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Algebraic and Dirac–Hestenes spinors and spinor fields

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Almost all presentations of Dirac theory in first or second quantization in physics (and mathematics) textbooks make use of covariant Dirac spinor fields. An exception is the presentation of that theory (first quantization) offered originally by Hestenes and now used by many authors. There, a new concept of spinor field (as a sum of nonhomogeneous even multivectors fields) is used. However, a careful analysis (detailed below) shows that the original Hestenes definition cannot be correct since it conflicts with the meaning of the Fierz identities. In this paper we start a program dedicated to the examination of the mathematical and physical basis for a comprehensive definition of the objects used by Hestenes. In order to do that we give a *preliminary* definition of algebraic spinor fields (ASF) and Dirac–Hestenes spinor fields (DHSF) on Minkowski space–time as some equivalence classes of pairs (Ξ_u, ψ_{Ξ_u}), where Ξ_u is a spinorial frame field and ψ_{Ξ_u} is an appropriate sum of multivectors fields (to be specified below). The necessity of our definitions are shown by a careful analysis of possible formulations of Dirac theory and the meaning of the set of Fierz identities associated with the bilinear covariants (on Minkowski space–time) made with ASF or DHSF. We believe that the present paper clarifies some misunderstandings (past and recent) appearing on the literature of the subject. It will be followed by a sequel paper where definitive definitions of ASF and DHSF are given as appropriate sections of a vector bundle called the *left* spin-Clifford bundle. The bundle formulation is essential in order to be possible to produce a coherent theory for the covariant derivatives of these fields on arbitrary Riemann–Cartan space–times. The present paper contains also Appendixes A–E which exhibits a truly useful collection of results concerning the theory of Clifford algebras (including many tricks of the trade) necessary for the intelligibility of the text. © *2004 American Institute of Physics.* [DOI: 10.1063/1.1757037]

I. INTRODUCTION

Physicists usually make first contact with Dirac spinors and Dirac spinor fields when they study relativistic quantum theory. At that stage they are supposed to have had contact with a good introduction to relativity theory and know the importance of the Lorentz and Poincaré groups. So, they are told that Dirac spinors are elements of a complex four-dimensional space $\mathbb{C}^4$, which are the carrier space of a particular representation of the Lorentz group. They are told that when you do Lorentz transformations Dirac spinors behave in a certain way, which is different from the way vectors and tensors behave under the same transformation. Dirac matrices are introduced as certain matrices on $\mathbb{C}(4)$ satisfying certain anticommutation rules and it is said that they close a particular Clifford algebra, known as Dirac algebra. The next step is to introduce Dirac wave functions. These are mappings, $\Psi:\mathcal{M}\rightarrow\mathbb{C}^4$, from Minkowski space–time $\mathcal{M}$ (at that stage often introduced as an affine space) to the space $\mathbb{C}^4$, which must have the structure of a Hilbert space. After that, Dirac equation, which is a first order partial differential equation is introduced for $\Psi(x)$. Physics come into play by interpreting $\Psi(x)$ as the quantum wave function of the electron. Problems with

[a)]Electronic mail: walrod@mpc.com.br or walrod@ime.unicamp.br this theory are discussed and it is pointed out that the difficulties can only be solved in relativistic quantum theory, where the Dirac spinor field, gains a new status. It is no more simply a mapping $\Psi:\mathcal{M}\rightarrow\mathbb{C}^4$, but a more complicated object [it becomes an operator valued distribution in a given Hilbert space (see, e.g., Ref. 162 for a correct characterization of these objects)] whose expectation values on certain one particle states can be represented by objects like Ψ. From a pragmatic point of view, only this knowledge is more than satisfactory. However, that approach, we believe, is not a satisfactory one to any scientist with an enquiring mind, in particular to one that is worried with the foundations of quantum theory. For such person the first questions which certainly occur are what is the geometrical meaning of the Dirac spinor wave function? From where did this concept come from?

Pure mathematicians, who study the theory of Clifford algebras, e.g., using Chevalley's classical books,[38,39] learn that spinors are elements of certain minimal *ideals* (do not worry if you did not know the meaning of this concept, it is not a difficult one and is introduced in Appendix B) in Clifford algebras. In particular Dirac spinors are the elements of a minimal ideal in a particular Clifford algebra, the Dirac algebra. Of course, the relation of that approach (*algebraic spinors*), with the one learned by physicists (covariant spinors) is known (see, e.g., Refs. 14, 67, and 68), but is not well known by the great majority of physicists, even for many which specialize in general relativity and more advanced theories, like string and M-theory.

Now, the fact is that the algebraic spinor concept (algebraic spinor fields on Minkowski space–time will be studied in details in what follows, and in Ref. 126 where the concept is introduced using fiber bundle theory on general Lorentzian manifolds) (as it is the case of the covariant spinor concept) fail to reveal the true geometrical meaning of spinor in general and Dirac spinors in particular.

In 1966, Hestenes[81] introduced a new definition of spinor field, that he called later *operator* spinor field. Objects in this class which in this paper, will be called Dirac–Hestenes spinor fields, have been introduced by Hestenes as mappings $\psi:\mathcal{M}\rightarrow\mathbb{R}^0_{1,3}$, where $\mathbb{R}^0_{1,3}$ is the even subalgebra of $\mathbb{R}_{1,3}$, a particular Clifford algebra, technically known as the *space–time* algebra. [$\mathbb{R}_{1,3}$ is not the original Dirac algebra, which is the Clifford algebra $\mathbb{R}_{4,1}$, but is closely related to it, indeed $\mathbb{R}_{1,3}$ is the even subalgebra of the Dirac algebra (see the Appendix B for details).] Hestenes in a series of remarkable papers[80,82–85,75] applied his new concept of spinor to the study of Dirac theory. He introduced an equation, now known as the Dirac–Hestenes equation, which does *not* contain (explicitly) imaginary numbers and obtained a very clever interpretation of that theory through the study of the geometrical meaning of the so-called bilinear covariants, which are the observables of the theory. He further developed an interpretation of quantum theory from his formalism,[88,89] that he called the *Zitterbewegung* interpretation. Also, he showed how his approach suggests a geometrical link between electromagnetism and the weak interactions, different from the original one of the standard model.[87]

Hestenes papers and his book with Sobczyk[86] have been the inspiration for a series of international conferences on "*Clifford Algebras and their Applications in Mathematical Physics*" which in 2002 has had its sixth edition. A consultation of the table of contents of the last two conferences[1,145,2] certainly will show that Clifford algebras and their applications generated a wider interest among many physicists, mathematicians, and even in engineering and computer sciences. (In what follows we quote some of the principal papers that we have had opportunity to study. We apologize to any author who thinks that his work is a worthy one concerning the subject and is not quoted in the present paper.) Physicists used Clifford algebras concepts and Hestenes methods, in many different applications. As some examples, we quote some developments in relativistic quantum theory as, e.g., Refs. 36, 37, 45, 46, 48–52, 56, 58, 74, and 70. The papers by De Leo and collaborators exhibit a close relationship between Hestenes methods and quaternionic quantum mechanics, as developed, e.g., by Adler,[4] a subject that is finding a renewed interest. Also, Clifford algebra methods have been used[102,135,149,151,152,165–168] to give an intuitive and geometrical clear picture of the dynamics of superparticles.[3,11,12,140,143,153,160,163] Also, that papers clarify the meaning of Grassmann variables and their calculus.[17] The relation with the *Zitterbewegung* model of Barut and collaborators[8–10] appears in a novel and less speculative way. Even more, in Ref. 151 it is shown that the concept of Dirac–Hestenes spinor field is closely related to the concepts of superfields as introduced by Witten.[169] Clifford algebras methods have also been used in disclosing a surprising connection between the Dirac and Maxwell and Seiberg–Witten[159] equations, as studied, e.g., in Refs. 155, 164, and 168, which suggest several physical developments. Applications of Clifford algebras methods in general relativity appeared also, e.g., in Refs. 35, 90, 54, 55, 57, 58, 62, 103–105, 119, 134, and 154, and suggest new ways for looking to the gravitational field. Clifford algebras methods, have been applied successfully also in quantum field theory, as, e.g., in Refs. 60 and 138 and more recently in string and p-brane theories, with noticeable results[25–34,136,137] which are worth being more carefully investigated.

Of course, Clifford algebras and Dirac operators are standard topics of research in Mathematics (see, e.g., Ref. 20), but we must say that Hestenes ideas have been an inspiring idea for mathematicians also. In particular, the concept of Clifford valued functions with domain in a manifold (the operator spinor fields are particular functions of this type) developed in a new, beautiful and powerful branch of mathematics.[47] Hestenes ideas, as we said, have found also their use in engineering and computer sciences, as in the study of neural circuits[91,92] and robotics and perception action systems.[18,19,99,100,42,59,101,125,161]

Having made all this propaganda, which we hope have awakened the reader's interest in studying Clifford algebras, we must remark, that (as often happens for every pioneer work) the concept of Dirac–Hestenes spinor field, as originally introduced by Hestenes, and used by many other researchers, is not a concept free of criticisms and objections from the mathematical point of view.

However, it is an important concept and one of the objectives of this paper and also of Ref. 126 is to give a presentation of the subject free of all previous criticisms, which are discussed in the next sections. The reader may ask if the enterprising for learning the theory presented below is worth the time. We think that the answer is yes, whether it be a physicist or mathematician. To encourage physicists, which may eventually become interested in the subject after reading the above propaganda, we say that the mathematical tools used, even if they may look complex at first sight, are indeed nothing more than easy additions to the contents of a linear algebra course. The main reward to someone that studies what follows is that they will start seeing some subjects that they thought were well known, under a new and (we believe) illuminating point of view. This hopefully may help anyone who is searching for new physical theories. For mathematicians, we say that the point of view developed here is somewhat new in relation to the original Chevalley's one and we believe, it is more satisfactory. In particular, the present paper serves as a preliminary step towards a rigorous theory of algebraic and Dirac–Hestenes spinor fields as sections of some well-defined fiber bundles, and the theory of the covariant derivatives of these fields. Having said all that, what is the present paper about?

We give definitions of algebraic spinor fields (ASF) and Dirac–Hestenes spinor fields (DHSF) living on Minkowski space–time and show how Dirac theory can be formulated in terms of these objects. [Minkowski space–time is parallelizable and as such admits a spin structure. In general, a spin structure does not exist for an arbitrary manifold equipped with a metric of signature (p,q). The conditions for existence of a spin structure in a general manifold are discussed in Refs. 93, 131, and 133. For the case of Lorentzian manifolds, see Ref. 72.] We start our presentation in Sec. II by studying a not-well-known subject, namely, the geometrical equivalence of representation modules of simple Clifford algebras $\mathcal{C}\ell(V,\mathbf{g})$. This concept, together with the concept of *spinorial frames* play a crucial role in our definition of algebraic spinors (AS) and of ASF. Once we grasp the definition of AS and particularly of Dirac AS we define Dirac–Hestenes spinors (DHS) in Sec. IV. Whereas AS may be associated to any real vector space of arbitrary dimension $n=p+q$ equipped with a nondegenerated metric of arbitrary signature (p,q), this is not the case for DHS. (ASF can be defined on more general manifolds called spin manifolds. This will be studied in Ref. 126. There, we show that the concept of Dirac–Hestenes spinor fields which exists for four-dimensional Lorentzian spin manifolds modeling a relativistic space–time, can be generalized for the case of general *spin* manifold of dimension $n=p+q$ [equipped with a metric of signature (p,q), only if the spinor bundle structure $P_{\mathrm{Spin}^e_{p,q}}M$ is trivial].) However, these objects exist for a four-dimensional vector space V equipped with a metric of Lorentzian signature and this fact makes them very much important mathematical objects for physical theories. Indeed, as we shall show in Sec. V it is possible to express Dirac equation in a consistent way using DHSF living on Minkowski space–time. Such equation is called the Dirac–Hestenes equation (DHE). In Sec. VII we express the Dirac equation using ASF. In Sec. IV we define Clifford fields and then ASF and DHSF. We observe here that our definitions of ASF and DHSF as some equivalence classes of pairs (Ξ_u, ψ_{Ξ_u}), where Ξ_u is a *spinorial coframe* field and ψ_{Ξ_u} is an appropriated Clifford field, i.e., a sum of multivector (or multiform) fields are not the usual ones that can be found in the literature. [Take notice that in this paper the term spinorial (co)frame field (defined below) is related, but distinct from the concept of a spin (co)frame, which is a section of a particular principal bundle called the spin (co)frame bundle (see Sec. IV and Ref. 126 for more details).] These definitions that, of course, come after the definitions of AS and DHS are essentially different from the definition of spinors given originally by Chevalley.[38,39] There, spinors are simply defined as elements of a minimal ideal carrying a modular representation of the Clifford algebra $\mathcal{C}\ell(V,\mathbf{g})$ associated to a structure $(V,\mathbf{g})$, where V is a real vector space of dimension $n=p+q$ and $\mathbf{g}$ is a metric of signature (p,q). And, of course, in that book there is no definition of DHS. Concerning DHS we mention that our definition of these objects is different also from the originally given in Refs. 79–81. [The definitions of AS, DHS, ASF, and DHSF given below are an improvement over preliminary tentative definitions of these objects given in Ref. 150. Unfortunately, that paper contains some equivocated results and errors (besides many misprints), which we correct here and in Ref. 126. We take the opportunity to apologize for any incovenience and misunderstandings that Ref. 150 may have caused. Some other papers where related (but not equivalent) material to the one presented in the present paper and in Ref. 126 can be found in Refs. 14–41, 44–69, 73–78, 93–109, 121–133, 144, and 146.] In view of these statements a justification for our definitions must be given and part of Sec. V and Sec. VI are devoted to such an enterprise. There it is shown that our definitions are the only ones compatible with the DHE and the meaning of the Fierz identities.[43,66] We discuss in Sec. VIII some misunderstandings resulting from the presentations of the standard Dirac equation when written with covariant Dirac spinors and also some misunderstandings concerning the DHE. It is important to emphasize here that the definitions of ASF, DHSF on Minkowski space–time and of the spin–Dirac operator given in Sec. V although correct are to be considered only as preliminaries. Indeed, these objects can be defined in a truly satisfactory way on a general Riemann–Cartan space–time only after the introduction of the concepts of the Clifford and the left (and right) spin–Clifford bundles. Moreover, a comprehensive formulation of Dirac equation on these manifolds requires a theory of connections acting on sections of these bundles. This nontrivial subject is studied in a forthcoming paper.[126] Section IX presents our conclusions. Finally we recall that our notations and some necessary results for the intelligibility of the paper are presented in Appendixes A–E. Although the appendixes contain known results, we decided to write them for the benefit of the reader, since the material cannot be found in a single reference. In particular Appendix A contains some of the ''tricks of the trade'' necessary to perform quickly calculations with Clifford algebras. If the reader needs more details concerning the theory of Clifford algebras and their applications than the ones provided by the Appendixes, the Refs. 14, 63, 64, 78, 86, 109, 141, 142 will certainly help. A final remark is necessary before we start our enterprise: the theory of the Dirac–Hestenes spinor fields of this (and the sequel paper[126]) does not contradict the standard theory of covariant Dirac spinor fields that is used by physicists and indeed it will be shown that the standard theory is no more than a matrix representation of theory described below.

Some acronyms are used in the present paper (to avoid long sentences) and they are summarized below for the reader's convenience:

AS, Algebraic spinor;

ASF, Algebraic spinor field;

CDS, Covariant Dirac spinor;

DHE, Dirac–Hestenes equation;

DHSF, Dirac–Hestenes spinor field.

II. ALGEBRAIC SPINORS

This section introduces the algebraic ideas that motivated the theory of ASF (which will be developed with full rigor in Ref. 126), i.e., we give a precise definition of AS. The algebraic side of the theory of DHSF, namely the concept of DHS is given in Sec. III. The justification for that definition will become clear in Secs. V and VI.

