Antonio Mario Sette
Elias H. Alves

ON THE EQUIVALENCE BETWEEN SOME SYSTEMS OF NON-CLASSICAL LOGIC

In [1], Loparić and da Costa define three systems of propositional logic, called β_0, β_1 and β_2.

In β_0 neither the principle of excluded middle $(A \vee \neg A)$, nor the principle of non-contradiction $(\neg(A \wedge \neg A))$ is valid, in general.

System β_1 is an extension of β_0, where the principle of non-contradiction is valid, but the principle of excluded-middle is not.

System β_2 is also an extension of β_0, where the principle of excluded middle is valid, but the principle of non-contradiction is not.

Systems such as β_2 are called *paraconsistent systems.*

Systems such as β_1 are called by Loparić and da Costa *paracomplete systems.*

In [1] it is mentioned that system β_2 is equivalent to system P_1, introduced by Sette in [3].

On the other hand, in [4], Sette and Carnielli study a system, called I_1, which is, according to them, *weakly-intuitionistic*, that is, where the law of excluded middle cannot be proved. (This corresponds to the notion of *paracompleteness* of Loparić and da Costa.)

According to Sette and Carnielli, system I_1 is a counterpart of the paraconsistent calculus P_1.

We will show, here, that system β_2 is, in fact, equivalent to P_1. (The proof of this fact appears in [2].) In addition, we will show that β_1 is equivalent to I_1.

The system β_1

The postulates of β_1 are the following:

1) $A \to (B \to A)$
2) $(A \to B) \to ((A \to (B \to C)) \to (A \to C))$

3) $A, A \to B/B$
4) $((A \to B) \to A) \to A$
5) $(A \wedge B) \to A$
6) $(A \wedge B) \to B$
7) $A \to (B \to (A \wedge B))$
8) $A \to (A \vee B)$
9) $B \to (A \vee B)$
10) $(A \to C) \to ((B \to C) \to ((A \vee B) \to C))$
11) $(\neg A \to B) \to ((\neg A \to \neg B) \to A)$, where A is molecular.

THEOREM 1. *If A is a classical tautology and we replace its propositional variables by molecular formulas, obtaining the formula A', then A' is provable in β_0 (and, therefore, in β_1 and β_2). (See [1], p. 75.)*

The system β_2

The postulates of β_2 are the same, 1 to 10, of β_1 plus the following:
11) $(\neg A \to B) \to ((\neg A \to \neg B) \to A)$, where B is molecular.

The system P_1

The postulates of P_1 are:
1) $A \to (B \to A)$
2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
3) $(\neg A \to \neg B) \to ((\neg A \to \neg\neg B) \to A)$
4) $(A \to B) \to \neg\neg(A \to B)$
5) $A, A \to B/B$.

THEOREM 2. *P_1 is complete relative to the following matrix:*
$\mathcal{M} =< \{T_0, T_1, F\}, \{T_0, T_1\}, \to, \neg >$, where $\{T_0, T_1\}$ are the distinguished values and $\to, \neg$ are defined by the tables:

$\to$	T_0	T_1	F
T_0	T_0	T_0	F
T_1	T_0	T_0	F
F	T_0	T_0	T_0

$\neg$	
T_0	F
T_1	T_0
F	T_0

PROOF. See [3], pp. 176–178.

The connectives $\wedge$ and $\vee$ are introduced by the following definitions:

$(A \wedge B) =_{df} (((A \to A) \to A) \to \neg((B \to B) \to B)) \to \neg(A \to \neg B)$

$(A \vee B) =_{df} (A \to \neg\neg A) \to (\neg A \to B)$

THEOREM 3. *In P_1 all the theorems and rules of positive classical logic are valid.*

PROOF. Using the characteristic matrix of P_1, defined in theorem 2.

The system I_1

The postulates of I_1 are:

1) $A \to (B \to A)$
2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$
3) $(\neg\neg A \to \neg B) \to ((\neg\neg A \to B) \to \neg A)$
4) $\neg\neg(A \to B) \to (A \to B)$
5) $A, A \to B/B$

THEOREM 4. *(Law of non-contradiction, negative form):*

$$\vdash_{I_1} (\neg A \to \neg B) \to ((\neg A \to B) \to \neg\neg A)$$

(see [4], p. 5).

