Omega = \Omega_{\min}$, determined by the experiment by Greytak,[13] one finds $g_4^{(2)}$ from the pole of $G_2^{lm}(|\vec{k}_1|, \Omega)$ in (17). The Green's function $G^{(0)}(k)$ leading to the result (17) is for $kT \ll \epsilon_{k_0}$ assumed to have the form

$$G^{(0)}(k) = Z_1(|\vec{k}|)/(\omega - \epsilon_{\vec{k}} + \tfrac{1}{2} i\Gamma). \tag{19}$$

The quantity Γ is the single-particle width, assumed to be energy independent, and $Z_1(|\vec{k}_1|)$ is a momentum-dependent normalization factor which is important in satisfying the sum rules.[24]

For simplicity we set $Z_1(|\vec{k}|) = 1$ for $k_1 < k < k_0 + (k_0 - k_1)$ and $Z_1(|\vec{k}|) = 0$ anywhere else. The above form of Z_1 will be taken in account from now on by introducing the proper integration limits in Eq. (18). We would like to point out here that this is a crude approximation for Z_1 which underestimates $g_4^{(2)}$ by at least a factor of 2.

Now let us compare the strength $g_4^{(2)}$ of the δ-function interaction with the interaction strength from our Hamiltonian. First note that for $\vec{K} = 0$ in (16) only BCS terms enter the computation of $G_2(k_1\Omega - k_1; k_3)$. Thus, the interaction between rotons of opposite momenta becomes

$$H_{\text{rot},B} = \frac{g}{2V} \sum_{\vec{k},\vec{p}}{}' \lambda^{(2)}(|\vec{k}|, |\vec{p}|) f^{(2)}(|\vec{k}|, |\vec{p}|) \times 5P_2(\cos\theta) \beta_{\vec{k}}^\dagger \beta_{-\vec{k}}^\dagger \beta_{\vec{p}} \beta_{-\vec{p}}. \tag{20}$$

Here we retained only the $l = 2$ component to account for the rotational symmetry of the quasiparticles. The quantity $\lambda^{(2)}(|\vec{k}|, |\vec{p}|)$, computed in Appendix B [in Appendix B we also present $\lambda^{(2)}(|\vec{k}|, |\vec{p}|)$ in a separable form], is

$$\lambda^{(2)}(|\vec{k}|,|\vec{p}|) = -2j_2(ka)j_2(pa) - [(k^2+p^2)/kp] \times\{\tfrac{1}{2}[j_0(|p-k|a) - j_0(|p+k|a)] - 3j_1(ka)j_1(pa)\} + \tfrac{1}{2}[j_0(|p+k|a) + j_0(|p-k|a)] - j_0(ka)j_0(pa)\,. \quad (21)$$

Here the j_l's are spherical Bessel functions.

Since we are interested in momenta k and p lying between $k_1 \approx 1$ Å^{-1} and $k_0 + (k_0 - k_1) \approx 3$ Å^{-1} and since $|U_k| \leq 0.25$ in this region,[8,9] it is a good approximation to set $f^{(2)}(|\vec{k}|,|\vec{p}|) = 1$. If we further assume that $|\vec{k}| = |\vec{p}|$ so that the two quasiparticles are treated as a two-body system, with center-of-mass motion zero, $\lambda^{(2)}(|\vec{k}|,|\vec{p}|)$ becomes a function of k alone shown in Fig. 2 (see also Appendix B). Remember that our calculations are based on a purely microscopic model with only one parameter, the diameter of a He atom.

We see from Fig. 2 that the interaction between quasiparticles is indeed attractive in the momentum region corresponding to the roton dip and also exhibits a minimum there. If we assume that most of the interaction takes place in a narrow momentum region around the roton dip, i.e., $|\vec{p}| \cong |\vec{k}| \cong k_0$, then (20) becomes

$$H_{\mathrm{rot},B} = \frac{g^{(2)}}{2V} \sum_{\vec{k},\vec{p}\ \mathrm{with}\ |\vec{k}|=|\vec{p}|=k_0} 5P_2(\cos\theta)\beta^\dagger_{\vec{k}}\beta^\dagger_{-\vec{k}}\beta_{\vec{p}}\beta_{-\vec{p}}\,, \quad (22)$$

which has the form postulated by RZ. Our interaction strength $g^{(2)}$ is defined by

$$g^{(2)} = g\lambda^{(2)}(k_0,k_0)\,. \quad (23)$$

Comparing our $g^{(2)}$ with the $g_4^{(2)}$ of RZ we find

$$g_4^{(2)} = -0.12\times10^{-38}\ \mathrm{erg\ cm}^3, \quad g^{(2)} = -2.1\times10^{-38}\ \mathrm{erg\ cm}^3\,. \quad (24)$$

One might think at first glance that the difference of 20 in interaction strength is a consequence of

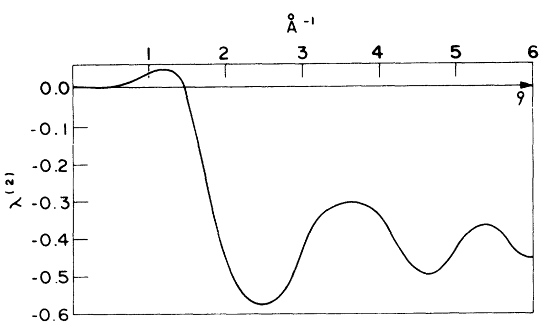

FIG. 2. Quasiparticle interaction resulting from our microscopic model.

our crude approximation $\lambda^{(2)}(|\vec{k}|,|\vec{p}|) \cong -\frac{1}{2}$ (see Fig. 2). Since for all practical purposes G_2 in (16) can be solved exactly with our potential, we investigated the validity of our approximation by actually computing the quantity $G_2^{lm}(|\vec{k}_1|,\Omega)$ (see Appendix A). Choosing g as a parameter in this computation we obtained the experimentally determined[13] roton-roton binding energy for a value of g which is 10 times smaller than the one given by our model $g = 4\pi a\hbar^2/m$.

If we choose in Eq. (18) a more accurate expression for $Z_1(k)$,[24] our potential is about 5 times too large or the Green's-function formalism in the present form is too simple minded, or both. A more careful analysis of this situation is the goal of this section. We proceed as follows.

Using a δ-function interaction and the two roton wave function proposed by Landau and Khalatnikov[25] (symmetrized planewave states) together with the experimental energy spectrum for rotons, Yau and Stephen[20] (YS) calculated the scattering matrix not only in the Born approximation[25] but to all orders. Their result for the binding energy is

$$E_B^{\mathrm{YS}} = p_{\mathrm{rel}}^2\hbar^2/(4\mu_0\sinh^2\lambda)\,,\quad \vec{p}_{\mathrm{rel}} = \tfrac{1}{2}(\vec{p}_1 - \vec{p}_2), \quad \lambda = \pi K\hbar^2/(\mu_0 k_0^2|V_0|)\,, \quad (25)$$

where μ_0 is the effective roton mass, $|V_0|$ the Landau-Khalatnikov interaction strength of the δ function, and $\vec{p}_1$ and $\vec{p}_2$ the momenta of the two rotons under consideration. For our case $p_{\mathrm{rel}} \cong k_0$ and $K \neq 0$ we find for

$$K < 1\ \text{Å}^{-1}:\ E_B^{\mathrm{YS}} = k_0^4\mu_0|V_0|^2k_0^2/4\pi^2\hbar^2K^2\,, \quad (26a)$$

$$K > 1\ \text{Å}^{-1}:\ E_B^{\mathrm{YS}} = 2(k_0^2\hbar^2/2\mu_0)\exp(-2\pi K\hbar^2/k_0^2\mu_0|V_0|)\,. \quad (26b)$$