A. Geometrical equivalence of representation modules of simple Clifford algebras $\mathcal{C}\ell(V,\mathbf{g})$

We start with the introduction of some notations and clarification of some subtleties.

(i) In what follows V is a n-dimensional vector space over the real field $\mathbb{R}$. The dual space of V is denoted V^*. Let

$$\mathbf{g}: V \times V \to \mathbb{R} \tag{1}$$

be a metric of signature (p,q).

(ii) Let $\mathrm{SO}(V,\mathbf{g})$ be the group of endomorphisms of V that preserves $\mathbf{g}$ and the space orientation. This group is isomorphic to $\mathrm{SO}_{p,q}$ (see Appendix C), but there is no natural isomorphism. We write $\mathrm{SO}(V,\mathbf{g}) \simeq \mathrm{SO}_{p,q}$. Also, the connected component to the identity is denoted by $\mathrm{SO}^e(V,\mathbf{g})$ and $\mathrm{SO}^e(V,\mathbf{g}) \simeq \mathrm{SO}^e_{p,q}$. In the case $p=1$, $q=3$, $\mathrm{SO}^e(V,\mathbf{g})$ preserves besides orientation also the time orientation. In this paper we are mainly interested in $\mathrm{SO}^e(V,\mathbf{g})$.

(iii) We denote by $\mathcal{C}\ell(V,\mathbf{g})$ the Clifford algebra of V associated to $(V,\mathbf{g})$ and by $\mathrm{Spin}^e(V,\mathbf{g})$ $(\simeq \mathrm{Spin}^e_{p,q})$ the connected component of the spin group $\mathrm{Spin}(V,\mathbf{g}) \simeq \mathrm{Spin}_{p,q}$ (see Appendix C for the definitions). [We reserve the notation $\mathbb{R}_{p,q}$ for the Clifford algebra of the vector space $\mathbb{R}^n$ equipped with a metric of signature (p,q), $p+q=n$. $\mathcal{C}\ell(V,\mathbf{g})$ and $\mathbb{R}_{p,q}$ are isomorphic, but there is no canonical isomorphism. Indeed, an isomorphism can be exhibited only after we fix an orthonormal basis of V.] Let $\mathbf{L}$ denote 2:1 homomorphism $\mathbf{L}: \mathrm{Spin}^e(V,\mathbf{g}) \to \mathrm{SO}^e(V,\mathbf{g})$, $u \mapsto \mathbf{L}(u) \equiv \mathbf{L}_u$. $\mathrm{Spin}^e(V,\mathbf{g})$ acts on V identified as the space of 1-vectors of $\mathcal{C}\ell(V,\mathbf{g}) \simeq \mathbb{R}_{p,q}$ through its adjoint representation in the Clifford algebra $\mathcal{C}\ell(V,\mathbf{g})$ which is related with the vector representation of $\mathrm{SO}^e(V,\mathbf{g})$ as follows [$\mathrm{Aut}(\mathcal{C}\ell(V,\mathbf{g}))$ denotes the (inner) automorphisms of $\mathcal{C}\ell(V,\mathbf{g})$]:

$$\mathrm{Spin}^e(V,g) \ni u \mapsto \mathrm{Ad}_u \in \mathrm{Aut}(\mathcal{C}\ell(V,\mathbf{g}))$$

$$\mathrm{Ad}_u|_V : V \to V, \mathbf{v} \mapsto u\mathbf{v}u^{-1} = \mathbf{L}_u \cdot \mathbf{v}. \tag{2}$$

In Eq. (2) $\mathbf{L}_u \cdot \mathbf{v}$ denotes the standard action $\mathbf{L}_u$ on $\mathbf{v}$ [see Eq. (5)] and where identified (without much ado) $\mathbf{L}_u \in \mathrm{SO}^e(V,\mathbf{g})$ with $\mathbf{L}_u \in \mathbf{V} \otimes \mathbf{V}^*$, $\mathbf{g}(\mathbf{L}_u \cdot \mathbf{v}, \mathbf{L}_u \cdot \mathbf{v}) = \mathbf{g}(\mathbf{v},\mathbf{v})$.

(iv) We denote by $\mathcal{C}\ell(V,\mathbf{g})$ the Clifford algebra of V associated to $(V,\mathbf{g})$ and by $\mathrm{Spin}^e(V,\mathbf{g})$ $(\simeq \mathrm{Spin}^e_{p,q})$ the connected component of the spin group $\mathrm{Spin}(V,\mathbf{g}) \simeq \mathrm{Spin}_{p,q}$ (see Appendix C for the definitions).

(v) Let $\mathcal{B}$ be the set of all oriented and time oriented orthonormal basis [we will call the elements of $\mathcal{B}$ (in what follows) simply by orthonormal basis] of V. Choose among the elements of $\mathcal{B}$ a basis $b_0 = \{\mathbf{E}_1, \ldots, \mathbf{E}_p, \mathbf{E}_{p+1}, \ldots, \mathbf{E}_{p+q}\}$, hereafter called the fiducial frame of V. With this choice, we define a 1–1 mapping

$$\Sigma : \mathrm{SO}^e(V,\mathbf{g}) \to \mathcal{B}, \tag{3}$$

given by

$$\mathbf{L}_u \mapsto \Sigma(\mathbf{L}_u) \equiv \Sigma_{\mathbf{L}_u} = \mathbf{L}_u b_0, \tag{4}$$

where $\Sigma_{\mathbf{L_u}} = \mathbf{L}_u b_0$ is a short for $\{\mathbf{e}_1, \ldots, \mathbf{e}_p, \mathbf{e}_{p+1}, \ldots, \mathbf{e}_{p+q}\} \in \mathcal{B}$, such that denoting the action of $\mathbf{L}_u$ on $\mathbf{E}_i \in b_0$ by $\mathbf{L}_u \cdot \mathbf{E}_i$ we have

$$\mathbf{e}_i = \mathbf{L}_u \cdot \mathbf{E}_i \equiv L_i^j \mathbf{E}_j, \quad i,j = 1,2, \ldots, n. \tag{5}$$

In this way, we can identify a given vector basis b of V with the isometry $\mathbf{L}_u$ that takes the fiducial basis b_0 to b. The fiducial basis b_0 will be also denoted by $\Sigma_{\mathbf{L}_0}$, where $\mathbf{L}_0 = e$, is the identity element of $\mathrm{SO}^e(V,\mathbf{g})$.

Since the group $\mathrm{SO}^e(V,\mathbf{g})$ is not simple connected their elements cannot distinguish between frames whose spatial axes are rotated in relation to the fiducial vector frame $\Sigma_{\mathbf{L}_0}$ by multiples of 2π or by multiples of 4π. For what follows it is crucial to make such a distinction. This is done by introduction of the concept of spinorial frames.

Definition 1: Let $b_0 \in \mathcal{B}$ be a fiducial frame and choose an arbitrary $u_0 \in \mathrm{Spin}^e(V,\mathbf{g})$. Fix once and for all the pair (u_0, b_0) with $u_0 = 1$ and call it the fiducial spinorial frame.

Definition 2: The space $\mathrm{Spin}^e(V,\mathbf{g}) \times \mathcal{B} = \{(u,b), ubu^{-1} = u_0 b_0 u_0^{-1}\}$ will be called the space of spinorial frames and denoted by Θ.

Remark 3: It is crucial for what follows to observe here that the definition 2 implies that a given $b \in \mathcal{B}$ determines two and only two spinorial frames, namely (u,b) and $(-u,b)$, since $\pm ub(\pm u^{-1}) = u_0 b_0 u_0^{-1}$.

(vi) We now parallel the construction in (v) but replacing $\mathrm{SO}^e(V,\mathbf{g})$ by its universal covering group $\mathrm{Spin}^e(V,\mathbf{g})$ and $\mathcal{B}$ by Θ. Thus, we define the 1–1 mapping

$$\Xi : \mathrm{Spin}^e(V,\mathbf{g}) \to \Theta,$$

$$u \mapsto \Xi(u) \equiv \Xi_u = (u,b), \tag{6}$$

where $ubu^{-1} = b_0$.

The fiducial spinorial frame will be denoted in what follows by Ξ_0. It is obvious from Eq. (6) that $\Xi(-u) = \Xi_{(-u)} = (-u,b) \neq \Xi_u$.

Definition 4: The natural right action of $a \in \mathrm{Spin}^e(V,\mathbf{g})$ denoted by $\cdot$ on Θ is given by

$$a \cdot \Xi_u = a \cdot (u,b) = (ua, \mathrm{Ad}_{a^{-1}} b) = (ua, a^{-1} b a). \tag{7}$$

Observe that if $\Xi_{u'} = (u',b') = u' \cdot \Xi_0$ and $\Xi_u = (u,b) = u \cdot \Xi_0$ then,

$$\Xi_{u'} = (u^{-1}u') \cdot \Xi_u = (u', u^{-1} u b u^{-1} u').$$

Note that there is a natural 2–1 mapping

$$\mathbf{s} : \Theta \to \mathcal{B}, \quad \Xi_{\pm u} \mapsto b = (\pm u^{-1}) b_0 (\pm u), \tag{8}$$

such that

$$\mathbf{s}((u^{-1}u') \cdot \Xi_u)) = \mathrm{Ad}_{(u^{-1}u')^{-1}}(\mathbf{s}(\Xi_u)). \tag{9}$$

Indeed, $\mathbf{s}((u^{-1}u') \cdot \Xi_u)) = \mathbf{s}((u^{-1}u') \cdot (u,b)) = u'^{-1} u b (u'^{-1} u)^{-1} = b' = \mathrm{Ad}_{(u^{-1}u')^{-1}} b = \mathrm{Ad}_{(u^{-1}u')^{-1}}(\mathbf{s}(\Xi_u))$. This means that the natural right actions of $\mathrm{Spin}^e(V,\mathbf{g})$, respectively, on Θ and $\mathcal{B}$, commute. In particular, this implies that the spinorial frames $\Xi_u, \Xi_{-u} \in \Theta$, which are, of course distinct, determine the same vector frame $\Sigma_{\mathbf{L}_u} = \mathbf{s}(\Xi_u) = \mathbf{s}(\Xi_{-u}) = \Sigma_{\mathbf{L}_{-u}}$. We have

$$\Sigma_{\mathbf{L}_u} = \Sigma_{\mathbf{L}_{-u}} = \mathbf{L}_{u^{-1}u_0} \Sigma_{\mathbf{L}_{u_0}}, \quad \mathbf{L}_{u^{-1}u_0} \in \mathrm{SO}^e_{p,q}. \tag{10}$$

Also, from Eq. (9), we can write explicitly

$$u_0\Sigma_{\mathbf{L}_{u_0}}u_0^{-1}=u\Sigma_{\mathbf{L}_u}u^{-1},\quad u_0\Sigma_{\mathbf{L}_{u_0}}u_0^{-1}=(-u)\Sigma_{\mathbf{L}_{-u}}(-u)^{-1},\quad u\in \mathrm{Spin}^e(V,\mathbf{g}),\tag{11}$$

where the meaning of Eq. (11) of course, is that if $\Sigma_{\mathbf{L}_u}=\Sigma_{\mathbf{L}_{-u}}=b=\{\mathbf{e}_1,\ldots,\mathbf{e}_p,\mathbf{e}_{p+1},\ldots,\mathbf{e}_q\}\in\mathcal{B}$ and $\Sigma_{\mathbf{L}_{u_0}}=b_0\in\mathcal{B}$ is the fiducial frame, then

$$u_0\mathbf{E}_j u_0^{-1}=(\pm u)\mathbf{e}_j(\pm u^{-1}).\tag{12}$$

In resume we can say that the space Θ of spinorial frames can be thought of as an extension of the space $\mathcal{B}$ of vector frames, where even if two vector frames have the same ordered vectors, they are considered distinct if the spatial axes of one vector frame is rotated by a odd number of 2π rotations relative to the other vector frame and are considered the same if the spatial axes of one vector frame is rotated by an even number of 2π rotations relative to the other frame. Even if this construction seems to be impossible at first sight, Aharonov and Susskind[6] warrants that it can be implemented physically.

(vii) Before we proceed an important digression on our notation used below is necessary. We recalled in Appendix B how to construct a minimum left (or right) ideal for a given real Clifford algebra once a vector basis $b\in\mathcal{B}$ for $V\hookrightarrow\mathcal{C}\ell(V,\mathbf{g})$ is given. That construction suggests to label a given primitive idempotent and its corresponding ideal with the subindex b. However, taking into account the above discussion of vector and spinorial frames and their relationship we find useful for what follows [especially in view of the definition 5 and the definitions of algebraic and Dirac–Hestenes spinors (see definitions 6 and 8 below)] to label a given primitive idempotent and its corresponding ideal with the subindex Ξ_u. Recall after all, that a given idempotent is according to definition 6 representative of a particular spinor in a given spinorial frame Ξ_u.

(viii) Next we recall Theorem 49 of Appendix B which says that a minimal left ideal of $\mathcal{C}\ell(V,\mathbf{g})$ is of the type

$$I_{\Xi_u}=\mathcal{C}\ell(V,\mathbf{g})e_{\Xi_u},\tag{13}$$

where e_{Ξ_u} is a primitive idempotent of $\mathcal{C}\ell(V,\mathbf{g})$.

It is easy to see that all ideals $I_{\Xi_u}=\mathcal{C}\ell(V,\mathbf{g})e_{\Xi_u}$ and $I_{\Xi_{u'}}=\mathcal{C}\ell(V,\mathbf{g})e_{\Xi_{u'}}$ such that

$$e_{\Xi_{u'}}=(u'^{-1}u)e_{\Xi_u}(u'^{-1}u)^{-1},\tag{14}$$

$u,u'\in\mathrm{Spin}^e(V,g)$ are isomorphic. We have the following.

Definition 5: Any two ideals $I_{\Xi_u}=\mathcal{C}\ell(V,\mathbf{g})e_{\Xi_u}$ *and* $I_{\Xi_{u'}}=\mathcal{C}\ell(V,\mathbf{g})e_{\Xi_{u'}}$ *such that their generator idempotents are related by Eq. (14) are said geometrically equivalent.*

But take care, no *equivalence relation* has been defined until now. We observe moreover that we can write

$$I_{\Xi_{u'}}=I_{\Xi_u}(u'^{-1}u)^{-1},\tag{15}$$

a equation that will play a key role in what follows.

B. Algebraic spinors of type I_{Ξ_u}

Let $\{I_{\Xi_u}\}$ be the set of all ideals geometrically equivalent to a given minimal $I_{\Xi_{u_0}}$ as defined by Eq. (15). Let

$$\mathfrak{T}=\{(\Xi_u,\Psi_{\Xi_u})\,|\,u\in\mathrm{Spin}^e(V,\mathbf{g}),\Xi_u\in\Theta,\Psi_{\Xi_u}\in I_{\Xi_u}\}.\tag{16}$$

Let $\Xi_u,\Xi_u\in\Theta$, $\Psi_{\Xi_u}\in I_{\Xi_u}$, $\Psi_{\Xi_{u'}}\in I_{\Xi_{u'}}$. We define an equivalence relation $\mathcal{R}$ on $\mathfrak{T}$ by setting

$$(\Xi_u,\Psi_{\Xi_u})\sim(\Xi_{u'},\Psi_{\Xi_{u'}})\tag{17}$$

if and only if $u\mathbf{s}(\Xi_u)u^{-1}=u'\mathbf{s}(\Xi_{u'})u'^{-1}$ and

$$\Psi_{\Xi_{u'}}u'^{-1}=\Psi_{\Xi_u}u^{-1}. \tag{18}$$

Definition 6: An equivalence class

$$\Psi_{\Xi_u}=[(\Xi_u,\Psi_{\Xi_u})]\in\mathfrak{T}/\mathcal{R} \tag{19}$$

is called an algebraic spinor of type I_{Ξ_u} for $\mathcal{C}\ell(V,\mathbf{g})$. $\psi_{\Xi_u}\in I_{\Xi_u}$ is said to be a representative of the algebraic spinor Ψ_{Ξ_u} in the spinorial frame Ξ_u.

