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Mass And Energy In General Relativity

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We consider the Denisov–Solov'ov example which shows that the inertial mass is not well defined in General Relativity. It is shown that the mathematical reason why this is true is a wrong application of the Stokes theorem. Then we discuss the role of the order of asymptotically flatness in the definition of the mass. In conclusion some comments on conservation laws in General Relativity are presented.

In any physical theory the notions of *mass* and *energy* play an important role. The related conservation laws are the corner stones of the theory. There is a huge amount of literature where authors describe these notions in the framework of the theory of General Relativity (GR) (see for instance Refs. 1–3). However some papers contain a serious criticism and doubts whether the mass and energy are well defined in GR, despite compatiblity at first sight with the theory of Special Relativity. The biggest criticism comes from Logunov and his collaborators [4]. The essential point of their argument is the following.

The *equivalence principle* states that the gravitational mass and the inertial mass are equal. It is a fundamental law in physics. Logunov and Mestvirishvili [4] agree that the gravitational mass is well defined in GR. However they point out that in GR there is no satisfactory definition of the

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inertial mass. The latter usually is defined in the following way (Refs. 5,4; in some parts our exposition is close to these works).

Let $T^{\mu\nu}$ be the energy-momentum tensor and $G^{\mu\nu}$ the Einstein tensor. The Einstein equations read

$$G^{\mu\nu} = T^{\mu\nu}. \tag{1}$$

Then fix a basis e_ν of 1-form fields and let $J^\mu = T^{\mu\nu} e_\nu$ and $G^\mu = G^{\mu\nu} e_\nu$ be respectively the energy-momentum and Einstein 1-form fields. The contracted second Bianchi identities $D * G^\mu = 0$ and Einstein equations (1) imply

$$D * J^\mu = d * J^\mu + \omega^\mu_\nu \wedge *J^\nu = 0\,, \tag{2}$$

where (ω^μ_ν) is the matrix of the connection 1-forms of the Levi-Civita connection D and $*$ is the Hodge star operator.

Now one looks for a "1-form" τ^μ such that

$$d * \tau^\mu = \omega^\mu_\nu \wedge *J^\nu. \tag{3}$$

From eqs. (2) and (3) we have the following conservation laws:

$$d * (J^\mu + \tau^\mu) = 0\,. \tag{4}$$

From (4) one concludes that there is an exact 3-form $-d * S^\mu$ such that (see Ref. 5)

$$*J^\mu + *\tau^\mu = -d * S^\mu. \tag{5}$$

The latter conclusion is not true for arbitrary 4-manifold since its third de Rham cohomology group could not be zero. In particular, this invalidates Thirring–Wallner's proof [5] that for a closed universe (with topology $\mathbb{R} \times S^3$) the total energy must be zero, since $H^3(S^3) = \mathbb{R}$ is non-trivial. However, we agree with eq. (5) if we are in $\mathbb{R}^4$ where every closed differential form is actually exact.

Further one integrates eq. (5) over a "certain finite three-dimensional volume", say a ball B, and then by the Stokes theorem

$$\int_B (*J^\mu + *\tau^\mu) = -\int_{\partial B} *S^\mu. \tag{6}$$

If we express $*S^\mu$ in eq. (6) in terms of a metric g_{ij}, the (inertial) mass is given by

$$m_i = \lim_{R\to\infty} \int_{\partial B} *S^0 = -\frac{1}{16\pi} \lim_{R\to\infty} \int_{\partial B} \frac{\partial}{\partial x^\beta} [g_{11} g_{22} g_{33} g^{\alpha\beta}] d\sigma_\alpha\,, \tag{7}$$

where $\partial B = S^2(R)$ is a 2-sphere of radius R, $(-1)n_\alpha$ its outward unit normal and $d\sigma_\alpha = -R^2 n_\alpha dA$. If the metric g_{ij} is asymptotically flat (see below), then eq. (7) is equivalent to

$$m_i = -\frac{1}{16\pi} \lim_{R\to\infty} \int_{S^2(R)} \sum_{\mu,\nu=1}^{3} \left(\frac{\partial g_{\mu\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\mu}}{\partial x^\nu} \right) d\sigma_\nu \,. \tag{8}$$

Logunov and Mestvirishvili claim that the inertial mass defined above depends on the spatial co-ordinates and therefore has no physical meaning. Indeed, Denisov and Solov'ov [6] (see also Ref. 4) have found an explicit change of variables for the Schwarzschild metric such that the mass in the new co-ordinates has a different value. Namely, consider the Schwarzschild metric in its usual form:

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin\theta^2 d\varphi^2). \tag{9}$$