By comparison, (RZ) obtained

$$K=0:\ E_B^{\mathrm{RZ}} = k_0^4\mu_0|g_4^{(2)}|^2/4\pi^2\hbar^2\,, \quad (27a)$$

$$K > 1\ \text{Å}^{-1}:\ E_B^{\mathrm{RZ}} = 2D\exp(-2\pi K\hbar^2/k_0^2\mu_0|g_4^{(2)}|)\,, \quad (27b)$$

where D is the energy difference between the peak and the dip of the excitation spectrum which is of order $(k_0\hbar)^2/2\mu_0$. The strong resemblance between (26) and (27) is due to the fact that both authors are essentially studying a simplified two quasiparticle system with δ-function interaction. The fact that the two expressions are not exactly the same is not a result of the different computational techniques used—RZ utilized Green's functions, Yau and Stephen scattering-matrix formalism—but rather a consequence of the different approximations employed. In fact the T-matrix approach will lead to the same binding energy as obtained by RZ provided that the same cutoff in momentum space is used. This has been shown by Iwamoto.[17]

Using δ-function interaction YS also investigated roton scattering. They found an upper bound on the collision frequency which is independent of the interaction strength, and four times smaller than the observed collision frequency. From this they concluded that the δ-function interaction is unphysical. For such a conclusion to be meaningful, it must be independent of the computational techniques used. Without deeper knowledge, the similarity of the results (26) and (27) would suggest that this is the case. Actually RZ have not mentioned the collision frequency in their initial papers[16] although it can be obtained straightforwardly from their results (in a later paper[26] they come to the same conclusion) and turns out to be exactly the same in form and magnitude as the one computed by YS. Thus the interaction strength $g_4^{(2)}$ found by Ruvalds and Zawadowsky is a parameter that is determined on one hand by the momentum space cutoff and on the other hand by matching the physical entities (Raman spectrum, collision frequency) to be calculated. Unfortunately, it is not possible to find a $g_4^{(2)}$ which gives the correct value of the collision frequency in the context of a theory based on δ-function interaction. Such an interaction strength is an effective one, and might be only remotely related to the interaction strength associated with a physical potential.

The question arising now is: Would our interaction potential lead to the correct results if it were employed in a more accurate computational scheme? For this purpose we shall compare our potential with what is considered a more physical potential than the δ-function interaction. Several authors[20-22] have proposed the form

$$V(\vec{r}) = (A\hbar^2/m)[3(\vec{p}_1\vec{r})(\vec{r}\vec{p}_2)/r^5 - \vec{p}_1\vec{p}_2/r^3], \qquad (28a)$$

the Fourier transform of which is

$$V(\vec{q}) = (4\pi A\hbar^2/ma^2)[\tfrac{1}{3}a^2\vec{p}_1\vec{p}_2 - a^2(\vec{p}_1\vec{q})(\vec{p}_2\vec{q})/q^2]. \qquad (28b)$$

Here $\vec{p}_1$ and $\vec{p}_2$ are the momenta of the two incoming rotons, $\vec{q}$ is the momentum transfer, m and a are the mass and the diameter, respectively, of a He atom, and $A = 3.8$ Å^3 is a constant which has been chosen in accordance with Ref. 20. The above potential represents, for example, the interaction due to the backflow fields of two spheres moving in a fluid far apart from each other.[21] It also can be understood from phonon-induced roton-roton interaction.[11,22] First we observe that the interaction strength $4\pi A\hbar^2/ma^2$ associated with (28) is of the same order of magnitude as ours, $g = 4\pi a\hbar^2/m$.

YS estimated the binding energy of two rotons interacting via potential (28). They took as variational function with d symmetry

$$\psi(\vec{r}_1 - \vec{r}_2) = \sum_{\vec{p}} \exp i\,\vec{p}(\vec{r}_1 - \vec{r}_2)h_p P_2(\cos\theta_p), \qquad (29)$$

and chose $h_p = [(p-k_0)^2 + \gamma^2]^{-1}$, where γ is a variational parameter. Assuming that potential (28) vanishes for $r < b$ corresponding to a breakdown in the hydrodynamic approximation, they found (after minimizing with respect to γ and choosing $b = 5$ Å) a binding energy of 0.4 °K in good agreement with the experiment. It is important to observe that potential (28) goes to zero at least as fast as r^{-3} for $r > b$. On the other hand the δ-function interaction is cut off in the momentum space. This always leads to spherical Bessel functions in configuration space, meaning that the potential actually used, behaves like r^{-1} for $r \to \infty$ rather than $\delta(r)$! This must be part of the reason why the interaction strength (28), although an order of magnitude larger than the one proposed by RZ, leads to approximately the same binding energy. On the other hand, as demonstrated by Toigo,[22] interaction (28) leads to a collision frequency which is an order of magnitude smaller than the observed one. From these observations YS concluded that the long-range attractive part leads to an explanation of the binding energy of two rotons, but is not important in the scattering of two rotons. They expect that roton-roton scattering must arise from the short-range interaction between rotons and, provided the short-range potential is not too weak, the upper bound of the collision frequency will be independent of the strength of the potential, but its range will be important. They tested the latter hypothesis with different types of potentials. For example, $V(r) = V_0$, $r < c$ and $V(r) = 0$, $r > c$, with $c \cong 2$ Å, and they found a collision frequency in good agreement with experiment. For different shapes of the short-range potentials the results did not vary significantly.

What can we learn from these facts? Since the binding energy is determined by the attractive long range behavior of the interaction, it is reasonable to assume that the dominant contribution to the bound states comes essentially from the forward scattering. Thus if we compare the BCS part of our potential (9) assuming $f^{(2)}(|\vec{k}|, |\vec{p}|) = 1$ with the BCS part of (28), we have for nearly forward scattering

$$V \cong (4\pi\hbar a/m)(1 - \tfrac{1}{3}a^2\vec{k}\vec{p}) \qquad (30a)$$

and

$$V \cong (4\pi A \hbar^2/ma^2)(-\tfrac{1}{3}a^2\vec{k}\vec{p}) , \tag{30b}$$

where $\vec{p}$ and $\vec{k}$ are incoming and outgoing momenta, respectively. Equation (30b) has been derived from Eq. (28a) casting it into second quantization by evaluating the integral

$$\frac{1}{2}\int dx_1^3\, dx_2^3\, \psi^\dagger(\vec{x}_1)\,\psi^\dagger(\vec{x}_2)\frac{\partial}{\partial \vec{x}_1} D(\vec{x}_1 - \vec{x}_2)\frac{\partial}{\partial \vec{x}_2}\psi(\vec{x}_1)\,\psi(\vec{x}_2)\,.$$

For $k \approx q \approx 2$ Å^{-1}, the region of interest, the two brackets in Eq. (30) lead to $-\frac{13}{3}$ and $-\frac{16}{3}$, respectively, which means that the resemblance of (30a) to the physically tested potential (30b) is a rather close one. The momentum dependence shown in (30a) and (30b) could be interpreted, if one is willing to adopt the notion of vortexlike entities for the quasiparticles, as the cancellation of the backflow of two opposite moving rings. This lowers the total energy of the two quasiparticles and leads to bound states.

Examining now the short-range scattering, responsible for the collision frequency, we observe that in the region of interest, where we can assume $f^{(i)}$ and $h \cong 1$, our interaction potential (9) and (10) has very much the same form as (3) which represents the pseudopotential for hard spheres of diameter $a = 1.2$ Å. Since the upper bound of the collision frequency is independent of the strength of the interaction and the hard-sphere size is close to the range used by YS our potential is bound to lead to correct collision times. This we shall show in the following Sec. IV.

The purpose of the above analysis was to reinforce the conjecture that our interaction, although apparently an order of magnitude larger than the one used by RZ, is actually quite reasonable. This conjecture will receive even stronger confirmation from the calculation of Sec. IV. The RZ interaction is an effective one, geared to give the right binding energy and to make the calculation of the interacting two particle density of states mathematically tractable. The result that with our large interaction strength, the correct binding energy of the two rotons with opposite momenta cannot be obtained correctly, seems to be a result of an over simplified use of the Green's-function approach. For example, the possibility of virtual emission and reabsorbtion of rotons leading to a renormalized interaction strength has not yet been taken into account. Thus, one should reinterpret $g_4^{(2)}$ as a renormalized vertex strength. An indication in favor of this interpretation is given by the work of Rajogopal, Bagchi, and Ruvalds[27] in which the authors arrive at the qualitative conclusion that a renormalization of the vertex function reduces the effective roton-roton interaction strength by an order of magnitude, which gives credence to our interaction strength. Furthermore, a Hartree-Fock calculation similar to the one in Ref. 16c with our potential Eq. (9) shows correctly a decrease of the roton gap with increasing temperature. To obtain quantitative agreement with the experiment we need an interaction strength that is approximately 3 times larger than ours, this is another indication for the correct order of magnitude of our potential.