We observe that the pairs (Ξ_u,Ψ_{Ξ_u}) and $(\Xi_{-u},-\Psi_{\Xi_{-u}})$ are equivalent, but the pairs (Ξ_u,Ψ_{Ξ_u}) and $(\Xi_{-u},-\Psi_{\Xi_{-u}})$ are not. This distinction is *essential* in order to give a structure of linear space (over the real field) to the set $\mathfrak{T}$. Indeed, a natural linear structure on $\mathfrak{T}$ is given by

$$a[(\Xi_u,\Psi_{\Xi_u})]+b[(\Xi_u,\Psi'_{\Xi_u})]=[(\Xi_u,a\Psi_{\Xi_u})]+[(\Xi_{u'},b\Psi'_{\Xi_u})],$$

$$(a+b)[(\Xi_u,\Psi_{\Xi_u})]=a[(\Xi_u,\Psi_{\Xi_u})]+b[(\Xi_u,\Psi_{\Xi_u})]. \tag{20}$$

The definition that we just gave is not a standard one in the literature.[38,39] However, the fact is that the standard definition (licit as it is from the mathematical point of view) is *not* adequate for a comprehensive formulation of the Dirac equation using algebraic spinor fields or Dirac–Hestenes spinor fields as we show in a preliminary way in Sec. V and in a rigorous and definitive way in a sequel paper.[126]

As observed on Appendix D a given Clifford algebra $\mathbb{R}_{p,q}$ may have minimal ideals that are not geometrically equivalent since they may be generated by primitive idempotents that are related by elements of the group $\mathbb{R}^{\star}_{p,q}$ which are not elements of $\mathrm{Spin}^e(V,\mathbf{g})$ (see Appendix C where different, nongeometrically equivalent primitive ideals for $\mathbb{R}_{1,3}$ are shown). These ideals may be said to be of different types. However, from the point of view of the representation theory of the real Clifford algebras (Appendix B) all these primitive ideals carry equivalent (i.e., isomorphic) modular representations of the Clifford algebra and no preference may be given to any one. (The fact that there are ideals that are algebraically, but not geometrically equivalent seems to contain the seed for new physics, see Refs. 123, and 124.) In what follows, when no confusion arises and the ideal I_{Ξ_u} is clear from the context, we use the wording algebraic spinor for any one of the possible types of ideals.

Remark 7: We observe here that the idea of definition of algebraic spinor fields as equivalent classes has it seed in a paper by Riez.[147] *However, Riez used in his definition simply orthonormal frames instead of the spinorial frames of our approach. As such, Riez defintion generates contradictions, as it is obvious from our discussion above.*

C. Algebraic Dirac spinors

These are the algebraic spinors associated with the Clifford algebra $\mathcal{C}\ell(\mathcal{M})\simeq\mathbb{R}_{1,3}$ (the space–time algebra) of Minkowski space–time $\mathcal{M}=(V,\boldsymbol{\eta})$, where V is a four-dimensional vector space over $\mathbb{R}$ and $\boldsymbol{\eta}$ is a metric of signature (1,3).

Some special features of this important case are as follows.

(a) The group $\mathrm{Spin}^e(\mathcal{M})$ is the universal covering of $\mathcal{L}_+^\uparrow$, the special and orthochronous Lorentz group that is isomorphic to the group $\mathrm{SO}^e(\mathcal{M})$ which preserves space–time orientation and also the time orientation[120] (see also Appendix B).

(b) $\mathrm{Spin}^e(\mathcal{M})\subset\mathcal{C}\ell^0(\mathcal{M})$, where $\mathcal{C}\ell^0(\mathcal{M})\simeq\mathbb{R}_{1,3}$ is the even subalgebra of $\mathcal{C}\ell(\mathcal{M})$ and is called the Pauli algebra (see Appendix C).

The most important property is a coincidence given by Eq. (21) below. It permits us to define a new kind of spinors.

III. DIRAC–HESTENES SPINORS (DHS)

Let $\Xi_u \in \Theta$ be a spinorial frame for $\mathcal{M}$ such that $\mathbf{s}(\Xi_u)=\{e_0,e_1,e_2,e_3\}\in\mathcal{B}$. Then, it follows from Eq. (D18) of Appendix D that

$$I_{\Xi_u}=\mathcal{C}\ell(\mathcal{M})e_{\Xi_u}=\mathcal{C}\ell^0(\mathcal{M})e_{\Xi_u}, \tag{21}$$

if

$$e_{\Xi_u}=\tfrac{1}{2}(1+e_0). \tag{22}$$

Then, each $\Psi_{\Xi_u}\in I_{\Xi_u}$ can be written as

$$\Psi_{\Xi_u}=\psi_{\Xi_u}e_{\Xi_u}, \quad \psi_{\Xi_u}\in\mathcal{C}\ell^0(\mathcal{M}). \tag{23}$$

From Eq. (18) we get

$$\psi_{\Xi_{u'}}u'^{-1}ue_{\Xi_u}=\psi_{\Xi_u}e_{\Xi_u}, \quad \psi_{\Xi_u},\psi_{\Xi_{u'}}\in\mathcal{C}\ell^0(\mathcal{M}). \tag{24}$$

A possible solution for Eq. (24) is

$$\psi_{\Xi_{u'}}u'^{-1}=\psi_{\Xi_u}u^{-1}. \tag{25}$$

Let $\Theta\times\mathcal{C}\ell(\mathcal{M})$ and consider an equivalence relation $\mathcal{E}$ such that

$$(\Xi_u,\phi_{\Xi_u})\sim(\Xi_{u'},\phi_{\Xi_{u'}}) \pmod{\mathcal{E}} \tag{26}$$

if and only if $\psi_{\Xi_{u'}}$ and ψ_{Ξ_u} are related by

$$\phi_{\Xi_{u'}}u'^{-1}=\phi_{\Xi_u}u^{-1}. \tag{27}$$

This suggests the following.

Definition 8: The equivalence classes $[(\Xi_u,\phi_{\Xi_u})]\in(\Theta\times\mathcal{C}\ell(\mathcal{M}))/\mathcal{E}$ *are the Hestenes spinors. Among the Hestenes spinors, an important subset is the one consisted of Dirac–Hestenes spinors where* $[(\Xi_u,\psi_{\Xi_u})]\in(\Theta\times\mathcal{C}\ell^0(\mathcal{M}))/\mathcal{E}$. *We say that* $\phi_{\Xi_u}(\psi_{\Xi_u})$ *is a representative of a Hestenes (Dirac–Hestenes) spinor in the spinorial frame* Ξ_u.

How to justify the above definitions of algebraic and Dirac–Hestenes spinors? The question is answered in the next section.

IV. CLIFFORD FIELDS, ASF AND DHSF

The objective of this section is to introduce the concepts of Dirac–Hestenes spinor fields (DHSF) and algebraic spinor fields (ASF) living on Minkowski space–time. A definitive theory of these objects that can be applied for arbitrary Riemann–Cartan space–times can be given only after the introduction of the Clifford and left (and right) spin–Clifford bundles and the theory of connections acting on these bundles. This theory will be presented in Ref. 126 and the presentation given below (which can be followed by readers that have only a rudimentary knowledge of the theory of fiber bundles) must be considered as a preliminary one.

Let $(M,\eta,\tau,\uparrow,\nabla)$ be Minkowski space–time, where M is diffeomorphic to $\mathbb{R}^4$, η is a constant metric field, ∇ is the Levi–Civita connection of η. M is oriented by $\tau\in\sec\Lambda^4M$ and is also time oriented by $\uparrow$ (Refs. 156–158).

Let ($P_{\mathrm{SO}^e_{1,3}}M$ is the orthonormal frame bundle, $\sec P_{\mathrm{SO}^e_{1,3}}M$ means a section of the frame bundle) $\{e_a\}\in\sec P_{\mathrm{SO}^e_{1,3}}M$ be an orthonormal (moving) frame, not necessarily a coordinate frame and let $\gamma^a \in \sec T^*M$ $(a=0,1,2,3)$ be such that the set $\{\gamma^a\}$ is dual to the set $\{e_a\}$, i.e., $\gamma^a(e_b) = \delta^a_b$. (Orthonormal moving frames are not to be confused with the concept of reference frames. The concepts are related, but distinct.[156–158])

The set $\{\gamma^a\}$ will be called also a (moving) frame. Let $\gamma_a = \eta_{ab}\gamma^b$, $a,b=0,1,2,3$. The set $\{\gamma_a\}$ will be called the reciprocal frame to the frame $\{\gamma^a\}$. Recall that [$\breve{\eta}$ is the metric of the contangent space and $\breve{\eta}(\gamma^a,\gamma^b)=\eta^{ab}=\eta_{ab}=\mathrm{diag}(1,-1,-1,-1)$] $(T_x^*M,\breve{\eta})\simeq\mathcal{M}$. We will denote $(T_x^*M,\breve{\eta})$ by $\mathcal{M}^*$. Now, due to the affine structure of Minkowski space–time we can identify all the cotangent spaces as usual. Consider then the Clifford algebra $\mathcal{C}\ell(\mathcal{M}^*)$ generated by the coframe $\{\gamma^a\}$, where now we can take $\gamma^a : x \mapsto \Lambda^1(\mathcal{M}^*) \subset \mathcal{C}\ell(\mathcal{M}^*)$. We have

$$\gamma^a(x)\gamma^b(x)+\gamma^b(x)\gamma^a(x)=2\eta^{ab}, \quad \forall x\in M. \tag{28}$$

Definition 9 (preliminary): A Clifford field is a mapping

$$C: M \ni x \mapsto C(x) \in \mathcal{C}\ell(\mathcal{M}^*). \tag{29}$$

In a coframe $\{\gamma^a\}$ the expression of a Clifford field is

$$\mathcal{C}=S+A_a\gamma^a+\frac{1}{2!}B_{ab}\gamma^a\gamma^b+\frac{1}{3!}T_{abc}\gamma^a\gamma^b\gamma^c+P\gamma^5, \tag{30}$$

where $S, A_a, B_{ab}, T_{abc}, P$ are scalar functions (the ones with two or more indices antisymmetric on that indices) and $\gamma^5=\gamma^0\gamma^1\gamma^2\gamma^3$ is the volume element. Saying with other words, a Clifford field is a sum of nonhomogeneous differential forms. [This result follows once we recall that as a vector space the Clifford algebra $\mathcal{C}\ell(\mathcal{M}^*)$ is isomorphic to the the Grassmann algebra $\Lambda(V^*) = \Sigma_{p=0}^4 \Lambda^p(V^*)$, where $\Lambda^p(V^*)$ is the space of p-forms. This is clear from the definition of Clifford algebra given in the Appendix A. Recall that $\mathcal{M}^*=(V^*\simeq T^*M,\breve{\eta})$.]

Here is the point where a minimum knowledge of the theory of fiber bundles is required. Minkowski space–time is parallelizable and admits a spin structure. See, e.g., Refs. 72, 131–139, and 126. This means that Minkowski space–time has a spin strucutre, i.e., there exists a principal bundle called the spin frame bundle and denoted by $P_{\mathrm{Spin}^e_{1,3}}M$ that is the double covering of $P_{\mathrm{SO}^e_{1,3}}M$, i.e., there is a 2:1 mapping $\rho: P_{\mathrm{Spin}^e_{1,3}}M \to P_{\mathrm{SO}^e_{1,3}}M$. The elements of $P_{\mathrm{Spin}^e_{1,3}}M$ are called the spin frame fields (when there is no possibility of confusion we abreviate spin frame field simply as spin frame), and if $F_u \in P_{\mathrm{Spin}^e_{1,3}}M$ then $\rho(F_u)=\{e_a\}\in P_{\mathrm{SO}^e_{1,3}}M$ (once we fix a spin frame and associate it to an arbitrary but fixed element of $u\in P_{\mathrm{Spin}^e_{1,3}}M$). This means, that as in Sec. I, we distinguish frames that differ from a 2π rotation. Besides $P_{\mathrm{SO}^e_{1,3}}M$, we introduce also $P'_{\mathrm{SO}^e_{1,3}}M$, the coframe orthonormal bundle, such that for $\{\gamma^a\}\in P'_{\mathrm{SO}^e_{1,3}}M$ there exists $\{e_a\}\in P_{\mathrm{SO}^e_{1,3}}M$, such that $\gamma^a(e_b)=\delta^a_b$. Note that $\{\gamma^a\}\in P'_{\mathrm{SO}^e_{1,3}}M$, but, as already observed, keep in mind that each $\gamma^a : x\mapsto \Lambda^1(\mathcal{M}^*)\subset\mathcal{C}\ell(\mathcal{M}^*)$. To proceed choose a fiducial coframe $\{\Gamma^a\}\in P'_{\mathrm{SO}^e_{1,3}}M$, dual to a fiducial frame $\rho(F_{u_0})\equiv\{E_a\}\in\sec P_{\mathrm{SO}^e_{1,3}}M$.

Now, let

$$u: x\mapsto u(x)\in \mathrm{Spin}^e(\mathcal{M}^*)\subset\mathcal{C}\ell^0(\mathcal{M}^*). \tag{31}$$

In complete analogy with Sec. I let $\Theta'_{\mathcal{M}}=\mathrm{Spin}^e(\mathcal{M}^*)\times P'_{\mathrm{SO}^e_{1,3}}M$ be the space of spinorial coframe fields. We define also the 1–1 mapping

$$\begin{aligned} \Xi &: \mathrm{Spin}^e(\mathcal{M}^*)\to\Theta'_{\mathcal{M}},\\ u &\mapsto \Xi(u)\equiv\Xi_u=(u,\{u^{-1}\Gamma_a u\}). \end{aligned} \tag{32}$$

Note that there is a 2–1 natural mapping

$$\begin{gathered}\mathbf{s}':\Theta'_{\mathcal{M}}\ni\Xi_u\mapsto\{\gamma^a\}\in P'_{\mathrm{SO}^e_{1,3}}M,\\ \gamma^a=u^{-1}\Gamma^a u.\end{gathered}\tag{33}$$

Also, denoting the action of $a(x)\in\mathrm{Spin}^e(\mathcal{M}^*)$ on $\Theta'_{\mathcal{M}}$ by $a\cdot\Xi_u=(ua,\{\gamma^a\})$ we have

$$\Xi_{u'}=(u^{-1}u')\cdot\Xi_u,\tag{34}$$

$$\mathbf{s}'((u^{-1}u')\cdot\Xi_u))=\mathrm{Ad}_{(u^{-1}u')^{-1}}(\mathbf{s}'(\Xi_u)).\tag{35}$$

As in the preceding section we have associated $1\in\mathrm{Spin}^e(\mathcal{M}^*)$ to the fiducial spinorial coframe field, but of course we could associate any other element $u_0;x\mapsto u_0(x)\in\mathrm{Spin}^e(\mathcal{M}^*)$ to the fiducial spinorial coframe. In this general case, writing Ξ_{u_0} for the fiducial spinorial coframe, we have $\mathbf{s}'(\Xi_{u_0})=\{\Gamma^a\}$.

Note that $\mathbf{s}'(\Xi_u)=\mathbf{s}'(\Xi_{(-u)})$ and that any other coframe field $\mathbf{s}'(\Xi_u)$ is then related to $\mathbf{s}'(\Xi_{u_0})$ by

$$u_0\mathbf{s}'(\Xi_{u_0})u_0^{-1}=\pm u\mathbf{s}'(\Xi_u)(\pm u^{-1})=\pm u\mathbf{s}'(\Xi_{(-u)})(\pm u^{-1}),\tag{36}$$

where the meaning of this equation is analogous to the one given to Eq. (11), through Eq. (12).