Then introduce Cartesian co-ordinates x_C^μ. The Schwarzschild metric becomes

$$ds^2 = g_{00} dt^2 + g_{\alpha\beta}\, dx_C^\alpha\, dx_C^\beta, \tag{10}$$

where

$$g_{00} = \frac{[1-(2m/4r)]^2}{[1+(2m/4r)]^2}\,, \qquad g_{\alpha\beta} = -\delta_{\alpha\beta}\left(1 + \frac{2m}{4r}\right)^4 . \tag{11}$$

Now make another change of the spatial co-ordinates,

$$x_C^\mu = x_N^\mu (1 + f(r_N))\,, \tag{12}$$

where $r_N = ((x_N^1)^2 + (x_N^2)^2 + (x_N^3)^2)^{1/2}$, f is an appropriate function such that f and f' have good behaviour at infinity and the Jacobian is positive. In fact

$$f(y) = a^2 \left(\frac{8m}{y}\right)^{1/2} (1 - \exp[-y]\,)\,, \tag{13}$$

where a is a non-zero constant. Clearly

$$f(y) \geq 0\,, \qquad \lim_{y\to\infty} f(y) = 0\,, \qquad \lim_{y\to\infty} y f'(y) = 0$$

and the Jacobian is greater than 1. After this change the metric has the form

$$g_{00} = \left(1 - \frac{2m}{4r_N(1+f)}\right)^2 \left(1 + \frac{2m}{4r_N(1+f)}\right)^{-2} \tag{14}$$

$$g_{\alpha\beta} = -\left(1 - \frac{2m}{4r_N(1+f)}\right)^4 \Big[\delta_{\alpha\beta}(1+f)^2 + x_\alpha^N x_\beta^N \Big[(f')^2 + \frac{2}{r_N} f'(1+f)\Big]\Big] \tag{15}$$

and the inertial mass m_i' in the new spatial co-ordinates x_N^μ is

$$m_i' = m(1+a^4).$$

Therefore the inertial mass changes and depends on the spatial co-ordinates even in the case of the well known Schwarzschild metric. Of course, this is not like the velocity dependence of mass in Special Relativity.

We have checked the above calculations and confirm their correctness. Here we would like to point out the mathematical reason why they are right, which seems to be unclear even to the authors that proposed this example. First let observe that the definition of τ^μ and $d * S^\mu$, $\mu = 0,1,2,3$, as can be seen from (3) and (5), depends upon a choice of basis $\{e_\nu,\ \nu = 0,1,2,3\}$ of 1-form fields. Hence it follows that τ^μ and $d * S^\mu$, $\mu = 0,1,2,3$, are not tensors. Above one applies the Stokes (Ostrogradskii–Gauss–Green) theorem to each one of the objects $d * S^\mu$, $\mu = 0,1,2,3$, in order to express $\int_B d * S^\mu$ as $\int_{\partial B} *S^\mu$. But the Stokes theorem concerns differential forms, which are anti-symmetric tensors. Therefore it cannot be applied to non-tensorial quantities like $d * S^\mu$, $\mu = 0,1,2,3$ and in particular to $d * S^0$.

The situation can be also viewed in local co-ordinates. Indeed, one has a decomposition of the energy-momentum tensor into two non-covariant quantities,

$$T^{\mu\nu} = \partial_\sigma h^{\mu\nu\sigma} + t^{\mu\nu},$$

where $t^{\mu\nu}$ is the so-called energy-momentum pseudo-tensor of the gravitational field [1,4]. In our case the corresponding symmetric "object" $t^{\mu\nu}$ is called the Landau–Lifshitz pseudo-tensor. In fact, there are many energy-momentum pseudo-tensors (see Refs. 1 or 4), which we shall not introduce here. Then the inertial mass

$$m_i = \lim_{R\to\infty} \int_{\partial B} h^{00\alpha} d\sigma_\alpha\,,$$

which is obtained by the Stokes theorem [as in eq. (6) and eq. (7)]. Actually, (7) is a consequence of the last formula. It is again clear that $\tau^0 = t^{0\nu} e_\nu$ and $d * S^0$, where

$$S_0 = \frac{1}{2} \frac{1}{2(-g)} g_{0\tau} [g(g^{\nu\tau} g^{\rho\beta} - g^{\rho\tau} g^{\nu\beta}]_{,\beta} e_\nu \wedge e_\rho$$