IV. ROTON-ROTON COLLISION FREQUENCY

Here we investigate whether the two-body potential Hamiltonian (2) leading to the roton-roton interaction will give rise to a reasonable value for the upper bound of the roton-roton collision frequency τ^{-1}. First we concentrate on the scattering of a roton with momentum $\vec{k}_3$ where $|\vec{k}_3| = k_0$ with all the other rotons present in the liquid. The collision frequency is given by

$$\tau^{-1} = \frac{\pi}{V}\sum_{\vec{k}_1,\vec{K}-\vec{k}_3;\vec{k}_3\text{ fixed}} |T(k_1 K - k_1; k_3)|^2 \times \delta(E_{\vec{k}_1} + E_{\vec{K}-\vec{k}_1} - E_{\vec{k}_3} - E_{\vec{K}-\vec{k}_3}) f(\vec{K}-\vec{k}_3)\,. \tag{31}$$

Here $T(k_1 K - k_1; k_3)$ represents the scattering matrix in the 4-momentum notation. T does not depend on ω_{k_1} nor ω_{k_3} but will depend on $\vec{k}_1, \vec{k}_2, \vec{K}, \Omega \equiv E$ ($\hbar = 1$). $f(\vec{K} - \vec{k}_3)$ is the number of thermally excited rotons per unit volume with momentum $\vec{K} - \vec{k}_3$. Because $f(\vec{K} - \vec{k}_3)$ peaks at k_0 we can restrict ourselves approximately to $|\vec{K} - \vec{k}_3| = k_0$. By use of the optical theorem,[28] τ^{-1} can be written

$$\tau^{-1} = -2 \sum_{\vec{K}-\vec{k}_3;\,\vec{k}_3\text{ fixed}} \mathrm{Im}\,T(k_3 K - k_3; k_3) f(\vec{K} - \vec{k}_3)\,. \tag{32}$$

Although the integral equation for T can be solved exactly in principle for our case of a separable potential (an integral equation of the same type as we have for T has been solved explicitly in Appendix A), we resort for the sake of simplicity, to an approximation procedure, the Lippmann-Schwinger variational method,[28,29] to solve for T, which gives

$$T(k_3 K - k_3; k_3) = \frac{2\hat{V}^2(0, \vec{k}_3)}{\hat{V}(0, \vec{k}_3) + \hat{F}(\vec{K}E)}, \tag{33}$$

where

$$\hat{F}(\vec{K}, E) = \frac{1}{V} \sum_{\vec{k}_3'} \frac{\hat{V}^2(\vec{k}_3' - \vec{k}_3, \vec{k}_3)}{E_{\vec{k}_3'} + E_{\vec{K}-\vec{k}_3'} - E - i\Gamma} \quad (34)$$

and

$$\hat{V}(\vec{k}_3' - \vec{k}_3, \vec{k}_3) \equiv g[\cos(|\vec{k}_3' - \vec{k}_3| a) - (a\vec{k}_3\vec{k}_3' / |\vec{k}_3' - \vec{k}_3|) \times j_1(|\vec{k}_3' - \vec{k}_3| a)]. \quad (35)$$

In Eqs. (31)–(33) $|\vec{k}_3| = k_0$. This means together with $|\vec{K} - \vec{k}_3| \cong k_0$ that $E \gtrsim 2E_0$. With latter condition $E \gtrsim 2E_0$ it follows, as explicitly shown in Appendix C, formula (C19), that for our purpose $\vec{k}_3'$ in (34) will be restricted to $|\vec{k}_3'| \cong k_0$. Thus we must add to the solution (33) and (34) the conditions

$$|\vec{k}_3| = k_0, \quad |\vec{k}_3'| \cong k_0, \quad |\vec{K} - \vec{k}_3| \cong k_0, \quad |\vec{K} - \vec{k}_3'| \cong k_0, \quad E \gtrsim 2E_{k_0}. \quad (36)$$

Before going any further, we must show where our interaction $\hat{V}(\vec{k}_3' - \vec{k}_3, \vec{k}_3)$ in (35) comes from.

The roton-roton interaction arising from the terms in H that have not yet been diagonalized, are $V^{\text{BCS}}(\vec{k}, \vec{q})$ and $V^{\text{HF}}(\vec{k}, \vec{q})$ shown in (9). $V^{\text{HF}}(\vec{k}, \vec{q})$ differs from $V^{\text{BCS}}(\vec{k}, \vec{q})$ only by the fact that $f^{(2)}(|\vec{k}|, |\vec{q}|)$ is replaced by $f^{(1)}(|\vec{k}|, |\vec{q}|)$. However as can be seen from the actual computation later in (47) and from Appendix C formula (C26) the contribution from these two terms to τ^{-1} are entirely negligible compared with the contribution of all possible nonpair terms in Eq. (10). Thus the interaction responsible for the roton collision frequency is given by $H_{r,\,\text{int}}$ (10). For our purposes it can be simplified further to

$$H_{\text{rot},c} = \frac{1}{2V} \sum_{\vec{p},\vec{q},\vec{k}} \hat{V}(\vec{k}, \vec{p}) \beta^\dagger_{\vec{p}} \beta^\dagger_{\vec{q}} \beta_{\vec{p}+\vec{k}} \beta_{\vec{q}-\vec{k}}, \quad (37)$$

where $\hat{V}(\vec{k}, \vec{p})$ is given by (35).

We did not impose any restriction on the summation in Eq. (37) in contrast with (10). The reason is the following: from (36) it follows that $|\vec{p}| \cong |\vec{q}| \cong |\vec{p} + \vec{k}| \cong |\vec{q} - \vec{k}| \cong k_0$, therefore we can drop the restrictions $\vec{p} \neq 0$, $\vec{q} \neq 0$, $\vec{p} + \vec{k} \neq 0$, $\vec{q} - \vec{k} \neq 0$. Furthermore $\vec{p} + \vec{q} = 0$ and $\vec{p} + \vec{k} = \vec{q}$ lead to exactly the $V^{\text{BCS}}(\vec{k}, \vec{q})$ and $V^{\text{HF}}(\vec{k}, \vec{q})$ terms discussed before, which means we can drop the restrictions $\vec{p} + \vec{q} \neq 0, \vec{p} + \vec{k} \neq q$. The only term that should be excluded is $\vec{k} = 0$. But since $\vec{k} = 0$ gives rise to $V^{\text{HF}}(0, \vec{q})$ and thus contributes negligibly to the evaluation of τ^{-1} we dropped the restriction $\vec{k} \neq 0$ in order to obtain a simple expression for $H_{\text{rot},c}$. Since $U_{k_0}^2 \leq 0.06$, it is a good approximation to set all $U_{k_0}^2$ and $U_{k_0}^4$ equal to zero and thus $h(\vec{p}, \vec{q}, \vec{p} + \vec{k}, \vec{q} - \vec{k}) = 1$. Expressing (37) in a more suitable form by replacing $\vec{p} \to \vec{p}_1$, $\vec{q} \to \vec{p}_2$, $\vec{p} + \vec{k} \to \vec{p}_1'$, and $\vec{q} - \vec{k} + \vec{p}_1 + \vec{p}_2 - \vec{p}_1'$ we obtain

$$H_{\text{rot},c} = \frac{1}{2V} \sum_{\vec{p}_1, \vec{p}_2, \vec{p}_1'} \hat{V}(\vec{p}_1' - \vec{p}_1, \vec{p}_1) \beta^\dagger_{\vec{p}_1} \beta^\dagger_{\vec{p}_2} \beta_{\vec{p}_1'} \beta_{\vec{p}_1 + \vec{p}_2 - \vec{p}_1'}, \quad (38)$$

with

$$|\vec{p}_1| = k_0, \quad |\vec{p}_1'| \cong k_0, \quad |\vec{p}_2| \cong k_0, \quad |\vec{p}_1 + \vec{p}_2 - \vec{p}_1'| \cong k_0. \quad (39)$$