Taking into account the results of the preceding sections and of the Appendixes A and B we are lead to the following definitions.

Let $\{I_{\Xi_u}\}$ be the set of all ideals geometrically equivalent to a given minimal $I_{\Xi_{u_0}}$ as defined by Eq. (15) where now u, u' are Clifford fields defined by mappings like the one defined in Eq. (31).

Let

$$\begin{gathered}\mathfrak{T}_{\mathcal{M}}=\{(x,(\Xi_u,\Psi_{\Xi_u}))|x\in M,u(x)\in\mathrm{Spin}^e(\mathcal{M}^*),\Xi_u\in\Theta'_{\mathcal{M}},\\ \Psi_{\Xi_u}:x\mapsto\Psi_{\Xi_u}(x)\in I_{\Xi_u},\Psi_{\Xi_{u'}}:x\mapsto\in\Psi_{\Xi_u}(x)\in I_{\Xi_{u'}}\}.\end{gathered}\tag{37}$$

Consider an equivalence relation $\mathcal{R}_{\mathcal{M}}$ on $\mathfrak{T}_{\mathcal{M}}$ such that

$$(x,(\Xi_u,\Psi_{\Xi_u}))\sim(y,(\Xi_{u'},\Psi_{\Xi_{u'}}))\tag{38}$$

if and only if $x=y$,

$$u(x)\mathbf{s}'(\Xi_{u(x)})u^{-1}(x)=u'(x)\mathbf{s}'(\Xi_{u'(x)})u'^{-1}(x)\tag{39}$$

and

$$\Psi_{\Xi_{u'}}u'^{-1}=\Psi_{\Xi_u}u^{-1}.\tag{40}$$

Definition 10 (preliminary): An algebraic spinor field (ASF) of type I_{Ξ_u} for $\mathcal{M}^*$ is an equivalence class $\Psi_{\Xi_u}=[(x,(\Xi_u,\Psi_{\Xi_u}))]\in\mathfrak{T}_{\mathcal{M}}/\mathcal{R}_{\mathcal{M}}$. We say that $\Psi_{\Xi_u}\in I_{\Xi_u}$ is a representative of the ASF Ψ_{Ξ_u} in the spinorial coframe field Ξ_u.

Consider an equivalence relation $\mathcal{E}_{\mathcal{M}}$ on the set $M\times\Xi_{\mathcal{M}}\times\mathcal{C}\ell(\mathcal{M}^*)$ such that [given $\psi_{\Xi_u}:x\mapsto\psi_{\Xi_u}(x)\in\mathcal{C}\ell(\mathcal{M}^*)$, $\psi_{\Xi_{u'}}:x\mapsto\in\psi_{\Xi_u}(x)\in\mathcal{C}\ell(\mathcal{M}^*)$] $((x,(\Xi_u,\psi_{\Xi_u})))$ and $((y,(\Xi_{u'},\psi_{\Xi_{u'}})))$ are equivalent if and only if $x=y$,

$$u(x)\mathbf{s}'(\Xi_{u(x)})u^{-1}(x)=u'(x)\mathbf{s}'(\Xi_{u'(x)})u'^{-1}(x)\tag{41}$$

and

$$\psi_{\Xi_{u'}} u'^{-1} = \psi_{\Xi_u} u^{-1}. \tag{42}$$

Definition 11 (preliminary): An equivalence class $\psi = [(x,(\Xi_u, \psi_{\Xi_u}))] \in M \times \Xi_{\mathcal{M}} \times \mathcal{C}\ell(\mathcal{M}^*)/\mathcal{E}_{\mathcal{M}}$ *is called a Hestenes spinor field for* $\mathcal{M}^*$. $\psi_{\Xi_u} \in \mathcal{C}\ell(\mathcal{M}^*)$ *is said to be a representative of the Hestenes spinor field* ϕ_{Ξ_u} *in the spinorial coframe field* Ξ_u. *When* $\psi_{\Xi_u}: x \mapsto \psi_{\Xi_u}(x) \in \mathcal{C}\ell^0(\mathcal{M}^*)$, $\psi_{\Xi_{u'}}: x \mapsto \in \psi_{\Xi_u}(x) \in \mathcal{C}\ell^0(\mathcal{M}^*)$ *we call the equivalence class a Dirac–Hestenes spinor field (DHSF).*

V. THE DIRAC–HESTENES EQUATION (DHE)

In our preliminary presentation of the Dirac equation (on Minkowski space–time) that follows we shall restrict our exposition to the case where any spinorial coframe field appearing in the equations that follows, e.g., $\mathbf{s}'(\Xi_u) = \{\gamma^a\}$ is teleparallel and constant. By this we mean that $\forall x, y \in M$ and $a = 0,1,2,3$,

$$\gamma^a(x) \equiv \gamma^a(y), \tag{43}$$

$$\nabla_{e_a} \gamma^b = 0. \tag{44}$$

Equation (43) has meaning due to the affine structure of Minkowski space–time which permits the usual identification of all tangent spaces (and of all cotangent spaces) of the manifold and Eq. (44), is the definition of a teleparallel frame. Of course, the unique solution for Eq. (44) is $\gamma^\mu = dx^\mu$, where $\{x^\mu\}$ are the coordinate functions of a global Lorentz chart of Minkowski space–time. Such a restriction is a necessary one in our elementary presentation, because otherwise we would need first to study the theory of the covariant derivative of spinor fields, a subject that simply cannot be appropriately introduced with the present formalism, thus clearly showing its limitation. Thus, to continue our elementary presentation we need some results of the general theory of the covariant derivatives of spinor fields studied in details in Ref. 126.

Using the results of the preceding sections and of the Appendixes we can show[80,148] that the usual Dirac equation[5,53] (which, as well known is written in terms of covariant Dirac spinor fields) for a representative of a DHSF in interaction with an electromagnetic potential $A: x \mapsto A(x) \in \Lambda^1(\mathcal{M}^*) \subset \mathcal{C}\ell(\mathcal{M}^*)$ is

$$\mathbf{D}^s \psi_{\Xi_u} \gamma_2 \gamma_1 - m \psi_{\Xi_u} \gamma_0 + qA \psi_{\Xi_u} = 0. \tag{45}$$

[Covariant Dirac spinor fields are defined in an obvious way once we take into account the definition of covariant Dirac spinors given by Eq. (E6) and Eq. (E7) of the Appendix E. See also Refs. 41, 131–133.]

Remark 12: It is important for what follows to have in mind that although each representative $\psi_{\Xi_u}: x \mapsto \psi_{\Xi_u}(x) \in \mathcal{C}\ell^0(\mathcal{M}^*)$ *of a* DHSF *is a sum of nonhomogeneous differential forms, spinor fields are not a sum of nonhomogeneous differential forms. Thus, they are mathematical objects of a nature different from that of Clifford fields*. (*Not* taking this difference into account can lead to misconceptions, as, e.g., some appearing in Ref. 71. See our comments in Ref. 155 on that paper.) *The crucial difference between a Clifford field, e.g., an electromagnetic potential A and a* DHSF *is that A is frame independent whereas a* DHSF *is frame dependent*.

In the DHE the spinor covariant derivative $\mathbf{D}^s$ is a first order differential operator, often called the spin–Dirac operator. [If we use more general frames, that are not Lorentzian coordinate frames, e.g., $\Xi_u = \{\gamma^a\}$ then $\mathbf{D}^s \psi_{\Xi_u}(x) = \gamma^a \nabla^s_{e_a} \psi_{\Xi_u}(x) = \gamma^a (e_a + \frac{1}{2}\omega_a) \psi_{\Xi_u}(x)$, where ω_a is a two form field associated with the spinorial connection, which is zero only for teleparallel frame fields, if they exist. Details in Ref. 126.] Let $\nabla^s_{f_a}$ be the spinor covariant derivative. We have the following representation for $\mathbf{D}^s$ in an arbitrary orthonormal frame $\{t^a\}$ dual of the frame $\{f_a\} \in P_{\mathrm{SO}^e_{1,3}}$,

$$\mathbf{D}^s = t^a \nabla^s_{f_a}. \tag{46}$$

In a teleparallel spin (co)frame $\mathbf{s}'(\Xi_u) = \{\gamma^\mu\}$ the above equation reduces to

$$\mathbf{D}^s = \mathrm{d}x^\mu \frac{\partial}{\partial x^\mu}. \tag{47}$$

The spin–Dirac operator in an arbitrary orthonormal frame acts on a product $(C\psi_{\Xi_u})$ where C is a Clifford field and ψ_{Ξ_u} a representative of a DHSF (or a Hestenes field) as a modular derivation,[20,126] i.e.,

$$\mathbf{D}^s(C\psi_{\Xi_u}) = t^a \nabla^s_{f_a}(C\psi_{\Xi_u}) = t^a[(\nabla_{f_a}C)\psi_{\Xi_u} + C(\nabla^s_{f_a}\psi_{\Xi_u})].$$

Also in Eq. (45) m and q are real parameters (mass and charge) identifying the elementary fermion described by that equation. (Note that we used natural unities in which the value of the velocity of light is $c=1$ and the value of Planck's constant is $\hbar = 1$.)

Now, from Eq. (42) we have

$$\psi_{\Xi_{u'}} = \psi_{\Xi_u} s^{-1}, \quad \Xi_{u'} = s \cdot \Xi_u, \tag{48}$$

$$A \mapsto A, \tag{49}$$

where $s: x \mapsto s(x) \in \mathrm{Spin}^e(\mathcal{M}^*) \subset \mathcal{C}\ell^0(\mathcal{M}^*)$ is to be considered a Clifford field. Consider the case where $s(x) = s(y) = s$, $\forall x, y \in M$. Such equation has a precise meaning due to our restriction to teleparallel frames. We see that the DHE is trivially covariant under this kind of transformation, which can be called a right gauge transformation.

Returning to the DHE we see also that the equation is covariant under active Lorentz gauge transformations, or left gauge transformations. Indeed, under an active left Lorentz gauge transformation (without changing the spinorial coframe field), we have

$$\begin{gathered} \psi_{\Xi_u} \mapsto \psi'_{\Xi_u} = s\psi_{\Xi_u}, \quad A \mapsto sAs^{-1}, \\ \mathbf{D}^s \psi_{\Xi_u} \mapsto \mathbf{D}'^s \psi'_{\Xi_u} = s\mathbf{D}^s \psi_{\Xi_u}. \end{gathered} \tag{50}$$

The justification for the active left Lorentz gauge transformation law $\mathbf{D}^s\psi_{\Xi_u} \mapsto \mathbf{D}'^s\psi'_{\Xi_u} = s\mathbf{D}^s\psi_{\Xi_u}$ is the following. (A study of active local left Lorentz gauge transformations will be presented elsewhere, for it needs the concept of gauge covariant derivatives.) The Dirac operator is a 1-form valued derivative operator $\mathbf{D}^s = \mathrm{d}x^\mu(\partial/\partial x^\mu)$. Then, under an active Lorentz gauge transformation s it must transform like a vector, i.e., $\mathbf{D}^s \mapsto \mathbf{D}'^s = s\,\mathrm{d}x^\mu s^{-1}(\partial/\partial x^\mu)$.

Note that ψ'_{Ξ_u} is a representative (in the spinorial coframe field Ξ_u) of a new spinor. Then, it follows, of course, that the representative of the new spinor in the spinorial coframe field $\Xi_{u'}$ is

$$\psi'_{\Xi_{u'}} = s\psi_{\Xi_u} s^{-1}. \tag{51}$$

We also recall that the DHE is invariant under simultaneous left and right (constants) gauge Lorentz transformations. In this case the relevant transformations are

$$\begin{gathered}\psi_{\Xi_u} \mapsto \psi'_{\Xi_u} = s\psi_{\Xi_u} s^{-1}, \\ A \mapsto sAs^{-1}, \quad \mathbf{D}'^{s}\psi'_{\Xi_{u'}} = s\mathbf{D}^{s}\psi_{\Xi_u} s^{-1}.\end{gathered} \tag{52}$$

VI. JUSTIFICATION OF THE TRANSFORMATION LAWS OF DHSF BASED ON THE FIERSZ IDENTITIES

We now give another justification for the definition of Dirac spinors and DHSF presented in the preceding sections. We start by recalling that a usual covariant Dirac spinor field determines a set of p-form fields, called bilinear covariants, which describe the physical contents of a particular solution of the Dirac equation described by that field. The same is true also for a DHSF.

In order to present the bilinear covariants using that fields, we introduce first the notion of the Hodge dual operator of a Clifford field $\mathcal{C}:M \ni x \mapsto \mathcal{C}(x) \in \mathcal{C}\ell(\mathcal{M}^*)$. We have the following.

Definition 13: The Hodge dual operator is the mapping

$$\star:\mathcal{C} \to \star\mathcal{C} = \tilde{\mathcal{C}}\gamma_5, \tag{53}$$

where $\tilde{\mathcal{C}}$ is the reverse of C [Eq. (A5), Appendix A].

Then, in terms of a representative of a DHSF in the spinorial frame field Ξ_u the bilinear covariants of Dirac theory reads (with $J = J_\mu \gamma^\mu$, $S = \frac{1}{2} S_{\mu\nu}\gamma^\mu\gamma^\nu$, $K = K_\mu \gamma^\mu$)

$$\begin{gathered}\psi_{\Xi_u}\tilde{\psi}_{\Xi_u} = \sigma + \star\omega, \quad \psi_{\Xi_u}\gamma^0\tilde{\psi}_{\Xi_u} = J, \\ \psi_{\Xi_u}\gamma^1\gamma^2\tilde{\psi}_{\Xi_u} = S, \quad \psi_{\Xi_u}\gamma^0\gamma^3\tilde{\psi}_{\Xi_u} = \star S, \\ \psi_{\Xi_u}\gamma^3\tilde{\psi}_{\Xi_u} = K, \quad \psi_{\Xi_u}\gamma^0\gamma^1\gamma^2\tilde{\psi}_{\Xi_u} = \star K.\end{gathered} \tag{54}$$

The so-called *Fierz identities* are

$$J^2 = \sigma^2 + \omega^2, \quad J\cdot K = 0, \quad J^2 = -K^2, \quad J \wedge K = -(\omega + \star K)S, \tag{55}$$

$$\begin{gathered}S \llcorner J = \omega K, \quad S \llcorner K = \omega J, \\ (\star S)\llcorner J = -\sigma K, \quad (\star S)\llcorner K = -\sigma J, \\ S\cdot S = \omega^2 - \sigma^2, \quad (\star S)\cdot S = -2\sigma\omega,\end{gathered} \tag{56}$$

$$\begin{gathered}JS = -(\omega + \star\sigma)K, \\ SJ = -(\omega - \star\sigma)K, \\ KS = -(\omega + \star\sigma)J, \\ SK = -(\omega - \star\sigma)J, \\ S^2 = \omega^2 - \sigma^2 - 2\sigma(\star\omega), \\ S^{-1} = -S(\sigma - \star\omega)^2/J^2 = KSK/J^4.\end{gathered} \tag{57}$$

The proof of these identities using the DHSF is almost a triviality and can be done in a few lines. This is not the case if you use covariant Dirac spinor fields (columns matrix fields). In this case you will need to perform several pages of matrix algebra calculations.

The importance of the bilinear covariants is due to the fact that we can recover from them the associate covariant Dirac spinor field (and thus the DHSF) except for a phase. This can be done with an algorithm due to Crawford[43] and presented in a very pedagogical way in Ref. 109.