THEOREM 5. *I_1 is complete relative to the following matrix:*

$\mathcal{M}' =< \{T, F_0, F_1\}, \{T\}, \to, \neg >$, where T is the only distinguished value and $\to, \neg$ are defined by the tables:

$\to$	T	F_1	F_0
T	T	F_0	F_0
F_1	T	T	T
F_0	T	T	T

$\neg$	
T	F_0
F_1	F_0
F_0	T

PROOF. See [4], pp. 11–15.

The connectives $\wedge$ and $\vee$ are introduced by the following definitions:

$(A \wedge B) =_{df} \neg(((A \to A) \to A) \to \neg((B \to B) \to B))$

$(A \vee B) =_{df} (\neg(B \to B) \to B) \to ((A \to A) \to A)$.

THEOREM 6. *In I_1 all the theorems and rules of positive classical logic are valid.*

PROOF. Using the characteristic matrix of I_1, defined in theorem 5.

We now show that β_2 is equivalent to P_1 and β_1 is equivalent to I_1.

β_2 equivalent to P_1

In view of theorem 3, it is enough to establish the following results:

$\vdash_{\beta_2} (A \rightarrow B) \rightarrow \neg\neg(A \rightarrow B)$.

PROOF. Consequence of theorem 1.

$\vdash_{\beta_2} (\neg A \rightarrow \neg B) \rightarrow ((\neg A \rightarrow \neg\neg B) \rightarrow A)$

PROOF. By axiom 11 of β_2.

$\vdash_{P_1} (\neg A \rightarrow B) \rightarrow ((\neg A \rightarrow \neg B) \rightarrow A)$, where B is molecular.

PROOF.
1) If B is $\neg C$, we have: $\vdash_{P_1} (\neg A \rightarrow \neg C) \rightarrow ((\neg A \rightarrow \neg\neg C) \rightarrow A)$, as a consequence of axiom 3 of P_1.
2) If B is $(C \rightarrow D)$, we need to prove that:
$\vdash_{P_1} (\neg A \rightarrow (C \rightarrow D)) \rightarrow ((\neg A \rightarrow \neg(C \rightarrow D)) \rightarrow A)$.

PROOF.
1) $\neg A \rightarrow (C \rightarrow D)$ Hyp.
2) $\neg A \rightarrow \neg(C \rightarrow D)$ Hyp.
3) $(C \rightarrow D) \rightarrow \neg\neg(C \rightarrow D)$ Ax. 5.
4) $\neg A \rightarrow \neg\neg(C \rightarrow D)$ By 1 and 3.
5) A By 2 and 4, Ax. 3.

β_1 equivalent to I_1

In view of theorem 6, it is enough to establish the following results:

$\vdash_{\beta_1} (\neg\neg A \rightarrow \neg B) \rightarrow ((\neg\neg A \rightarrow B) \rightarrow \neg A$.

PROOF. By axiom 11 of β_1.

$\vdash_{\beta_1} \neg\neg(A \to B) \to (A \to B)$.

PROOF. Consequence of theorem 1.

$\vdash_{I_1} (\neg A \to B) \to ((\neg A \to \neg B) \to A)$, where A is molecular.

PROOF.
1) If A is $\neg C$, we have: $\vdash_{I_1} ((\neg\neg C \to B) \to ((\neg\neg C \to \neg B) \to \neg C)$, as a consequence of axiom 3 of I_1.
2) If A is $(C \to D)$, we have:
$\vdash_{I_1} (\neg(C \to D) \to B) \to ((\neg(C \to D) \to \neg\neg(C \to D))$, by theorem 4. And, by axiom 4 of I_1, we obtain:
$\vdash_{I_1} (\neg(C \to D) \to B) \to ((\neg(C \to D) \to \neg B(C \to D))$.

References

[1] A. Loparić and C. A. da Costa, *Paraconsistency, Paracompleteness and Induction*, **Logique et Analyse** 113 (1986), pp. 73–80.

[2] E. G. Boscaino, **Os cálculos paraconsistentes** P_1 *e* β_2, Master's Thesis, Pontifícia Católica, São Paulo, Brazil, 1992.

[3] A. M. Sette, *On the propositional calculus* P_1, **Mathematica Japonicae**, vol. 18, no 3 (1973).

[4] A. M. Sette and W. A. Carnielli, *Maximal weakly-intuistionistic logics*, **Studia Logica** 55 (1995), pp. 181–203.

Center for Logic, Epistemology and History of Science
CLE/UNICAMP
P.O. Box 6133
13081–970 Campinas, S.P.
Brazil