Now we return to the calculation of τ^{-1}. From (33) we obtain

$$-\text{Im}T(k_3K - k_3; k_3) = \frac{2\hat{F}_2(\vec{K}, E)\hat{V}^2(0, \vec{k}_3)}{[\hat{V}(0, \vec{k}_3) + \hat{F}_1(\vec{K}, E)]^2 + \hat{F}_2^2(\vec{K}, E)}, \quad (40)$$

where $\hat{F}_1$ and $\hat{F}_2$ are the real and imaginary parts of $\hat{F}$. An upper bound for $-\text{Im}T(k_3K - k_3; k_3)$ certainly is given by

$$-\text{Im}T^{\max}(k_3K - k_3; k_3) = 2\hat{V}^2(0, \vec{k}_3)/\hat{F}_2(\vec{K}, E). \quad (41)$$

Thus with (32) and (41) $\tau_{\min}^{-1}$ is given by

$$\tau_{\min}^{-1} = 4 \sum_{\vec{p}_2} \hat{V}^2(0, \vec{p}_1) f(\vec{p}_2) [\hat{F}_2(\vec{K}, E)]^{-1}, \quad (42)$$

where from (34) $\hat{F}_2(\vec{K}, E)$ becomes

$$\hat{F}_2(\vec{K}E) = \frac{\pi}{(2\pi)^3} \int dp_1'^3 \, \hat{V}^2(\vec{p}_1' - \vec{p}_1, \vec{p}_1) \times \delta(E_{\vec{p}_1'} + E_{\vec{K} - \vec{p}_1'} - E). \quad (43)$$

In the case of a δ-function interaction (see Appendix C)

$$\hat{V}(\vec{p}_1' - \vec{p}_1, \vec{p}_1) \to \hat{V}_0 \equiv \hat{V}(0, k_0^2) = g(1 - \tfrac{1}{3}a^2k_0^2)$$

and (43) becomes as shown by Solona *et al.* (see appendix A of Ref. 26):

$$\hat{F}_2(\vec{K}, E) = 4k_0^2 \mu_0 \pi^3 \hat{V}_0^2 / [|\vec{K}| (2\pi)^3]. \quad (44)$$

These authors[26] did not employ the T-matrix formalism, but computed the integral (43) in a different context. For this particular interaction it is an exact result that $\hat{F}_2$ does not depend on E! From (42) and (44) we then obtain

$$\tau_{\min}^{-1} = \frac{8}{k_0^2 \mu_0} \sum_{\vec{p}_2} |\vec{K}| f(\vec{p}_2), \quad (45)$$

in accordance with YS[20] and Solona *et al.*[26] With our potential (35), we have shown in Appendix C that $\hat{F}_2(\vec{K}, E)$ approximately becomes

$$\hat{F}_2(\vec{K}, E) = \mu_0 k_0^2 \hat{V}_0^2 / \alpha a |\vec{p}| |\vec{K}|, \quad (46)$$

which again has the same structure as the improved results of YS[20] for the short-range potentials. The E dependence is buried in the relative momentum $\vec{p}$. Equation (42) thus becomes

$$\tau_{\min}^{-1} = 4\,\frac{\alpha a}{\mu_p k_0^2}\sum_{\vec{p}_2} |\vec{p}||\vec{K}| f(\vec{p}_2)\,. \tag{47}$$

We also observe that the $\vec{K}=0$ contribution to this integral has zero weight. This is the contribution of the BCS terms in the interaction, which ultimately justifies the earlier remark that these terms do not contribute anything. Integration of (47) is now easily performed by remembering that $|\vec{p}_1| = k_0$, $\vec{K} = \vec{p}_1 + \vec{p}_2$, $\vec{p} = \frac{1}{2}(\vec{p}_1 - \vec{p}_2)$, and $f(\vec{p}_2)$ strongly peaked around $|\vec{p}_2| = k_0$.

Writing Eq. (47) more explicitly we have

$$\begin{aligned}\tau_{\min}^{-1} &= \frac{4\alpha a}{\mu_0 k_0^2}\,\frac{V}{(2\pi)^3}\,2\pi\tfrac{1}{2}k_0^2\int_0^\infty p_2^2\,dp_2 f(p_2)\\ &\quad\times\int_0^\pi d\theta\,2\sin\tfrac{1}{2}\theta\,2\cos\tfrac{1}{2}\theta\\ &= \frac{\alpha a\pi}{\mu_0}\,\frac{V}{(2\pi)^3}\int_0^\infty 4\pi p_2^2 f(p_2)\,dp_2\,,\end{aligned} \tag{48}$$

and defining n_r, the total number of rotons per unit volume, by

$$n_r = \frac{V}{(2\pi)^3}\int_0^\infty 4\pi p_2^2 f(\vec{p}_2)\,dp_2\,, \tag{49}$$

we finally arrive at

$$\tau_{\min}^{-1} = (\alpha a\pi/\mu_0)n_r\,,\quad \alpha = \tfrac{10}{3}. \tag{50}$$

Quantitatively and qualitatively, this result is in reasonably good agreement with the experiments by Brewer and Edwards[19] as pointed out by Yau and Stephen[20] who obtained similar expressions using several different trial Hamiltonians.

V. CONCLUSIONS

We have shown that the hard-sphere model of superfluid ^{4}He leads to a roton-roton interaction potential which contains to a sufficient degree the hard-core short-range features necessary to produce a satisfactory roton-roton collision frequency. Furthermore the size of our interaction is compatible with other roton binding-energy computations.[15,27,30] The only parameter required to fix the interaction is the hard-sphere radius a. Notice that short-range repulsive properties of the two-body potential Hamiltonian (2) result mainly from the nonpair terms as the computation of the roton collision frequency reveals. This is consistent with the failure to obtain a good pair correlation function based on the pair terms alone as pointed out in Ref. 9. It also indicates that the inclusion of the nonpair terms in such a computation should improve the pair correlation function as mentioned in the same reference.

APPENDIX A

The integral equation to be solved is [see Eq. (16)]

$$\begin{aligned}G_2(k_1K-k_i;k_3) &= i(2\pi)^4[\delta^4(K-k_1-k_3)+\delta^4(k_1-k_3)]\,G(k_1)G(K-k_1) + i(2\pi)^{-4}G(k_1)G(K-k_1)\\ &\quad\times\int dq_3^4\,V(k_1K-k_1;q_3K-q_3)G_2(q_3K-q_3;k_3).\end{aligned} \tag{A1}$$

We only need to find $G_2^{lm}(|\vec{k}_1|,\Omega)$ [see Eq. (15)]. The potential is of the form

$$V(\vec{q}_1-\vec{q}_2) = \sum_{lm} 4\pi g\lambda^{(l)}(|\vec{q}_1|,|\vec{q}_2|)Y_{lm}^*(\hat{q}_1)Y_{lm}(\hat{q}_2)\,. \tag{A2}$$