Let us consider, e.g., the equation $\psi_{\Xi_u}\gamma_0\tilde{\psi}_{\Xi_u}=J$ in (54). Now, $J(x)\in\Lambda^1(\mathcal{M}^*)\subset\mathcal{C}\ell(\mathcal{M}^*)$ is an intrinsic object on Minkowski space–time and according to the accepted first quantization interpretation theory of the Dirac equation it is proportional to the electromagnetic current generated by an elementary fermion. The expression of J in terms of the representative of a DHSF in the spinorial coframe $\Xi_{u'}$ is (of course)

$$\psi_{\Xi_{u'}}\gamma_0'\tilde{\psi}_{\Xi_{u'}}=J. \tag{58}$$

Now, since

$$\gamma_0'=(u'^{-1}u)\gamma_0(u'^{-1}u)^{-1}, \tag{59}$$

we see that we must have

$$\psi_{\Xi_{u'}}=\psi_{\Xi_u}(u'^{-1}u)^{-1}, \tag{60}$$

which justifies the definition of DHSF given above [see Eq. (40)].

We observe also that if $\psi_{\Xi_u}\tilde{\psi}_{\Xi_u}=\sigma+\star\omega\neq 0$, then we can write

$$\psi_{\Xi_u}=\rho^{1/2}e^{1/2\,\beta\gamma^5}R, \tag{61}$$

where $\forall x\in M$,

$$\begin{aligned}
&\rho(x)\in\Lambda^0(\mathcal{M}^*)\subset\mathcal{C}\ell(\mathcal{M}^*),\\
&\beta(x)\in\Lambda^0(\mathcal{M}^*)\subset\mathcal{C}\ell(\mathcal{M}^*),\\
&R\in\mathrm{Spin}^e_{1,3}(\mathcal{M}^*)\subset\mathcal{C}\ell(\mathcal{M}^*).
\end{aligned} \tag{62}$$

With this result the current J can be written

$$J=\rho v \tag{63}$$

with $v=R\gamma^0R^{-1}$. Equation (63) discloses the secret geometrical meaning of DHSF. These objects rotate and dilate vector fields, this being the reason why they are sometimes called operator spinors.[80–86,109]

VII. DIRAC EQUATION IN TERMS OF ASF

We recall from Eq. (D2) of Appendix D that

$$e'_{\Xi_u}=\tfrac{1}{2}(1+\gamma_3\gamma_0) \tag{64}$$

is also a primitive idempotent field (here understood as a Hestenes spinor field) that is algebraically, but not geometrically equivalent to the idempotent field $e_{\Xi_u}=\frac{1}{2}(1+\gamma_0)$. Let $I'_{\Xi_u}=\mathcal{C}\ell(\mathcal{M}^*)e'_{\Xi_u}$ be a minimal left ideal generated by e'_{Ξ_u}. Now, multiply the DHE [Eq. (45)] on the left, first by the primitive idempotent e_{Ξ_u} and then by the primitive idempotent e'_{Ξ_u}. We get after some algebra

$$\mathbf{D}^s\Phi_{\Xi_u}-m\Phi_{\Xi_u}(\star 1)+qA\Phi_{\Xi_u}=0, \tag{65}$$

where $\star 1 = \gamma_5$ is the oriented volume element of Minkowski space–time and

$$\Phi_{\Xi_u} = \psi_{\Xi_u} e_{\Xi_u} e'_{\Xi_u} \in I'_{\Xi_u} = \mathcal{C}\ell(\mathcal{M}^*) e'_{\Xi_u}. \tag{66}$$

Equation (65) is one of the many faces of the original equation found by Dirac in terms of ASF and using teleparallel orthonormal frames.

Of course, Eq. (65), as it is the case of the DHE [Eq. (45)] is compatible with the transformation law of ASF that follows directly from the transformation law of AS given in Sec. II. In contrast to the DHE, in Eq. (65) there seems to be no explicit reference to elements of a spinorial coframe field (except for the indices Ξ_u) since $\star 1$, the volume element is invariant under (Lorentz) gauge transformations. We emphasize also that the transformation law for ASF is compatible with the presentation of Fierz identities using these objects, as the interested reader can verify without difficulty.

VIII. MISUNDERSTANDINGS CONCERNING COORDINATE REPRESENTATIONS OF THE DIRAC AND DIRAC–HESTENES EQUATIONS

We investigate now some subtleties of the Dirac and Dirac–Hestenes equations. We start by pointing out and clarifying some misunderstandings that often appears in the literature of the subject of the DHE when that equation is presented in terms of a representative of a DHSF in a global coordinate chart (M,φ) of the maximal atlas of M with Lorentz coordinate functions $\langle x^\mu \rangle$ associated to it (see, e.g., Ref. 156). In that case, $\mathbf{s}'(\Xi_u) = \{\gamma^\mu = \mathrm{d}x^\mu\}$. After that we study the (usual) matrix representation of Dirac equation and show how it hides many features that are only visible in the DHE.

Let $\{e_\mu = \partial/\partial x^\mu\}$ and $\{e'_\mu = \partial/\partial x'^\mu\}$. The spinorial coframe fields Ξ_u and $\Xi_{u'}$ (as defined in the preceding section) are associated to the coordinate bases (dual basis) $\mathbf{s}'(\Xi_u) = \{\gamma^\mu = \mathrm{d}x^\mu\}$ and $\mathbf{s}'(\Xi_{u'}) = \{\gamma'^\mu = dx'^\mu\}$, corresponding to the global Lorentz charts (M,φ) and (M,φ'). The DHE is written in the charts $\langle x^\mu \rangle$ and $\langle x'^\mu \rangle$ as

$$\begin{gathered}
\gamma^\mu \left(\frac{\partial}{\partial x^\mu} \Psi_{\Xi_u} + q A_\mu \Psi_{\Xi_u} \gamma_1 \gamma_2 \right) \gamma_2 \gamma_1 - m \Psi_{\Xi_u} \gamma_0 = 0, \\
\gamma'^\mu \left(\frac{\partial}{\partial x'^\mu} \Psi'_{\Xi_{u'}} + q A'_\mu \Psi_{\Xi_{u'}} \gamma'_1 \gamma'_2 \right) \gamma'_2 \gamma'_1 - m \Psi_{\Xi_{u'}} \gamma'_0 = 0,
\end{gathered} \tag{67}$$

where $\mathbf{D}^s = \gamma^\mu (\partial/\partial x^\mu) = \gamma'^\mu (\partial/\partial x'^\mu)$ and where (Ψ_{Ξ_u}, A_μ) and $(\Psi_{\Xi_{u'}}, A'_\mu)$ are the coordinate representations of (ψ_{Ξ_u}, A) and $(\psi_{\Xi_{u'}}, A)$, i.e., for any $x \in M$, we have

$$\begin{gathered}
A = A'_\mu(x'^\mu) \mathrm{d}x'^\mu = A_\mu(x^\mu) \mathrm{d}x^\mu, \\
A'_\mu(x'^0, x'^1, x'^2, x'^3) = \mathbf{L}^\nu_\mu A_\nu(x^0, x^1, x^2, x^3), \\
(\Psi_{\Xi_{u'}} U'^{-1})|_{(x'^0(x), x'^1(x), x'^2(x), x'^3(x))} = (\Psi_{\Xi_u} U^{-1})|_{(x^0(x), x^1(x), x^2(x), x^3(x))},
\end{gathered} \tag{68}$$

with U and U' the coordinate representations of u and u' [see Eq. (42)] and L^ν_μ is an appropriate Lorentz transformation.

Now, taking into account that the complexification of the algebra $\mathcal{C}\ell(\mathcal{M}^*)$, i.e., $\mathbb{C} \otimes \mathcal{C}\ell(\mathcal{M}^*)$ is isomorphic to the Dirac algebra $\mathbb{R}_{4.1}$ (Appendix C), we can think of all the objects appearing in Eqs. (67) as having values also in $\mathbb{C} \otimes \mathcal{C}\ell(\mathcal{M}^*)$. Multiply then, both sides of each one of the Eqs. (67) by the following primitive idempotents fields [considered as complexified Hestenes spinor fields (see Definition 8)] of $\mathbb{C} \otimes \mathcal{C}\ell(\mathcal{M}^*)$ [see Eq. (D14) of Appendix D]

$$
\begin{aligned}
f_{\Xi_u} &= \tfrac{1}{2}(1+\gamma^0)\tfrac{1}{2}(1+i\gamma^1\gamma^2),\\
f_{\Xi_{u'}} &= \tfrac{1}{2}(1+\gamma'^0)\tfrac{1}{2}(1+i\gamma'^1\gamma'^2).
\end{aligned}
\tag{69}
$$

Next, look for a matrix representation in $\mathbb{C}(4)$ of the resulting equations. We get (using the notation of Appendix D)

$$\overline{\gamma}^{\mu}\left(i\frac{\partial}{\partial x^{\mu}}\right)+qA_{\mu}(x^{\mu})\Psi(x^{\mu})-m\Psi(x^{\mu})=0, \tag{70}$$

$$\underline{\gamma}^{\mu}\left(i\frac{\partial}{\partial x'^{\mu}}\right)+qA'_{\mu}(x'^{\mu})\Psi'(x'^{\mu})-m\Psi'(x'^{\mu})=0, \tag{71}$$

where $\Psi(x^{\mu}),\Psi'(x'^{\mu})$ are the matrix representations [Eq. (D15), Appendix D] of Ψ_{Ξ_u} and $\Psi_{\Xi_{u'}}$. The matrix representations of the spinors are related by an equation analogous to Eq. (E2) of Appendix E, except that now, these equations refer to fields. The $\{\underline{\gamma}^{\mu}\}$, $\mu=0,1,2,3$ is the set of Dirac matrices given by Eq. (D13) of Appendix D. Of course, we arrived at the usual form of the Dirac equation, except for the irrelevant fact that in general the Dirac spinor is usually represented by a column spinor field, and here we end with a 4×4 matrix field, which however has non-null elements only in the first column. [The reader can verify without great difficulty that Eq. (65) also has a matrix representation analogous to Eq. (71) but with a set of gamma matrices differing from the set $\{\underline{\gamma}^{\mu}\}$ by a similarity transformation.]

Equation (70), that is the usual presentation of Dirac equation in Physics textbooks, hides several important facts. First, it hides the basic dependence of the spinor fields on the spinorial frame field, since the spinorial frames Ξ_u, $\Xi_{u'}$ are such that $\mathbf{s}'(\Xi_u)=\{\gamma^{\mu}\}$ and $\mathbf{s}'(\Xi_{u'})=\{\gamma'^{\mu}\}$ are mapped on the same set of matrices, namely $\{\underline{\gamma}^{\mu}\}$. Second, it hides an obvious geometrical meaning of the theory, as first disclosed by Hestenes.[80,81] Third, taking into account the discussion in a preceding section, we see that the usual presentation of the Dirac equation does not leave clear at all if we are talking about passive or active Lorentz gauge transformations. Finally, since diffeomorphisms on the world manifold are in general erroneous associated with coordinate transformations in many Physics textbooks, Eq. (70) suggests that spinors must change under diffeomorphisms in a way different from the true one, for indeed Dirac spinor fields (and also, DHSF) are scalars under diffeomorphisms, an issue that we will discuss in another publication.

IX. CONCLUSIONS

In this paper we investigated how to define algebraic and Dirac–Hestenes spinor fields on *Minkowski* space–time. We showed first, that in general, algebraic spinors can be defined for any real vector space of any dimension and equipped with a nondegenerated metric of arbitrary signature, but that is not the case for *Dirac–Hestenes spinors*. These objects exist for a four-dimensional real vector space equipped with a metric of Lorentzian signature. It is this fact that makes them very important objects (and gave us the desire to present a rigorous mathematical theory for them), since as shown in Secs. V and VII the Dirac equation can be written in terms of *Dirac–Hestenes spinor fields* or *algebraic spinor fields*. We observe that our definitions of algebraic and Dirac–Hestenes spinor fields as some equivalence classes in appropriate sets are not the standard ones and the core of the paper was to give genuine motivations for them. We observe moreover that the definitions of Dirac–Hestenes spinor fields and of the spin–Dirac operator given in Sec. V although correct are to be considered only as preliminaries. The reason is that any rigorous presentation of the theory of the spin–Dirac operator (an in particular, on a general Riemann–Cartan space–time) can only be given after the introduction of the concepts of Clifford and spin–Clifford bundles over these space–times. This is studied in a sequel paper.[126] In Ref. 155

we show some nontrivial applications of the concept of Dirac–Hestenes spinor fields by proving (mathematical) Maxwell–Dirac equivalences of the first and second kinds and showing how these equivalences can eventually put some light on a possible physical interpretation of the famous Seiberg–Witten equations for Minkowski space–time.

Noted added: After we finished the writing of the present paper and of Ref. 126, we learned about the very interesting papers by Marchuck.[110–118] There, a different point of view concerning the writing of the Dirac equation using tensor fields is developed. (Reference 110, indeed, uses a particular case of objects that we called extensors in a recent series of papers.[63–65,127–130]) We will discuss Marchuck papers elsewhere.

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APPENDIX A: SOME FEATURES ABOUT REAL AND COMPLEX CLIFFORD ALGEBRAS

In this appendix we fix the notations that we used and introduce the main ideas concerning the theory of Clifford algebras necessary for the intelligibility of the paper.

1. Definition of the Clifford algebra $\mathcal{C}\ell(\mathbf{V},\mathbf{b})$

In this paper we are interested only in Clifford algebras of a vector space (we reserve the notation V for real vector spaces) $\mathbf{V}$ of finite dimension n over a field $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$. Let $\mathbf{q}:\mathbf{V}\to\mathbb{F}$ be a nondegenerate quadratic form over $\mathbf{V}$ with values in $\mathbb{F}$ and $\mathbf{b}:\mathbf{V}\times\mathbf{V}\to\mathbb{F}$ the associated bilinear form (which we call a metric in the case $\mathbb{F}=\mathbb{R}$). We use the notation

$$x\cdot y=\mathbf{b}(x,y)=\tfrac{1}{2}(\mathbf{q}(x+y)-\mathbf{q}(x)-\mathbf{q}(y)). \tag{A1}$$

Let $\Lambda\mathbf{V}=\Sigma_{i=0}^{n}\Lambda^i\mathbf{V}$ be the exterior algebra of $\mathbf{V}$ where $\Lambda^i\mathbf{V}$ is the $\binom{n}{i}$ dimensional space of the i-vectors. $\Lambda^0\mathbf{V}$ is identified with $\mathbb{F}$ and $\Lambda^1\mathbf{V}$ is identified with $\mathbf{V}$. The dimension of $\Lambda\mathbf{V}$ is 2^n. A general element $X\in\Lambda\mathbf{V}$ is called a multivector and can be written as

$$X=\sum_{i=0}^{n}\langle X\rangle_i,\quad \langle X\rangle_i\in\Lambda^i\mathbf{V}, \tag{A2}$$

where

$$\langle\ \ \rangle_i:\Lambda\mathbf{V}\to\Lambda^i\mathbf{V} \tag{A3}$$

is the projector in $\Lambda^i\mathbf{V}$, also called the i-part of X.

Definition 14: The main involution or grade involution is an automorphism

$$\hat{}:\Lambda\mathbf{V}\ni\mathbf{X}\mapsto\hat{\mathbf{X}}\in\Lambda\mathbf{V} \tag{A4}$$

such that

$$\hat{X}=\sum_{k=0}^{n}(-1)^k\langle X\rangle_k. \tag{A5}$$

$\hat{X}$ is called the grade involution of X or simply the involuted of X.

Definition 15: The reversion operator is the anti-automorphism

$$\tilde{}: \Lambda \mathbf{V} \ni \mathbf{X} \mapsto \tilde{\mathbf{X}} \in \Lambda \mathbf{V} \tag{A6}$$

such that

$$\tilde{X} = \sum_{k=0}^{n} (-1)^{1/2\,k(k-1)} \langle X \rangle_k, \tag{A7}$$

$\tilde{X}$ is called the reverse of X.