(see Ref. 5), do not transform as tensors. Compare this with a paragraph in the Ref. 1, p.465, where a comment correctly notes that all objects like τ^μ are co-ordinate dependent: "All the quantities $H^{\mu\alpha\nu\beta}$, $T^{\mu\nu}_{\text{eff}}$ and $t^{\mu\nu}$ depend for their definition and existence on the choice of co-ordinates; they have no existence independent of co-ordinates; they are not components of tensors or of any other geometric object. Correspondingly, the equations (20.14) and (20.19) involving $T^{\mu\nu}_{\text{eff}}$ and $t^{\mu\nu}$ have no geometric, co-ordinate-free significance; they are not "covariant tensor equations"." However, the comments that follow are wrong, because they claim that the integral given by our equation (8) is co-ordinate independent. The Denisov–Solov'ov–Logunov–Mestvirishvili example shows the opposite.

Another purpose of the present paper is to discuss how the definition of the mass notion is intimately related with the concept of asymptotically flat metric. The definition actually states two things: (i) existence of special co-ordinates and (ii) the behaviour of the metric at infinity is of the form

$$g = \delta + O(r^{-k}). \tag{16}$$

In the earlier paper [7] by Schoen and Yau, $k = 2$, while in the next one [8] $k = 1$, a weaker condition. Note that Schoen and Yau consider metrics on three-dimensional manifolds. Therefore in (16) by g we mean the spatial part $g_{\alpha\beta}$ of a Lorentzian metric, which in the considered example is negative-definite. The latter resulted in the minus sign in the definition of m_i.

In all papers on the positive mass conjecture (e.g. Refs. 7,8), the mass is defined for asymptotically flat metrics and then it is shown that it is non-negative if the scalar curvature is non-negative. However, Bando et al. [9] emphasize that it is not absolutely certain that the mass (= inertial mass, note) is not independent of the co-ordinates. They prove a sufficient condition for the existence of asymptotically flat co-ordinates. They also mention a paper by Bartnik [10], where he proves that the mass is a "geometric invariant". In view of the example above this is not true. And anyway, we would say, it should be independent only of the asymptotically flat co-ordinates, in which it is actually defined. Bartnik himself cites the paper by Denisov and Solov'ov [6] saying that they have found an extremal example. In fact, we see that what happens essentially depends on the order k of asymptotical flatness. If $k > \frac{1}{2}$, then $m'_i = m$, if $k < \frac{1}{2}$ then m'_i is infinite. In the new co-ordinates in the example considered $k = \frac{1}{2}$ [see (14) and (15)]. Therefore these are not the asymptotically flat co-ordinates used by Schoen and Yau, who need co-ordinates in which the metric has $k = 1$. Hence we are *not* saying that Schoen-Yau result is not true. Simply, what they call mass is not good for physics since it depends on the spatial co-ordinates, and the co-ordinates do not have physical meaning. The above problems cannot be overcome also by the global definition of asymptotical flatness [2,11]. Indeed, problem 2 of Ref. 2, p.295, reduces the definition of asymptotically flatness at spatial infinity to the class of metrics which satisfy (16) with $k = 1$. Then the definition of (inertial) mass (Ref. 2, p.293), is the same as that given by the integral (8), up to the sign convention we have already mentioned. It is also worth emphasizing that LeBrun's counter example to the generalized positive action conjecture [12] provides a good metric with negative mass. We think that this is quite significant.

Before concluding this short note, we would like to comment briefly on the conservation laws in General Relativity (GR). In his paper [13] Dalton arrives at the right conclusion that in GR we can have only conservation in *infinitesimal* regions of the spacetime and that this conservation is expressed by the vanishing of the covariant derivative of the energy-momentum tensor. What surprises is that, according to [13], there is not a real problem in GR due to the lack of integral conservation laws for energy-momentum and angular momentum in this theory. This point has been emphasized by Vargas and Torr [14]. They use vector-valued differential forms and correctly obtain the result that local conservation of the vector-valued differential forms $\Pi = \Pi^\mu e_\mu$ is represented by the vanishing of the exterior covariant derivative of the form Π^μ, i.e., $D\Pi^\mu = d\Pi^\mu + \omega^\mu_\nu \wedge \Pi^\nu = 0$. Another analysis of the possibility for genuine conservation laws in a general field theory where gravitation (and possible other fields) are geometrized has been proposed by Benn [15]. However, using the words of Ferraris and Francaviglia [16], the problems of conserved quantities are "problems still to be satisfactorily solved in General Relativity" [16]. In such attempts one must always keep in mind that global conservation laws for energy, momentum and angular momentum depend on the existence of appropriate Killing vector fields in the spacetime manifold. Such vector fields in general do not exist in an arbitrary Lorentzian (or Riemann–Cartan) manifold.

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