The vertex function in (A1) becomes

$$\begin{aligned}V(k_1K-k_1;q_3K-q_3) &= V(\vec{k}_1\vec{K}-\vec{k}_1;\vec{q}_3\vec{K}-\vec{q}_3)\\ &= \frac{4\pi}{4}g\Big(\sum_{lm}Y_{lm}^*(\hat{k}_1)[\lambda^{(l)}(|\vec{k}_1|,|\vec{q}_3|)Y_{lm}(\hat{q}_3)+\lambda^{(l)}(|\vec{k}_1|,|\vec{K}-\vec{q}_3|)Y_{lm}(K\hat{-}q_3)]\\ &\quad+\sum_{lm}Y_{lm}^*(K\hat{-}k_1)[\lambda^{(l)}(|\vec{K}-\vec{k}_1|,|\vec{q}_3|)Y_{lm}(\hat{q}_3)\lambda^{(l)}(|\vec{K}-\vec{k}_1|,|\vec{K}-\vec{q}_3|\,Y_{lm}(K\hat{-}q_3)]\Big),\end{aligned} \tag{A3}$$

where the symbols $K\hat{-}q$ mean $(\vec{K}-\vec{q})/(|\vec{K}-\vec{q}|)$. Thus for $\vec{K}=0$ we have

$$V(\vec{k}_1-\vec{k}_1;\vec{q}_3-\vec{q}_3) = \frac{4\pi}{4}g\sum_{lm}\lambda^{(l)}(|\vec{k}_1|,|\vec{q}_3|)[Y_{lm}^*(\hat{k}_1)+Y_{lm}^*(-\hat{k}_1)][Y_{lm}(\hat{q}_3)+Y_{lm}(-\hat{q}_3)]\,. \tag{A4}$$

Equations (A1), (A4), and definition (15) lead to

$$G_2^{lm}(|\vec{k}_1|,\Omega)=i\frac{4\pi}{(2\pi)^4}\int G(k_1)G(\Omega-k_1)Y_{lm}(\hat{k}_1)[Y^*_{lm}(\hat{k}_1)+Y^*_{lm}(-\hat{k}_1)]d\omega_{k1}d\Omega_{\vec{k}1}$$
$$+i\frac{4\pi}{2(2\pi)^4}\int G(k_1)G(\Omega-k_1)Y_{lm}(\hat{k}_1)[Y^*_{lm}(\hat{k}_1)+Y^*_{lm}(-\hat{k}_1)]\,d\omega_{k1}\,d\Omega_{\vec{k}1}$$
$$\times\frac{4\pi}{2(2\pi)^8}\int g\lambda^{(l)}(|\vec{k}_1|,|\vec{k}_4|)\left(\int G_2(k_4\Omega-k_4;k_3)\,Y^*_{lm}(\hat{k}_3)[Y_{lm}(\hat{k}_4)+Y_{lm}(-\hat{k}_4)]dk_3^4\,d\omega_{k4}\,d\Omega_{\vec{k}4}\right)k_4^2\,dk_4\,. \tag{A5}$$

Since

$$Y_{lm}(\hat{k})=(-1)^l Y_{lm}(-\hat{k}), \tag{A6}$$

we obtain for $l=1,3,5,\ldots$

$$G_2^{lm}(|\vec{k}_1|,\Omega)=0\,, \tag{A7}$$

and for $l=0,2,4,\ldots$

$$G_2^{lm}(|\vec{k}_1|,\Omega)=G_2^{(0)lm}(|\vec{k}_1|,\Omega)+\tfrac{1}{2}G_2^{(0)lm}(|\vec{k}_1|,\Omega)\times\int g\lambda^{(l)}(|\vec{k}_1|,|\vec{k}_4|)G_2^{lm}(|\vec{k}_4|,\Omega)k_4^2\,dk_4, \tag{A8}$$

where $G_2^{(0)lm}(|\vec{k}_1|,\Omega)$ is given in analogy with Eq. (15) by

$$G_2^{(0)lm}(|\vec{k}_1|,\Omega)=i\frac{4\pi}{(2\pi)^4}\int d\omega_{k_1}\,d\Omega_{\vec{k}1}\times G(k_1)G(\Omega-k_1)Y_{lm}(\hat{k}_1)[Y^*_{lm}(\hat{k}_1)+Y^*_{lm}(-\hat{k}_1)]\,. \tag{A9}$$

The decomposition in integral equations for each angular momentum component is a considerable simplification of the problem and is a result of the fact, that for $\vec{K}=0$ our problem is of spherical nature and therefore l and m are good quantum numbers to describe the situation. We shall solve (A8) with a separable kernel of the form

$$\lambda^{(l)}(|\vec{k}_1|,|\vec{k}_4|)=\sum_{i=1}^{M_l}\alpha_i f_i^{(l)}(|\vec{k}_1|)f_i^{(l)}(|\vec{k}_4|);\quad M_l<\infty\,. \tag{A10}$$

For the simplest case where $M_l=1$, we set

$$g\lambda^{(l)}(|\vec{k}_1|,|\vec{k}_4|)=g^{(l)}V(|\vec{k}_1|)V(|\vec{k}_4|)\,, \tag{A11}$$

and obtain from (A8)

$$G_2^{lm}(|\vec{k}_1|,\Omega)=G_2^{(0)lm}(|\vec{k}_1|,\Omega)+G_2^{(0)lm}(|\vec{k}_1|,\Omega)g^{(l)}\times V(|\vec{k}_1|)\,{}^{(1)}F^{lm}(\Omega)/[1-g^{(l)}\,{}^{(2)}F^{lm}(\Omega)]\,, \tag{A12}$$

where ${}^{(s)}F^{lm}(\Omega)$ for $s=0,1,2$ is defined by

$${}^{(s)}F^{lm}(\Omega)=\frac{4\pi i}{(2\pi)^4}\int_{k_1}^{2k_0-k_1}[V(|\vec{k}|)]^s G(k)G(\Omega-k)\times Y_{lm}(\hat{k})Y^*_{lm}(\hat{k})\,dk^4\,. \tag{A13}$$

The above cutoff has been discussed in the text around Eq. (19). We observe that the expression ${}^{(2)}F^{lm}(\Omega)$ in (A12) only exists without employing any cutoff, if $\lambda^{(l)}(|\vec{k}|,|\vec{k}|)$ falls off faster than $1/k$. Thus in the RZ model (δ-function interaction) one cannot even obtain such an expression for the binding energy without employing a cutoff.

Integrating (A12) over $k_1^2dk_1$ leads to the quantity

$$G_2^{lm}(\Omega)=2\,{}^{(0)}F^{lm}(\Omega)+2g^{(l)}({}^{(1)}F^{lm}(\Omega))^2/[1-g^{(l)}\,{}^{(2)}F^{lm}(\Omega)] \tag{A14}$$

actually investigated by RZ. We see that the momentum cutoff becomes absolutely necessary, independent of the form of the potential employed. With the above cutoff the RZ result is finally obtained by assuming a δ-function interaction, i.e., $V(k)\equiv 1$. Eq. (A13) gives

$$F(\Omega)\equiv{}^{(s)}F^{lm}(\Omega)\quad\text{for } s=0,1,2, \tag{A15}$$

and thus (A14) leads to

$$G_2^{lm}(\Omega)=2F(\Omega)/[1-g^{(l)}F(\Omega)]\quad\text{for } l=0,2,4,\ldots\,. \tag{A16}$$

For the case where $M_l>1$ we shall drop the subscripts lm for simplicity and write p and p' for $|\vec{k}_1|$ and $|\vec{k}_4|$, respectively.

From (A8) and (A10)

$$G_2(p,\Omega)=G_2^{(0)}(p,\Omega)+\tfrac{1}{2}gG_2^{(0)}(p,\Omega)\sum_{i=1}^{N}\alpha_i f_i(p)\times\int_0^\infty f_i(p')G_2(p',\Omega)p'^2\,dp'. \tag{A17}$$

We define

$$A_i(\Omega)=\int_0^\infty f_i(p)G_2(p,\Omega)p^2\,dp \tag{A18}$$

and rewrite (A17)

$$G_2(p,\Omega) = G_2^{(0)}(p,\Omega) + \tfrac{1}{2} g G_2^{(0)}(p,\Omega) \sum_{i=1}^{N} \alpha_i f_i(p) A_i(\Omega). \tag{A19}$$

Inserting (A19) into (A18) leads to N coupled algebraic linear equations for the N unknowns $A_i(\Omega)$ of the form

$$A_i(\Omega) = \int_0^\infty f_i(p) G_2^{(0)}(p,\Omega) p^2\, dp + \tfrac{1}{2} g \sum_{j=1}^{N} \alpha_j A_j(\Omega) \times \int_0^\infty f_i(p) G_2^{(0)}(p,\Omega) f_j(p) p^2\, dp, \tag{A20}$$

where

$$\tfrac{1}{2} G_2^{(0)}(p,\Omega) = [4\pi/(2\pi)^3](\Omega - 2\epsilon_p + i\Gamma)^{-1}. \tag{A21}$$