The composition of the grade evolution with the reversion operator, denote by $-$ is called by some authors (e.g., Refs. 109, 141, 142) the conjugation and, $\bar{X}$ is called the conjugate of X. We have $\bar{X} = (\tilde{\hat{X}}) = (\hat{\tilde{X}})$.

Since the grade and reversion operators are involutions on the vector space of multivectors, we have that $\hat{\hat{X}} = X$ and $\tilde{\tilde{X}} = X$. both involutions commute with the k-part operator, i.e., $\widehat{\langle X \rangle_k} = \langle \hat{X} \rangle_k$ and $\widetilde{\langle X \rangle_k} = \langle \tilde{X} \rangle_k$, for each $k = 0,1,\ldots,n$.

Definition 16: The exterior product of multivectors X and Y is defined by

$$\langle X \wedge Y \rangle_k = \sum_{j=0}^{k} \langle X \rangle_j \wedge \langle Y \rangle_{k-j}, \tag{A8}$$

for each $k = 0,1,\ldots,n$. Note that on the right-hand side there appears the exterior product of j-vectors and $(k-j)$-vectors with $0 \leqslant j \leqslant n$. (We assume that the reader is familiar with the exterior algebra. We only caution that there are some different definitions of the exterior product in terms of the tensor product differing by numerical factors. This may lead to some confusions, if care is not taken. Details can be found in Refs. 63 and 64.)

This exterior product is an internal composition law on $\Lambda \mathbf{V}$. It is associative and satisfies the distributives laws (on the left and on the right).

Definition 17: The vector space $\Lambda \mathbf{V}$ endowed with this exterior product Λ is an associative algebra called the exterior algebra of multivectors.

We recall now some of the most important properties of the exterior algebra of multivectors.

For any $\alpha, \beta \in \mathbb{F}$, $X \in \Lambda \mathbf{V}$,

$$\begin{aligned} &\alpha \wedge \beta = \beta \wedge \alpha = \alpha\beta \quad (\text{product of } \mathbb{F} \text{ numbers}), \\ &\alpha \wedge X = X \wedge \alpha = \alpha X \quad (\text{multiplication by scalars}). \end{aligned} \tag{A9}$$

For any $X_j \in \Lambda^j \mathbf{V}$ and $Y_k \in \Lambda^k \mathbf{V}$

$$X_j \wedge Y_k = (-1)^{jk} Y_k \wedge X_j. \tag{A10}$$

For any $X, Y \in \Lambda \mathbf{V}$

$$\begin{aligned} \widehat{X \wedge Y} &= \hat{X} \wedge \hat{Y}, \\ \widetilde{X \wedge Y} &= \tilde{X} \wedge \tilde{Y}. \end{aligned} \tag{A11}$$

2. Scalar product of multivectors

Definition 18: A scalar product between the multivectors $X,Y \in \Lambda\mathbf{V}$ is given by

$$X \cdot Y = \sum_{i=0}^{n} \langle X\rangle_i \cdot \langle Y\rangle_i , \tag{A12}$$

where $\langle X\rangle_0 \cdot \langle Y\rangle_0 = \langle X\rangle_0 \langle Y\rangle_0$ is the multiplication in the field $\mathbb{F}$ and $\langle X\rangle_i \cdot \langle Y\rangle_i$ is given by Eq. (A2), and writing

$$\begin{aligned} \langle X\rangle_k &= \frac{1}{k!} X^{i_1 i_2 \cdots i_k} b_{i_1} \wedge b_{i_2} \cdots b_{i_k}, \\ \langle Y\rangle_k &= \frac{1}{k!} Y^{i_1 i_2 \cdots i_k} b_{i_1} \wedge b_{i_2} \cdots b_{i_k} \end{aligned} \tag{A13}$$

where $\{b_k\}, k=1,2,\ldots,n$ is an arbitrary basis of $\mathbf{V}$ we have

$$\langle X\rangle_k \cdot \langle Y\rangle_k = \frac{1}{(k!)^2} X^{i_1 i_2 \cdots i_k} Y^{j_1 j_2 \cdots j_k} (b_{i_1} \wedge b_{i_2} \cdots b_{i_k}) \cdot (b_{j_1} \wedge b_{j_2} \cdots b_{j_k}), \tag{A14}$$

with

$$(b_{i_1} \wedge b_{i_2} \cdots b_{i_k}) \cdot (b_{j_1} \wedge b_{j_2} \cdots b_{j_k}) = \begin{vmatrix} b_{i_1} \cdot b_{j_1} & \cdots & \cdots & b_{i_1} \cdot b_{j_k} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ b_{i_k} \cdot b_{j_1} & \cdots & \cdots & b_{i_k} \cdot b_{j_k} \end{vmatrix}. \tag{A15}$$

It is easy to see that for any $X,Y \in \Lambda\mathbf{V}$,

$$\begin{aligned} \hat{X} \cdot Y &= X \cdot \hat{Y}, \\ \tilde{X} \cdot Y &= X \cdot \tilde{Y}. \end{aligned} \tag{A16}$$

Remark 19: Observe that the definition of the scalar product given in this paper by Eq. (A12) differs by a signal from the scalar product of multivectors defined, e.g., in Ref. 79. Our definition is a natural one if we start the theory with the euclidean Clifford algebra of multivectors of a real vector space $\mathbf{V}$. *The euclidean Clifford algebra is fundamental for the construction of the theory of extensors and extensor fields.*[63–65,127–130]

3. Interior algebras

Definition 20: We define two different contracted products *for arbitrary multivectors $X,Y \in \Lambda\mathbf{V}$ by*

$$\begin{aligned} (X \lrcorner Y) \cdot Z &= Y(\tilde{X} \wedge Z), \\ (X \llcorner Y) &= X \cdot (Z \wedge \tilde{Y}), \end{aligned} \tag{A17}$$

where $Z \in \Lambda\mathbf{V}$. The internal composition rules $\lrcorner$ *and* $\llcorner$ *will be called, respectively, the left and the right contracted product*.

These contracted products $\lrcorner$ and $\llcorner$ are internal laws on $\Lambda\mathbf{V}$. Both contract products satisfy the distributive laws (on the left and on the right) but they are *not* associative.

Definition 21: The vector space $\Lambda\mathbf{V}$ *endowed with each one of these contracted products (either* $\lrcorner$ *or* $\llcorner$*) is a nonassociative algebra. They are called the interior algebras of multivectors.*

We present now some of the most important properties of the interior products:

(a) For any $\alpha,\beta\in\mathbb{F}$, and $X\in\Lambda\mathbf{V}$,

$$\alpha\lrcorner\beta=\alpha\llcorner\beta=\alpha\beta \text{ (product in } \mathbb{F}),$$
$$\alpha\lrcorner X=X\llcorner\alpha=\alpha X \quad \text{(multiplication by scalars)}. \tag{A18}$$

(b) For any $X_j\in\Lambda^j\mathbf{V}$ and $Y_k\in\Lambda^k\mathbf{V}$ with $j\leqslant k$,

$$X_j\lrcorner Y_k=(-1)^{j(k-j)}Y_k\llcorner X_j. \tag{A19}$$

(c) For any $X_j\in\Lambda^j\mathbf{V}$ and $Y_k\in\Lambda^k\mathbf{V}$,

$$X_j\lrcorner Y_k=0, \text{ if } j>k,$$
$$X_j\llcorner Y_k=0, \text{ if } j<k. \tag{A20}$$

(d) For any $X_k,Y_k\in\Lambda^k\mathbf{V}$

$$X_j\lrcorner Y_k=X_j\llcorner Y_k=\tilde{X}_k\cdot Y_k=X_k\cdot\tilde{Y}_k. \tag{A21}$$

(e) For any $v\in\mathbf{V}$ and $X,Y\in\Lambda\mathbf{V}$

$$v\lrcorner(X\wedge Y)=(v\lrcorner X)\wedge Y+\hat{X}\wedge(v\lrcorner Y). \tag{A22}$$

4. Clifford algebra $\mathcal{C}\ell(\mathbf{V},\mathbf{b})$

Definition 22: The Clifford product of multivectors X and Y (denoted by juxtaposition) is given by the following axiomatic:

(i) For all $\alpha\in\mathbb{F}$ and $X\in\Lambda\mathbf{V}$: $\alpha X=X\alpha$ equals multiplication of multivector X by scalar α.
(ii) For all $v\in\mathbf{V}$ and $X\in\Lambda\mathbf{V}$: $vX=v\lrcorner X+v\wedge X$ and $Xv=X\llcorner v+X\wedge v$.
(iii) For all $X,Y,Z\in\Lambda\mathbf{V}$: $X(YZ)=(XY)Z$.

The Clifford product is an internal law on $\Lambda\mathbf{V}$. It is associative [by the axiom (iii)] and satisfies the distributives laws (on the left and on the right). The distributive laws follow from the corresponding distributive laws of the contracted and exterior products.

Definition 23: The vector space of multivectors over $\mathbf{V}$ *endowed with the Clifford product is an associative algebra with unity called* $\mathcal{C}\ell(\mathbf{V},\mathbf{b})$.

5. Relation between the exterior and the Clifford algebras and the tensor algebra

Modern algebra books give the

Definition 24: The exterior algebra of $\mathbf{V}$ *is the quotient algebra* $\Lambda\mathbf{V}=T(\mathbf{V})/I$, *where* $T(\mathbf{V})$ *is the tensor algebra of* $\mathbf{V}$ *and* $I\subset T(\mathbf{V})$ *is the bilateral ideal generated by the elements of the form* $\mathbf{x}\otimes\mathbf{x}$, $\mathbf{x}\in\mathbf{V}$.

Definition 25: The Clifford algebra of $(\mathbf{V},\mathbf{b})$ *is the quotient algebra* $\mathcal{C}\ell(\mathbf{V},\mathbf{b})=T(\mathbf{V})/I_b$, *where* I_b *is the bilateral ideal generated by the elements of the form* $\mathbf{x}\otimes\mathbf{x}-2\mathbf{b}(\mathbf{x},\mathbf{x})$, $\mathbf{x}\in\mathbf{V}$.

We can show that this definition is equivalent to the one given above. [When the exterior algebra is defined as $\Lambda\mathbf{V}=T(\mathbf{V})/I$ and the Clifford algebra as $\mathcal{C}\ell(\mathbf{V},\mathbf{b})=T(\mathbf{V})/I_{\mathbf{b}}$, the (associative) exterior product of the multivectors in the terms of the tensor product of these multivectors is fixed once and for all. We have, e.g., that for $x,y\in\mathbf{V}$, $x\wedge y=\frac{1}{2}(x\otimes y-y\otimes x)$. However, keep in mind that it is possible to define an (associative) exterior product in $\Lambda\mathbf{V}$ differing from the above one by numerical factors, and indeed in Refs. 63–65, 127–130 we used another choice. When reading a text on the subject it is a good idea to have in mind the definition used by the author, for otherwise confusion may result.] The space $\mathbf{V}$ is naturally *embedded* on $\mathcal{C}\ell(\mathbf{V},\mathbf{b})$, i.e.,

$$\mathbf{V}\overset{i}{\hookrightarrow}T(\mathbf{V})\overset{j}{\rightarrow}T(\mathbf{V})/I_{\mathbf{b}}=\mathcal{C}\ell(\mathbf{V},\mathbf{b}),$$
$$\text{and}\quad \mathbf{V}\equiv j\circ i(\mathbf{V})\subset\mathcal{C}\ell(\mathbf{V},\mathbf{b}). \tag{A23}$$

Let $\mathcal{C}\ell^0(\mathbf{V},\mathbf{b})$ and $\mathcal{C}\ell^1(\mathbf{V},\mathbf{b})$ be, respectively, the j-images of $\oplus_{i=0}^{\infty}T^{2i}(\mathbf{V})$ and $\oplus_{i=0}^{\infty}T^{2i+1}(\mathbf{V})$ in $\mathcal{C}\ell(\mathbf{V},\mathbf{b})$. The elements of $\mathcal{C}\ell^0(\mathbf{V},\mathbf{b})$ form a subalgebra of $\mathcal{C}\ell(\mathbf{V},\mathbf{b})$ called the even subalgebra of $\mathcal{C}\ell(\mathbf{V},\mathbf{b})$. Also, there is a canonical vector isomorphism $\Lambda\mathbf{V}\rightarrow\mathcal{C}\ell(\mathbf{V},\mathbf{b})$, which permits to speak of the embeddings $\Lambda^p\mathbf{V}\subset\mathcal{C}\ell(\mathbf{V},\mathbf{b})$, $0\leqslant p\leqslant n$, where n is the dimension of $\mathbf{V}$ (Ref. 20). [The isomorphism is compatible with the filtrations of the filtered algebra $\Lambda\mathbb{V}$, i.e., $(\Lambda^r\mathbb{V})\wedge(\Lambda^s\mathbb{V})\subseteq\Lambda^{r+s}\mathbb{V}$.]

6. Some useful properties of the real Clifford algebras $\mathcal{C}\ell(V,\mathbf{g})$

We now collect some useful formulas which hold for a real Clifford algebra $\mathcal{C}\ell(V,\mathbf{g})$ and which has been used in calculations in the text and Appendixes. (As the reader can verify, many of these properties are also valid for the complex Clifford algebras.)

For any $v\in V$ and $X\in\Lambda V$,

$$v\lrcorner X=\tfrac{1}{2}(vX-\overline{X}v)\quad\text{and}\quad X\llcorner v=\tfrac{1}{2}(Xv-v\overline{X}),$$
$$v\wedge X=\tfrac{1}{2}(vX+\overline{X}v)\quad\text{and}\quad X\wedge v=\tfrac{1}{2}(Xv+v\overline{X}).\tag{A24}$$

For any $X,Y\in V$,

$$X\cdot Y=\langle\tilde{X}Y\rangle_0=\langle X\tilde{Y}\rangle_0.\tag{A25}$$

For any $X,Y,Z\in V$,

$$(XY)\cdot Z=Y\cdot(\tilde{X}Z)=X\cdot(Z\tilde{Y}),$$
$$X\cdot(YZ)=(\tilde{Y}X)\cdot Z=(X\tilde{Z})\cdot Y.\tag{A26}$$

For any $X,Y\in V$,

$$\overline{XY}=\overline{X}\overline{Y},$$
$$\widetilde{XY}=\tilde{Y}\tilde{X}.\tag{A27}$$

Let $I\in\Lambda^nV$ then for any $v\in V$ and $X\in\Lambda V$,

$$I(v\wedge X)=(-1)^{n-1}v\lrcorner(IX).\tag{A28}$$

Equation (A22) is sometimes called the duality identity and plays an important role in the applications involving the Hodge dual operator [see Eq. (53)].

For any $X,Y,Z\in V$,

$$X\lrcorner(Y\wedge Z)=(X\wedge Y)\lrcorner Z,$$
$$(X\llcorner Y)\llcorner Z=X\llcorner(Y\wedge Z).\tag{A29}$$

For any $X,Y\in V$,

$$X\cdot Y=\langle\tilde{X}Y\rangle_0.\tag{A30}$$

For $X_r\in\Lambda^rV,\ Y_s\in\Lambda^sV$ we have

$$X_rY_s=\langle X_rY_s\rangle_{|r-s|}+\langle X_rY_s\rangle_{|r-s|+2}+\cdots+\langle X_rY_s\rangle_{r+s}.\tag{A31}$$

(We observe also that when $K=\mathbb{R}$ and the quadratic form is Euclidean then $X\cdot Y$ is positive definite.)

APPENDIX B: REPRESENTATION THEORY OF THE REAL CLIFFORD ALGEBRAS $\mathbb{R}_{p,q}$

The real Clifford algebras $\mathbb{R}_{p,q}$ are associative algebras and they are simple or semisimple algebras. For the intelligibility of the present paper, it is then necessary to have in mind some results concerning the presentation theory of associative algebras, which we collect in what follows, without presenting proofs.

1. Some results from the representation theory of associative algebras

Let $\mathbf{V}$ be a set and $\mathbb{K}$ a division ring. Give to the set $\mathbf{V}$ a structure of finite-dimensional linear space over $\mathbb{K}$. Suppose that $\dim_{\mathbb{K}}\mathbf{V}=n$, where $n\in\mathbb{Z}$. We are interested in what follows in the cases where $\mathbb{K}=\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. When $\mathbb{K}=\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, we call $\mathbf{V}$ a vector space over $\mathbb{K}$. When $\mathbb{K}=\mathbb{H}$ it is necessary to distinguish between right or left $\mathbb{H}$-linear spaces and in this case $\mathbf{V}$ will be called a right or left $\mathbb{H}$-module. Recall that $\mathbb{H}$ is a division ring (sometimes called a noncommutative field or a skew field) and since $\mathbb{H}$ has a natural vector space structure over the real field, then $\mathbb{H}$ is also a division algebra.

Let $\dim_{\mathbb{R}}\mathbf{V}=2m=n$. In this case it is possible to give the following.

Definition 26: A linear mapping

$$\mathbf{J}:\mathbf{V}\to\mathbf{V}, \tag{B1}$$

such that

$$\mathbf{J}^2=-\mathrm{Id}_{\mathbf{V}}, \tag{B2}$$

is called a complex structure mapping.

Definition 27: The pair $(\mathbf{V},\mathbf{J})$ *will be called a complex vector space structure and denote by* $\mathbf{V}_{\mathbb{C}}$ *if the following product holds. Let* $\mathbb{C}\ni z=a+ib$ *and let* $\mathbf{v}\in\mathbf{V}$*. Then*

$$z\mathbf{v}=(a+ib)\mathbf{v}=a\mathbf{v}+b\mathbf{J}\mathbf{v}. \tag{B3}$$

It is obvious that $\dim_{\mathbb{C}}=m/2$.

Definition 28: Let $\mathbf{V}$ *be a vector space over* $\mathbb{R}$*. A* complexification *of* $\mathbf{V}$ *is a complex structure associated with the real vector space* $\mathbf{V}\oplus\mathbf{V}$*. The resulting complex vector space is denoted by* $\mathbf{V}^{\mathbb{C}}$*. Let* $\mathbf{v},\mathbf{w}\in\mathbf{V}$*. Elements of* $\mathbf{V}^{\mathbb{C}}$ *are usually denoted by* $\mathbf{c}=\mathbf{v}+i\mathbf{w}$*, and if* $\mathbb{C}\ni z=a+ib$ *we have*

$$z\mathbf{c}=a\mathbf{v}-b\mathbf{w}+i(a\mathbf{w}+b\mathbf{v}). \tag{B4}$$

Of course, we have that $\dim_{\mathbb{C}}\mathbf{V}^{\mathbb{C}}=\dim_{\mathbb{R}}\mathbf{V}$.

Definition 29: A $\mathbb{H}$*-module is a real vector space* $\mathbf{V}$ *carrying three linear transformation,* $\mathbf{I}$, $\mathbf{J}$, *and* $\mathbf{K}$ *each one of them satisfying*

$$\begin{gathered}\mathbf{I}^2=\mathbf{J}^2=-\mathbf{I}\mathrm{d}_{\mathbf{S}},\\ \mathbf{IJ}=-\mathbf{JI}=\mathbf{K},\quad \mathbf{JK}=-\mathbf{KJ}=\mathbf{I},\quad \mathbf{KI}=-\mathbf{IK}=\mathbf{J}.\end{gathered} \tag{B5}$$

Definition 30: Any subset $I\subseteq\mathcal{A}$ *such that*

$$\begin{gathered}a\psi\in I,\forall a\in\mathcal{A},\forall\psi\in I,\\ \psi+\phi\in I,\forall\psi,\phi\in I\end{gathered} \tag{B6}$$

is called a left ideal of $\mathcal{A}$.

Remark 31: An analogous definition holds for right ideals where Eq. (B6) reads $\psi a \in I, \forall a \in \mathcal{A}$, $\forall \psi \in I$, *for* bilateral *ideals where in this case Eq. (B6) reads* $a\psi b \in I, \forall a,b \in \mathcal{A}$, $\forall \psi \in I$.

Definition 32: An associative $\mathcal{A}$ *algebra on the* field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$) *is* simple *if the only bilateral ideals are the zero ideal and* $\mathcal{A}$ *itself.*

We give without proofs the following theorems.

Theorem 33: *All minimal left (respectively, right) ideals of* $\mathcal{A}$ *are of the form* $J = \mathcal{A}e$ (*respectively,* $e\mathcal{A}$), *where* e *is a primitive idempotent of* $\mathcal{A}$.

Theorem 34: *Two minimal left ideals of* $\mathcal{A}$, $J = \mathcal{A}e$ *and* $J = \mathcal{A}e'$ *are* isomorphic *if and only if there exist a non-null* $X' \in J'$ *such that* $J' = JX'$.

We recall that $e \in \mathcal{A}$ *is an* idempotent *element if* $e^2 = e$. *An idempotent is said to be* primitive *if it cannot be written as the sum of two nonzero annihilating (or orthogonal) idempotent,* i.e., $e \neq e_1 + e_2$, *with* $e_1 e_2 = e_2 e_1 = 0$ *and* $e_1^2 = e_1$, $e_2^2 = e_2$.

Not all algebras are simple and in particular semisimple algebras are important for our considerations. A definition of semisimple algebras requires the introduction of the concepts of nilpotent ideals and radicals. To define these concepts adequately would lead us to a long incursion on the theory of associative algebras, so we avoid to do that here. We only quote that semisimple algebras are the direct sum of simple algebras. Then, the study of semisimple algebras is reduced to the study of simple algebras.

Now, let $\mathcal{A}$ be an associative and simple algebra on the field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$), and let $\mathbf{S}$ be a finite-dimensional linear space over a division ring $\mathbb{K} \subseteq \mathbb{F}$.

Definition 35: *A* representation *of* $\mathcal{A}$ *in* $\mathbf{S}$ *is a* $\mathbb{K}$ *algebra homomorphism* [we recall that a $\mathbb{K}$-algebra homomorphism is a $\mathbb{K}$-linear map ρ such that $\forall X, Y \in \mathcal{A}$, $\rho(XY) = \rho(X)\rho(Y)$] $\rho: \mathcal{A} \rightarrow \mathbf{E} = \mathrm{End}_{\mathbb{K}}\mathbf{S}$ ($\mathbf{E} = \mathrm{End}_{\mathbb{K}}\mathbf{S} = \mathrm{Hom}_{\mathbb{K}}(\mathbf{S},\mathbf{S})$ *is the endomorphism algebra of* $\mathbf{S}$) *which maps the unit element of* $\mathcal{A}$ *to* $\mathrm{Id}_{\mathbf{E}}$. *The dimension* $\mathbb{K}$ *of* $\mathbf{S}$ *is called the* degree *of the representation.*

The addition in $\mathbf{S}$ together with the mapping $\mathcal{A} \times \mathbf{S} \rightarrow \mathbf{S}$, $(a,x) \mapsto \rho(a)x$ turns $\mathbf{S}$ in a left $\mathcal{A}$-module, called the left *representation module*. [We recall that there are left and right modules, so we can also define right modular representations of $\mathcal{A}$ by defining the mapping $\mathbf{S} \times \mathcal{A} \rightarrow \mathbf{S}$, $(x,a) \mapsto x\rho(a)$. This turns $\mathbf{S}$ in a right $\mathcal{A}$-module, called the right *representation module*.]

Remark 36: It is important to recall that when $\mathbb{K} = \mathbb{H}$ *the usual recipe for* $\mathrm{Hom}_{\mathbb{H}}(\mathbf{S},\mathbf{S})$ *to be a linear space over* $\mathbb{H}$ *fails and in general* $\mathrm{Hom}_{\mathbb{H}}(\mathbf{S},\mathbf{S})$ *is considered as a linear space over* $\mathbb{R}$, *which is the center of* $\mathbb{H}$.

Remark 37: We also have that if $\mathcal{A}$ *is an algebra over* $\mathbb{F}$ *and* $\mathbf{S}$ *is an* $\mathcal{A}$*-module, then* $\mathbf{S}$ *can always be considered as a vector space over* $\mathbb{F}$ *and if* $e \in \mathcal{A}$, *the mapping* $\chi: a \rightarrow \chi_a$ *with* $\chi_a(\mathbf{s}) = a\mathbf{s}$, $\mathbf{s} \in \mathbf{S}$, *is a homomorphism* $\mathcal{A} \rightarrow \mathbf{E} = \mathrm{End}_{\mathbb{F}}\mathbf{S}$, *and so it is a representation of* $\mathcal{A}$ *in* $\mathbf{S}$. *The study of* $\mathcal{A}$ *modules is then equivalent to the study of the* $\mathbb{F}$ *representations of* $\mathcal{A}$.

Definition 38: A representation ρ *is faithful if its kernel is zero, i.e.,* $\rho(a)x = 0, \forall x \in \mathbf{S} \Rightarrow a = 0$. *The kernel of* ρ *is also known as the* annihilator *of its module.*

Definition 39: ρ *is said to be* simple *or irreducible if the only invariant subspaces of* $\rho(a)$, $\forall a \in \mathcal{A}$, *are* $\mathbf{S}$ *and* $\{0\}$.

Then, the representation module is also simple. That means that it has no proper submodules.

Definition 40: ρ *is said to be semisimple, if it is the direct sum of simple modules, and in this case* $\mathbf{S}$ *is the direct sum of subspaces which are globally invariant under* $\rho(a)$, $\forall a \in \mathcal{A}$.

When no confusion arises $\rho(a)x$ may be denoted by $a \cdot x$, $a * x$ or ax.

Definition 41: Two $\mathcal{A}$*-modules* $\mathbf{S}$ *and* $\mathbf{S}'$ *(with the exterior multiplication being denoted, respectively, by* $\cdot$ *and* $*$*) are* isomorphic *if there exists a bijection* $\varphi: \mathbf{S} \rightarrow \mathbf{S}'$ *such that*

$$\varphi(x+y) = \varphi(x) + \varphi(y), \quad \forall x,y \in \mathbf{S},$$

$$\varphi(a \cdot x) = a * \varphi(x), \quad \forall a \in \mathcal{A},$$

and we say that representation ρ *and* ρ' *of* $\mathcal{A}$ *are equivalent if their modules are isomorphic.*

This implies the existence of a $\mathbb{K}$-linear isomorphism $\varphi: \mathbf{S} \rightarrow \mathbf{S}'$ such that $\varphi \circ \rho(a) = \rho'(a) \circ \varphi$, $\forall a \in \mathcal{A}$ or $\rho'(a) = \varphi \circ \rho(a) \circ \varphi^{-1}$. If $\dim \mathbf{S} = n$, then $\dim \mathbf{S}' = n$.

TABLE I. Representation of the Clifford algebras $\mathbb{R}_{p,q}$ as matrix algebras.

$p-q$ mod 8	0	1	2	3	4	5	6	7
$\mathbb{R}_{p,q}$	$\mathbb{R}(2^{[n/2]})$	$\mathbb{R}(2^{[n/2]}) \oplus \mathbb{R}(2^{[n/2]})$	$\mathbb{R}(2^{[n/2]})$	$\mathbb{C}(2^{[n/2]})$	$\mathbb{H}(2^{[n/2]-1})$	$\mathbb{H}(2^{[n/2]-1}) \oplus \mathbb{H}(2^{[n/2]-1})$	$\mathbb{H}(2^{[n/2]-1})$	$\mathbb{C}(2^{[n/2]})$

Definition 42: A complex representation *of $\mathcal{A}$ is simply a real representation $\rho:\mathcal{A} \to \mathrm{Hom}_{\mathbb{R}}(\mathbf{S},\mathbf{S})$ for which*

$$\rho(X)\circ\mathbf{J}=\mathbf{J}\circ\rho(X), \quad \forall X\in\mathcal{A}. \tag{B7}$$

This means that the image of ρ commutes with the subalgebra generated by $\{\mathbf{I}\mathrm{d}_{\mathbf{S}}\}\sim\mathbb{C}$.

Definition 43: A quaternionic representation of $\mathcal{A}$ is a representation $\rho:\mathcal{A}\to\mathrm{Hom}_{\mathbb{R}}(\mathbf{S},\mathbf{S})$ such that

$$\rho(X)\circ\mathbf{I}=\mathbf{I}\circ\rho(X), \quad \rho(X)\circ\mathbf{J}=\mathbf{J}\circ\rho(X), \quad \rho(X)\circ\mathbf{K}=\mathbf{K}\circ\rho(X), \quad \forall X\in\mathcal{A}. \tag{B8}$$

This means that the representation ρ has a commuting subalgebra isomorphic to the quaternion ring.

The following theorem[61,109] is crucial.

Theorem 44 (Wedderburn): *If $\mathcal{A}$ is simple algebra over $\mathbb{F}$ then $\mathcal{A}$ is isomorphic to $\mathbb{D}(m)$, where $\mathbb{D}(m)$ is a matrix algebra with entries in $\mathbb{D}$ (a division algebra), and m and $\mathbb{D}$ are unique (modulo isomorphisms).*

Now, it is time to specialize our results to the Clifford algebras over the field $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$. We are particularly interested in the case of real Clifford algebras. In what follows we take $(\mathbf{V},\mathbf{b})=(\mathbb{R}^n,\mathbf{g})$. We denote by $\mathbb{R}^{p,q}$ a real vector space of dimension $n=p+q$ endowed with a nondegenerate metric $\mathbf{g}:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$. Let $\{E_i\}$, $(i=1,2,\ldots,n)$ be an orthonormal basis of $\mathbb{R}^{p,q}$,

$$\mathbf{g}(E_i,E_j)=g_{ij}=g_{ji}=\begin{cases} +1, & i=j=1,2,\ldots,p \\ -1, & i=j=p+1,\ldots,p+q=n \\ 0, & i\neq j. \end{cases} \tag{B9}$$

Definition 45: The Clifford algebra $\mathbb{R}_{p,q}=\mathcal{C}\ell(\mathbb{R}^{p,q})$ is the Clifford algebra over $\mathbb{R}$, generated by 1 and the $\{E_i\}$ $(i=1,2,\ldots,n)$, such that $E_i^2=\mathbf{q}(E_i)=\mathbf{g}(E_i,E_i)$, $E_iE_j=-E_jE_i$ $(i\neq j)$, and $E_1E_2\ldots E_n\neq\pm1$.

$\mathbb{R}_{p,q}$ is obviously of dimension 2^n and as a vector space it is the direct sum of vector spaces $\Lambda^k\mathbb{R}^n$ of dimensions $\binom{n}{k}$,$0\leqslant k\leqslant n$. The canonical basis of $\Lambda^k\mathbb{R}^n$ is given by the elements $e_A=E_{\alpha_1}\cdots E_{\alpha_k}$, $1\leqslant\alpha_1<\cdots<\alpha_k\leqslant n$. The element $e_J=E_1\cdots E_n\in\Lambda^k\mathbb{R}^n\subset\mathbb{R}_{p,q}$ commutes (n odd) or anticommutes (n even) with all vectors $E_1\cdots E_n\in\Lambda^1\mathbb{R}^n\equiv\mathbb{R}^n$. The center $\mathcal{C}\ell_{p,q}$ is $\Lambda^0\mathbb{R}^n\equiv\mathbb{R}$ if n is even and it is the direct sum $\Lambda^0\mathbb{R}^n\oplus\Lambda^0\mathbb{R}^n$ if n is odd.

All Clifford algebras are semisimple. If $p+q=n$ is even, $\mathbb{R}_{p,q}$ is simple and if $p+q=n$ is odd we have the following possibilities.