Equation (A20) can be written in a more comprehensive way by defining the following quantities:

$$F_i(\Omega) \equiv \frac{4\pi}{(2\pi)^3} \int_0^\infty \frac{f_i(p) p^2\, dp}{\Omega - 2\epsilon_p + i\Gamma}, \tag{A22}$$

$$g\alpha_j F_{ij}(\Omega) \equiv g\alpha_j \frac{4\pi}{(2\pi)^3} \int_0^\infty \frac{f_i(p) f_j(p) p^2\, dp}{\Omega - 2\epsilon_p + i\Gamma} \equiv R_{ij}(\Omega), \tag{A23}$$

and

$$\bar{a} \equiv A_i(\Omega), \quad \bar{F} \equiv F_i(\Omega), \quad R(\Omega) \equiv R_{ij}(\Omega). \tag{A24}$$

Equation (A20) then becomes

$$\bar{a} = 2\bar{F} + R(\Omega)\bar{a} \tag{A25}$$

and the binding energy is given by

$$\mathrm{Det}[I - R(\Omega)] = 0. \tag{A26}$$

For $N=1$ and $\alpha_1 = g^{(1)}/g$ (A26) leads to

$$1 - g^{(1)} \frac{4\pi}{(2\pi)^3} \int_0^\infty \frac{f_1^2(p) p^2\, dp}{\Omega - 2\epsilon_p + i\Gamma} \tag{A27}$$

in complete agreement with (A12) and (A16). In order to gain some familiarity with the general case we calculate (A26) for $N=3$ which contains already all the features of the general case that we are interested in. We arrive at

$$\begin{aligned} 0 = \mathrm{Det}[I - R(\Omega)] = &- R_{11} - R_{22} - R_{33} + (R_{22}R_{33} - R_{23}R_{32}) \\ &+ (R_{11}R_{22} - R_{12}R_{21}) + (R_{11}R_{33} - R_{13}R_{31}) + (R_{32}R_{32} - R_{22}R_{33})R_{11} \\ &+ (R_{21}R_{33} - R_{23}R_{31})R_{12} + (R_{31}R_{22} - R_{21}R_{32})R_{13}. \end{aligned} \tag{A28}$$

If we approximate $R_{ij}(\Omega)$ by choosing all $f_i(p) \equiv f_i(k_0)$ which together with the momentum cutoff represents the RZ model, (A28) becomes

$$0 = \mathrm{Det}[I - R(\Omega)] = 1 - R_{11} - R_{22} - R_{33}. \tag{A29}$$

All the brackets in (A28) are zero as can easily be seen with the definition (A23). Thus we obtain

$$0 = 1 - \frac{4\pi}{(2\pi)^3} \int_{k_0}^{2k_0 - k_1} \frac{p^2\, dp}{\Omega - 2\epsilon_p + i\Gamma}\, g \sum_{i=1}^{N=3} \alpha_i f_i^2(k_0). \tag{A30}$$

Above consideration can obviously be generalized to any N. In particular to $N=\infty$. Since

$$g \sum_{i=1}^{\infty} \alpha_i f_i^2(k_0) = g\lambda^{(1)}(k_0, k_0) = g^{(1)}, \tag{A31}$$

Eq. (A30) reads

$$1 - g^{(1)} F(\Omega) = 0, \tag{A32}$$

which leads to the same solution as (A16). This is expected because the formalism developed here is valid for any separable potential. The important point therefore is to estimate how much the terms in the brackets of (A28) deviate from zero, once $f_i(p)$ is not held constant any more. This has been done numerically. We used the separable form of $\lambda^{(2)}(|\vec{k}|, |\vec{p}|)$ given in Appendix B (B14). Calculating the determinant in (A28) to the same order of accuracy to which Wong and Huang calculated their excitation energy involves a 12×12 matrix for $R_{ij} \equiv gF_{ij}(\Omega)\alpha_j$. Since $\Gamma = 0.07\,^\circ$K and $2\epsilon_{k_0} - \Omega \cong 0.4\,^\circ$K we can neglect the imaginary part in our computation. Keeping the binding energy, $2\epsilon_{k_0} - \Omega \cong 0.4\,^\circ$K, constant we find the g that satisfies (A28) is about two times smaller than the one needed in the δ-function case (A29)–(A32) to satisfy (A32).

APPENDIX B

For the calculation of $\lambda^{(2)}(|\vec{k}|, |\vec{q}|) \equiv \lambda^{(2)}(k,q)$ in (21) we recall the following definitions[31]:

$$P_n^m(\mu) = (1-\mu^2)^{m/2} \frac{d^m}{d\mu^m} P_n(\mu), \quad \frac{d^m}{d\mu^m} P_n = T_{n-m}^m,$$

$$T_l^0 = P_l^0 = P_l, \quad P_0^0 = T_0^0 = T_0^1 = 1. \tag{B1}$$

The expressions $\cos(|\vec{k} - \vec{q}|a)$ and $j_1(|\vec{k} - \vec{q}|a)$ can then be decomposed[31] into

$$\cos(|\vec{k}-\vec{q}|a) = \sum_{l=0}^{\infty}(2l+1)T_l^0(\mu)j_l(ka)j_l(qa) - \frac{|\vec{k}-\vec{q}|^2}{kq}\sum_{l=0}^{\infty}(2l+3)T_l^1(\mu)j_{l+1}(ka)j_{l+1}(qa), \quad \text{(B2)}$$

$$j_1(|\vec{k}-\vec{q}|a) = \frac{|\vec{k}-\vec{q}|}{a\vec{k}\vec{q}}\sum_{l=0}^{\infty}(2l+3)T_l^1(\mu)j_l(ka)j_l(qa). \quad \text{(B3)}$$

Therefore the integral to be calculated is

$$2\lambda^{(2)}(k,q) = \int_{-1}^{1} d\mu\, P_2(\mu) \times\left(\cos(|\vec{q}-\vec{k}|a) - \frac{a\vec{k}\vec{q}}{|\vec{q}-\vec{k}|}j_1(|\vec{q}-\vec{k}|a)\right). \quad \text{(B4)}$$

With the substitution

$$|\vec{q}-\vec{k}| = (q^2+k^2-2qk\mu)^{1/2}, \quad \mu\equiv\cos\theta, \quad \text{(B5)}$$

and Eqs. (B2) and (B3) we arrive at

$$\int_{-1}^{1} d\mu\, P_2(\mu)\left(\sum_{l=0}(2l+1)T_l^0(\mu)j_l(ka)j_l(qa) - \frac{|\vec{k}-\vec{q}|^2}{kq}\sum_{l=0}(2l+3)T_l^1(\mu)j_{l+1}(ka)j_{l+1}(qa) - \frac{\vec{k}\vec{q}}{kq}\sum_{l=0}(2l+3)T_l^1(\mu)j_{l+1}(ka)j_{l+1}(qa)\right)$$

$$= \int_{-1}^{1} d\mu\, P_2(\mu)\left(\sum_0 (2l+1)P_l(\mu)j_l(ka)j_l(qa) + \frac{k^2+q^2-2kq\mu}{kq}\sum_{l=0}(2l+3)\frac{\partial}{\partial\mu}P_{l+1}(\mu)j_{l+1}(ka)j_{l+1}(qa) - P_1(\mu)\sum_{l=0}(2l+3)\frac{\partial}{\partial\mu}P_{l+1}(\mu)j_{l+1}(ka)j_{l+1}(qa)\right), \quad \text{(B6)}$$

$$= 2j_2(ka)j_2(qa) - \frac{k^2+q^2}{kq}\sum_{l=0}(2l+3)j_{l+1}(ka)j_{l+1}(qa)\int_{-1}^{1} d\mu\, P_2(\mu)\frac{\partial}{\partial\mu}P_{l+1}(\mu) + \sum_{l=0}(2l+3)j_{l+1}(ka)j_{l+1}(qa)\int_{-1}^{1} d\mu\, P_2(\mu)P_1(\mu)\frac{\partial}{\partial\mu}P_{l+1}(\mu). \quad \text{(B7)}$$

Partial integration then finally leads to a separable form given by

$$\frac{-2(k^2+q^2)}{kq}\sum_{l=1}(4l+3)j_{2l+1}(ka)j_{2l+1}(qa) + 2\sum_{l=1}(4l+1)j_{2l}(ka)j_{2l}(qa) - 4j_2(ka)j_2(qa). \quad \text{(B8)}$$

With the help of[32]

$$(Z)^{-1}j_{n+1}(Z) = (2n+3)^{-1}j_{n+2}(Z) + (2n+3)^{-1}j_n(Z), \quad \text{(B9)}$$

we transform the first term in (B8) into

$$\frac{-2a(k^2+q^2)}{q}\sum_{l=1}j_{2l+1}(qa)[j_{2l+2}(ka)+j_{2l}(ka)], \quad \text{(B10)}$$

$$\frac{-2a(k^2+q^2)}{q}\left(\sum_{l=0}j_{2l+1}(qa)[j_{2l+2}(ka)+j_{2l}(ka)] - j_1(qa)[j_2(ka)+j_0(ka)]\right). \quad \text{(B11)}$$

With the index transformation $l\to l+1$ (B11) can be rewritten

$$-\frac{2a(k^2+q^2)}{q}\left(\sum_{l=1}j_{2l-1}(qa)[j_{2l}(ka)+j_{2l-2}(ka)] - j_1(qa)[j_2(ka)+j_0(ka)]\right). \quad \text{(B12)}$$

Combining the $j_{2l}(ka)$ terms in (B11) and (B12) we obtain

$$-\frac{2a(k^2+q^2)}{q}\left(\sum_{l=1}j_{2l}(ka)[j_{2l-1}(qa)+j_{2l+1}(qa)] + j_1(qa)j_0(ka) - j_1(qa)[j_2(ka)+j_0(ka)]\right). \quad \text{(B13)}$$

Using (B9) again and inserting the result in (B8) we finally obtain an expression that essentially only contains spherical Bessel functions of even order:

$$-4j_2(ka)j_2(qa) + (2a/q)(k^2+q^2)j_1(qa)j_2(ka) - \frac{2k^2}{q^2}\sum_{l=1}(4l+1)j_{2l}(ka)j_{2l}(qa). \quad \text{(B14)}$$

Expression (B14) was used for numerical computations. With

$$j_0(|x\pm y|) = \sum_{l=0}(2l+1)(\mp 1)^l j_l(x)j_l(y), \quad \text{(B15)}$$

which leads to

$$\tfrac{1}{2}[j_0(|x+y|)+j_0(|x-y|)] = \sum_{l=0} (4l+1) j_{2l}(x) j_{2l}(y), \tag{B16}$$

$$\tfrac{1}{2}[j_0(|x-y|)-j_0(|x+y|)] = \sum_{l=0} (4l+3) j_{2l+1}(x) j_{2l+1}(y), \tag{B17}$$

equation (B8) can easily be transformed into the expression used in the text

$$\begin{aligned} 2\lambda^{(2)}(k,q) = &-4 j_2(ka) j_2(qa) \\ &+2\{\tfrac{1}{2}[j_0(|q+k|a)+j_0(|q-k|a)] \\ &-j_0(ka) j_0(qa)\} \\ &-2(k^2+q^2)/kq\,\{\tfrac{1}{2}[j_0(|q-k|a)-j_0(|q+k|a)] \\ &-3 j_1(ka) j_1(qa)\} \end{aligned} \tag{B18}$$

and in the case that $q=k$, we find

$$\begin{aligned} 2\lambda^{(2)}(k,k) = &-1+3 j_0(2ka) - 2 j_0^2(ka) \\ &+12 j_1^2(ka) - 4 j_2^2(ka). \end{aligned} \tag{B19}$$

APPENDIX C

Using the approximations discussed in Sec. IV, we shall calculate here the expression [see Eq. (43)]

$$\begin{aligned} \hat{F}_2(\vec{K},E) = &\frac{\pi}{(2\pi)^3}\int dp_1'^3\, \hat{V}^2(|\vec{p}-\vec{p}_1'|, \vec{p}_1\vec{p}_1') \\ &\times\delta(E_{\vec{p}_1'}+E_{\vec{K}+\vec{p}_1'}-E), \end{aligned} \tag{C1}$$

where as a matter of convenience we have used a slightly different notation for the potential, given by

$$\begin{aligned} \hat{V}(|\vec{p}_1-\vec{p}_1'|, \vec{p}_1\vec{p}_1') = g[&\cos(|\vec{p}_1-\vec{p}_1'|a) \\ &-(a\vec{p}_1\vec{p}_1'/|\vec{p}_1-\vec{p}_1'|) j_1(|\vec{p}_1-\vec{p}_1'|a)]. \end{aligned} \tag{C2}$$

Since $|\vec{p}_1| = k_0$ in our problem, we obtain for zero momentum exchange

$$\hat{V}_0 \equiv \hat{V}(0,k_0^2) = g(1-\tfrac{1}{3}a^2k_0^2). \tag{C3}$$

Figures 3(a) and 3(b) define the symbols we shall use during our computation. With $dp_1'^3 = p_1'^2\, dp_1'\, d\cos\theta\, d\varphi$ and the usual approximation

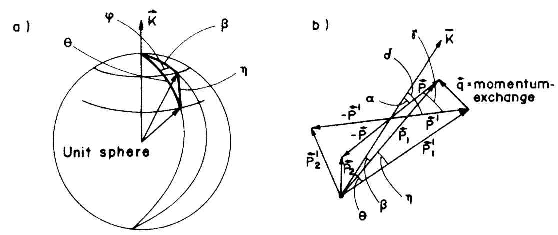

FIG. 3. (a) and (b) Illustrations to geometrical identities used in the text.

$$E_k = \Delta_0 + (k-k_0)^2/2\mu_0 \tag{C4}$$

for the excitation spectrum in the roton region we obtain for Eq. (C1), using the notation $|\vec{p}_1| \equiv p_1$, etc.,

$$\begin{aligned} \hat{F}_2 = &\frac{2\mu_0}{(2\pi)^3}\,\pi\int p_1'^2 dp_1' d\cos\theta\, d\varphi\, \hat{V}^2(|\vec{p}_1-\vec{p}_1'|, \vec{p}_1\vec{p}_1') \\ &\times\delta(2\mu_0(2\Delta_0-E)+(p_1'-k_0)^2 \\ &+[(K^2+p_1'^2-2Kp_1'\cos\theta)^{1/2}-k_0]^2). \end{aligned} \tag{C5}$$

For the $d\cos\theta$ integration we follow Solona *et al.*[26] (Appendix A) and obtain

$$\begin{aligned} \hat{F}_2^{\pm} = &\frac{\mu_0\pi}{K(2\pi)^3}\int_{p_{1\,\min}'}^{p_{1\,\max}'} d\varphi\, dp_1'\, p_1'\, \hat{V}^2(|\vec{p}_1-\vec{p}_1'|, \vec{p}_1\vec{p}_1') \\ &\times|1 \pm k_0[2\mu_0(E-2\Delta_0)-(p_1'-k_0)^2]^{-1/2}|, \end{aligned} \tag{C6}$$

with the restrictions

$$\left.\begin{matrix} p_{1\min}' \\ p_{1\max}' \end{matrix}\right\} \equiv k_0 \mp \alpha^{1/2}, \quad \alpha \equiv 2\mu_0(E-2\Delta_0), \tag{C7}$$

$$2k_0 - [4\mu_0(E-2\Delta_0)]^{1/2} > K > [4\mu_0(E-2\Delta_0)]^{1/2}. \tag{C8}$$

The direction of $\vec{p}_1'$ in $\hat{V}^2$ of (C6) is now, as far as the angle θ is concerned a function of K, p_1', and E. For $p_{1\min}' < p_1' < p_{1\max}'$ the inequality

$$1 < k_0|[2\mu_0(E-2\Delta_0)-(p_1'-k_0)^2]^{-1/2}| \tag{C9}$$

holds. Thus with

$$\hat{F}_2 = \hat{F}_2^+ + \hat{F}_2^- \tag{C10}$$

we arrive at

$$\hat{F}_2 = 2k_0\frac{\mu_0\pi}{K(2\pi)^3}\int_{p_{1\min}'}^{p_{1\max}'} d\varphi\, dp_1'\, p_1'\frac{\hat{V}^2(|\vec{p}_1-\vec{p}_1'|, \vec{p}_1\vec{p}_1')}{[2\mu_0(E-2\Delta_0)-(p_1'-k_0)^2]^{1/2}}. \tag{C11}$$

Now we investigate the $d\varphi$ integration. From Figs. 3(a) and 3(b) we have

$$q^2 = p_1'^2 + p_1^2 - 2p_1p_1'\cos\eta, \tag{C12}$$

$$\cos\eta = \cos\theta\cos\beta + \sin\theta\sin\beta\cos\varphi, \tag{C13}$$

and defining

$$A=-2p_1p_1'\sin\theta\sin\beta, \tag{C14}$$

$$B=p_1^2+p_1'^2-2p_1p_1'\cos\theta\cos\beta, \tag{C15}$$

we obtain emphasizing the φ dependence only

$$q^2=A\cos\varphi+B. \tag{C16}$$

Thus (C11) becomes

$$\hat{F}_2=\frac{2k_0\mu_0\pi}{K(2\pi)^3}\int_{p'_{1\min}}^{p'_{1\max}}d\varphi\,dp_1'\,p_1'\frac{\hat{V}^2[(A\cos\varphi+B)^{1/2},\vec{p}_1\vec{p}_1']}{[2\mu_0(E-2\Delta_0)-(p_1'-k_0)^2]^{1/2}}. \tag{C17}$$

As pointed out in Sec. IV (36) it is a good approximation to choose

$$E\gtrsim 2\Delta_0\equiv 2E(k_0), \tag{C18}$$

with (C7), this immediately leads to

$$|\vec{p}_1'|\cong k_0. \tag{C19}$$

The dp_1' integration can be performed easily by noticing that for any function $f(p_1')$ that is well behaved at $p_1'\cong k_0$. We have

$$\lim_{\alpha=0}\int_{k_0-\alpha^{1/2}}^{k_0+\alpha^{1/2}}\frac{f(p_1')\,dp_1'}{[2\mu_0(E-2\Delta_0)-(p_1'-k_0)^2]^{1/2}}=\pi f(k_0), \tag{C20}$$

therefore (C17) becomes

$$\hat{F}_2=\frac{2k_0^2\mu_0\pi^2}{K(2\pi)^3}\int_{-\pi}^{\pi}d\varphi\,\hat{V}^2[(A\cos\varphi+B)^{1/2},\vec{p}_1\vec{p}_1']. \tag{C21}$$

Furthermore, inserting (C18) and (C19) into the δ function in (C5) we obtain

$$\cos\theta\cong K/2k_0. \tag{C22}$$

With (C19) and (C22) we learn from Fig. 3(b) that the angle $\delta=\frac{1}{2}\pi$. And since $|\vec{p}_1|=k_0$ we also have $\alpha=\frac{1}{2}\pi$, implying $\beta\equiv\theta$. From these facts, we immediately obtain

$$\vec{p}_1\vec{p}_1'=\tfrac{1}{4}K^2+p^2\cos\varphi, \tag{C23}$$

where $p\equiv|\vec{p}|$ is the relative momentum defined in Fig. 3(b). Furthermore A and B under these conditions, $\delta=\alpha=\frac{1}{2}\pi$ and $\beta\equiv\theta$ are given by

$$A=-2p^2,\quad B=2p^2, \tag{C24}$$

leading to a momentum exchange

$$(A\cos\varphi+B)^{1/2}=2p\sin\tfrac{1}{2}\varphi, \tag{C25}$$

with (C3), (C23), and (C25) we obtain for (C21)

$$\hat{F}_2=\frac{4k_0^2\mu_0\pi^2}{K(2\pi)^3}\hat{V}_0^2\times\int_0^{\pi}d\varphi\left(\cos(2ap\sin\tfrac{1}{2}\varphi)^2-\frac{a^2(\frac{1}{4}K^2+p^2\cos\varphi)}{2ap\sin\frac{1}{2}\varphi}j_1(2ap\sin\tfrac{1}{2}\varphi)\right)^2\times(1-\tfrac{1}{3}a^2k_0^2)^{-2}. \tag{C26}$$

It is nice to observe that for the hard-core parameter $a\to 0$ (δ-function interaction) (C26) becomes

$$\hat{F}_2=[4k_0^2\mu_0\pi^3/K(2\pi)^3]\hat{V}_0^2, \tag{C27}$$

which is identical with expression (44). By comparison of integral (C26) with Fig. 3(b) we see that

$$\lim_{\epsilon=0}\int_0^{\epsilon}d\varphi[\quad]^2$$

and

$$\lim_{\epsilon=0}\int_{\pi-\epsilon}^{\pi}d\varphi[\quad]^2$$

constitute the contributions of the terms $V^{\rm HF}(0,\vec{q})$ and $V^{\rm HF}(\vec{k},\vec{q})$ and therefore have the measure zero as mentioned in Sec. IV. We proceed with an approximate $d\varphi$ integration in order to obtain a qualitative picture of $\hat{F}_2$. Setting

$$ap\varphi=x_1,\quad \sin\tfrac{1}{2}\varphi\cong\tfrac{1}{2}\varphi,\quad j_1(x)/x\cong\tfrac{1}{3}j_0(x), \tag{C28}$$

we arrive at

$$\int_0^{\pi}d\varphi\cdots=\frac{(1-\frac{1}{3}a^2k_0^2)^{-2}}{ap}\int_0^{\pi ap}dx\{\cos x-\tfrac{1}{3}j_0(x)a^2[\tfrac{1}{4}K^2+p^2\cos(x/ap)]\}^{-2}. \tag{C29}$$

Since $a=2.17$ Å and $p=\sin\beta k_0$ we can put for a relatively wide range of β, $\cos(x/ap)\cong 1$ as long as $x<\pi$. With $(\frac{1}{2}K)^2+p^2=k_0^2$ we obtain therefore

$$\int_0^{\pi} d\varphi \cdots = \frac{(1-\frac{1}{3}a^2k_0^2)^{-2}}{ap} \int_0^{\pi a p} dx\,[\cos x - \tfrac{1}{3}j_0(x)a^2k_0^2]^2, \tag{C30}$$

$$= \frac{(1-\frac{1}{3}a^2k_0^2)^{-2}}{ap} \int_0^{\pi a p} dx \left(\cos^2 x + \frac{a^4k_0^4}{9} j_0^2(x) - \tfrac{2}{3}a^2k_0^2 \cos(x) j_0(x)\right). \tag{C31}$$

A good approximation of above integral for a wide range of β is

$$\frac{(1-\frac{1}{3}a^2k_0^2)^{-2}}{ap}\left(\int_0^{\pi a p} dx \cos^2(x) - \tfrac{1}{2}\pi\tfrac{1}{3}a^2k_0^2(1-\tfrac{1}{3}a^2k_0^2)\right), \tag{C32}$$

with $k_0 = 1.95$ Å^{-1}, $a = 2.17$ Å, and $p \leq k_0$ the cos-integral in (C32) is practically negligible. A functionally good approximation of Eq. (C32) is therefore

$$\int_0^{\pi} d\varphi \cdots = \frac{3\pi}{5ap}. \tag{C33}$$

Defining $\alpha = \frac{10}{3}$ and substituting result (C33) together with α in expression (C26) we finally obtain

$$\hat{F}_2 = (\mu_0 k_0^2/\alpha a p K)\hat{V}_0^2. \tag{C34}$$

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