(a) $\mathbb{R}_{p,q}$ is simple $\leftrightarrow c_J^2=-1\leftrightarrow p-q\neq1 \pmod 4\leftrightarrow$center of $\mathbb{R}_{p,q}$ is isomorphic to $\mathbb{C}$;

(b) $\mathbb{R}_{p,q}$ is not simple (but is a direct sum of two simple algebras)$\leftrightarrow c_J^2=+1\leftrightarrow p-q=1 \pmod 4\leftrightarrow$center of $\mathbb{R}_{p,q}$ is isomorphic to $\mathbb{R}\oplus\mathbb{R}$.

Now, for $\mathbb{R}_{p,q}$ the division algebras $\mathbb{D}$ are the division rings $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. The explicit isomorphism can be discovered with some hard but not difficult work. It is possible to give a general classification off all real (and also the complex) Clifford algebras and a classification table can be found, e.g., in Refs. 141 and 142. Table I is reproduced and $[n/2]$ means the integer part of $n/2$.

Now, to complete the classification we need the following theorem.[141]

Theorem 46 (Periodicity):

$$\mathbb{R}_{n+8}=\mathbb{R}_{n,0}\otimes\mathbb{R}_{8,0}, \quad \mathbb{R}_{0,n+8}=\mathbb{R}_{0,n}\otimes\mathbb{R}_{0,8},$$
$$\mathbb{R}_{p+8,q}=\mathbb{R}_{p,q}\otimes\mathbb{R}_{8,0}, \quad \mathbb{R}_{p,q+8}=\mathbb{R}_{p,q}\otimes\mathbb{R}_{0,8}. \tag{B10}$$

Remark 47: We emphasize here that since the general results concerning the representations of simple algebras over a field $\mathbb{F}$ applies to the Clifford algebras $\mathbb{R}_{p,q}$ we can talk about real, complex or quaternionic representation of a given Clifford algebra, even if the natural matrix identification is not a matrix algebra over one of these fields. A case that we shall need is that $\mathbb{R}_{1,3}\simeq\mathbb{H}(2)$. But it is clear that $\mathbb{R}_{1,3}$ has a complex representation, for any quaternionic representation of $\mathbb{R}_{p,q}$ is automatically complex, *once we restrict $\mathbb{C}\subset\mathbb{H}$ and of course, the complex dimension of any $\mathbb{H}$-module must be even. Also, any complex representation of $\mathbb{R}_{p,q}$ extends automatically to a representation of $\mathbb{C}\otimes\mathbb{R}_{p,q}$.*

Remark 48: *Now, $\mathbb{C}\otimes\mathbb{R}_{p,q}$ is an abbreviation for the* complex *Clifford algebra $\mathcal{C}\ell_{p+q}=\mathbb{C}\otimes\mathbb{R}_{p,q}$, i.e., it is the tensor product of the algebras $\mathbb{C}$ and $\mathbb{R}_{p,q}$, which are subalgebras of the finite-dimensional algebra $\mathcal{C}\ell_{p+q}$ over $\mathbb{C}$.*

For the purposes of the present paper we must keep in mind that

$$\begin{gathered}\mathbb{R}_{0,1}\simeq\mathbb{C},\\ \mathbb{R}_{0,2}\simeq\mathbb{H},\\ \mathbb{R}_{3,0}\simeq\mathbb{C}(2),\\ \mathbb{R}_{1,3}\simeq\mathbb{H}(2),\\ \mathbb{R}_{3,1}\simeq\mathbb{R}(4),\\ \mathbb{R}_{4,1}\simeq\mathbb{C}(4).\end{gathered} \tag{B11}$$

$\mathbb{R}_{3,0}$ is called the Pauli algebra, $\mathbb{R}_{1,3}$ is called the *space–time* algebra, $\mathbb{R}_{3,1}$ is called *Majorana* algebra and $\mathbb{R}_{4,1}$ is called the *Dirac* algebra. Also the following particular results have been used in the text and below:

$$\mathbb{R}^0_{1,3}\simeq\mathbb{R}^0_{3,1}=\mathbb{R}_{3,0}, \quad \mathbb{R}^0_{4,1}\simeq\mathbb{R}_{1,3},$$
$$\mathbb{R}_{4,1}\simeq\mathbb{C}\otimes\mathbb{R}_{3,1}, \quad \mathbb{R}_{4,1}\simeq\mathbb{C}\otimes\mathbb{R}_{3,1}, \tag{B12}$$

which means that the Dirac algebra is the complexification of both the space–time or the Majorana algebras.

Equation (B11) show moreover, in view of Remark 7 that the space–time algebra has a complexification matrix representation in $\mathbb{C}(4)$. Obtaining such a representation is fundamental for the present work and it is given in Appendix D.

2. Minimal lateral ideals of $\mathbb{R}_{p,q}$

It is important for the objectives of this paper to know some results concerning the minimal lateral ideals of $\mathbb{R}_{p,q}$. The identification table of these algebras as matrix algebras helps a lot. Indeed, we have[61] the following theorem.

Theorem 49: *The maximum number of pairwise orthogonal idempotents in $\mathbb{K}(m)$ (where $\mathbb{K}=\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$) is m.*

The decomposition of $\mathbb{R}_{p,q}$ into minimal ideals is then characterized by a spectral set $\{e_{pq,j}\}$ of idempotents elements of $\mathbb{R}_{p,q}$ such that

(a) $\Sigma_{i=1}^{n}e_{pq,i}=1$,
(b) $e_{pq,j}e_{pq,k}=\delta_{jk}e_{pq,j}$,

(c) the rank of $e_{pq,j}$ is minimal and nonzero, i.e., is primitive.

By rank of $e_{pq,j}$ we mean the rank of the $\Lambda\mathbb{R}^{p,q}$ morphism, $e_{pq,j}:\phi\mapsto\phi e_{pq,j}$. Conversely, any $\phi\in\mathbf{I}_{pq,j}$ can be characterized by an idempotent $e_{pq,j}$ of minimal rank $\neq 0$, with $\phi=\phi e_{pq,j}$.

We now need to know the following theorem.[109]

Theorem 50: *A minimal left ideal of $\mathbb{R}_{p,q}$ is of the type*

$$\mathbf{I}_{pq}=\mathbb{R}_{p,q}e_{pq},$$

where

$$e_{pq}=\tfrac{1}{2}(1+e_{\alpha_1})\cdots\tfrac{1}{2}(1+e_{\alpha_k}) \tag{B13}$$

is a primitive idempotent of $\mathbb{R}_{p,q}$ and were $e_{\alpha_1},\cdots,e_{\alpha_k}$ are commuting elements in the canonical basis of $\mathbb{R}_{p,q}$ generated in the standard way through the elements of the basis Σ such that $(e_{\alpha_i})^2=1$, $(i=1,2,\ldots,k)$ generate a group of order 2^k, $k=q-r_{q-p}$ and r_i are the Radon–Hurwitz numbers, defined by the recurrence formula $r_{i+8}=r_i+4$ and

i	0	1	2	3	4	5	6	7
r_i	0	1	2	2	3	3	3	3

(B14)

Recall that $\mathbb{R}_{p,q}$ is a ring and the minimal lateral ideals are modules over the ring $\mathbb{R}_{p,q}$. They are *representation modules* of $\mathbb{R}_{p,q}$, and indeed we have (recall the table above) the following theorem.[141]

Theorem 51: *If $p+q$ is even or odd with $p-q\neq 1$ (mod 4), then*

$$\mathbb{R}_{p,q}=\mathrm{Hom}_{\mathbb{K}}(I_{pq})\simeq\mathbb{K}(m), \tag{B15}$$

where (as we already know) $\mathbb{K}=\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Also,

$$\dim_{\mathbb{K}}(I_{pq})=m \tag{B16}$$

and

$$\mathbb{K}\simeq e\mathbb{K}(m)e, \tag{B17}$$

where e is the representation of $\mathbf{e}_{pq}$ in $\mathbb{K}(m)$.

If $p+q=n$ is odd, with $p-q=1$ (mod 4), then

$$\mathbb{R}_{p,q}=\mathrm{Hom}_{\mathbb{K}}(I_{pq})\simeq\mathbb{K}(m)\oplus\mathbb{K}(m), \tag{B18}$$

with

$$\dim_{\mathbb{K}}(I_{pq})=m \tag{B19}$$

and

$$e\mathbb{K}(m)e\simeq\mathbb{R}\oplus\mathbb{R}$$

$$\text{or} \tag{B20}$$

$$e\mathbb{K}(m)e\simeq\mathbb{H}\oplus\mathbb{H}.$$

With the above isomorphisms we can immediately identify the minimal left ideals of $\mathbb{R}_{p,q}$ with the column matrices of $\mathbb{K}(m)$.

Algorithm for finding primitive idempotents of $\mathbb{R}_{p,q}$: With the ideas introduced above it is now a simple exercise to find primitive idempotents of $\mathbb{R}_{p,q}$. First we look at Table I and find the matrix algebra to which our particular Clifford algebra $\mathbb{R}_{p,q}$ is isomorphic. Suppose $\mathbb{R}_{p,q}$ is simple. (Once we know the algorithm for a simple Clifford algebra it is straightfoward to devise an algorithm for the semisimple Clifford algebras.) Let $\mathbb{R}_{p,q} \simeq \mathbb{K}(m)$ for a particular $\mathbb{K}$ and m. Next we take an element $e_{\alpha_1} \in \{e_A\}$ from the canonical basis $\{e_A\}$ of $\mathbb{R}_{p,q}$ such that

$$e_{\alpha_1}^2 = 1, \tag{B21}$$

then construct the idempotent $e_{pq} = (1 + e_{\alpha_1})/2$ and the ideal $\mathbf{I}_{pq} = \mathbb{R}_{p,q} e_{pq}$ and calculate $\dim_{\mathbb{K}}(I_{pq})$. If $\dim_{\mathbb{K}}(I_{pq}) = m$, then e_{pq} is primitive. If $\dim_{\mathbb{K}}(I_{pq}) \neq m$, we choose $e_{\alpha 2} \in \{e_A\}$ such that $e_{\alpha 2}$ commutes with e_{α_1} and $e_{\alpha_2}^2 = 1$ [see Theorem 39 and construct the idempotent $e'_{pq} = (1 + e_{\alpha_1})(1 + e_{\alpha_1})/4$]. If $\dim_{\mathbb{K}}(I'_{pq}) = m$, then e'_{pq} is primitive. Otherwise we repeat the procedure. According to the Theorem 39 the procedure is finite.

These results will be used in Appendix D in order to obtain necessary results for our presentation of the theory of algebraic and Dirac–Hestenes spinors (and spinors fields).

APPENDIX C: $\mathbb{R}_{p,q}^{\star}$, CLIFFORD, PINOR AND SPINOR GROUPS

The set of the invertible elements of $\mathbb{R}_{p,q}$ constitutes a non-Abelian group which we denote by $\mathbb{R}_{p,q}^{\star}$. It acts naturally on $\mathbb{R}_{p,q}$ as an algebra homomorphism through its adjoint representation

$$\mathrm{Ad}: \mathbb{R}_{p,q}^{\star} \to \mathrm{Aut}(\mathbb{R}_{p,q}); u \mapsto \mathrm{Ad}_u, \text{ with } \mathrm{Ad}_u(x) = uxu^{-1}. \tag{C1}$$

Definition 52: The Clifford–Lipschitz *group is the set*

$$\Gamma_{p,q} = \{u \in \mathbb{R}_{p,q}^{\star} | \forall x \in \mathbb{R}^{p,q}, uxu^{-1} \in \mathbb{R}^{p,q}\}. \tag{C2}$$

Definition 53: The set $\Gamma_{p,q}^{+} = \Gamma_{p,q} \cap \mathbb{R}_{p,q}$ *is called* special *Clifford–Lipshitz group.*

Definition 54: The Pinor group $\mathrm{Pin}_{p.q}$ *is the subgroup of* $\Gamma_{p,q}$ *such that*

$$\begin{gathered} \mathrm{Pin}_{p,q} = \{u \in \Gamma_{p,q} | N(u) = \pm 1\}, \\ N: \mathbb{R}_{p,q} \to \mathbb{R}_{p,q}, N(x) = \langle \bar{x} x \rangle_0 . \end{gathered} \tag{C3}$$

Definition 55: The Spin group $\mathrm{Spin}_{p,q}$ *is the set*

$$\mathrm{Spin}_{p,q} = \{u \in \Gamma_{p,q} | N(u) = \pm 1\}. \tag{C4}$$

It is easy to see that $\mathrm{Spin}_{p,q}$ is not connected.

Definition 56: The group $\mathrm{Spin}_{p,q}^{e}$ *is the set*

$$\mathrm{Spin}_{p,q}^{e} = \{u \in \Gamma_{p,q} | N(u) = +1\}. \tag{C5}$$

The superscript e, means that $\mathrm{Spin}_{p,q}^{e}$ is the connected component to the identity. We can prove that $\mathrm{Spin}_{p,q}^{e}$ is connected for all pairs (p,q) with the exception of $\mathrm{Spin}^{e}(1,0) \simeq \mathrm{Spin}^{e}(0,1)$.

We recall now some classical results[120] associated with the pseudo-orthogonal groups $\mathrm{O}_{p,q}$ of a vector space $\mathbb{R}^{p,q}$ $(n = p + q)$ and its subgroups.

Let $\mathbf{G}$ be a diagonal $n \times n$ matrix whose elements are

$$G_{ij} = \mathrm{diag}(1,1, \ldots, -1,-1, \ldots -1), \tag{C6}$$

with p positive and q negative numbers.

Definition 57: $O_{p,q}$ is the set of $n\times n$ real matrices $\mathbf{L}$ *such that*

$$\mathbf{LGL}^T=\mathbf{G},\quad \det\mathbf{L}^2=1. \tag{C7}$$

Equation (C7) shows that $\mathrm{O}_{p,q}$ is not connected.

Definition 58: $SO_{p,q}$, the special (proper) pseudo-orthogonal group is the set of $n\times n$ real matrices $\mathbf{L}$ *such that*

$$\mathbf{LGL}^T=\mathbf{G},\quad \det\mathbf{L}=1. \tag{C8}$$

When $p=0$ $(q=0)$ $\mathrm{SO}_{p,q}$ is connected. However, $\mathrm{SO}_{p,q}$ is not connected and has two connected components for $p,q\geqslant 1$. The group $\mathrm{SO}^e_{p,q}$, the connected component to the identity of $\mathrm{SO}_{p,q}$ will be called the special *orthocronous* pseudo-orthogonal group. [This nomenclature comes from the fact that $\mathrm{SO}^e(1,3)=\mathcal{L}_+^\uparrow$ is the special (proper) orthochronous Lorentz group. In this case the set is easily defined by the condition $L_0^0\geqslant+1$. For the general case see Ref. 120.]

Theorem 59: $Ad_{|\mathrm{Pin}_{p,q}}:\mathrm{Pin}_{p,q}\to\mathrm{O}_{p,q}$ *is onto with kernel* $\mathbb{Z}_2$. $Ad_{|\mathrm{Spin}_{p,q}}:\mathrm{Spin}_{p,q}\to\mathrm{SO}_{p,q}$ *is onto with kernel* $\mathbf{Z}_2$. $Ad_{|\mathrm{Spin}^e_{p,q}}:\mathrm{Spin}^e_{p,q}\to\mathrm{SO}^e_{p,q}$ *is onto with kernel* $\mathbf{Z}_2$. *We have*

$$\mathrm{O}_{p,q}=\frac{\mathrm{Pin}_{p,q}}{\mathbb{Z}_2},\quad \mathrm{SO}_{p,q}=\frac{\mathrm{Spin}_{p,q}}{\mathbb{Z}_2},\quad \mathrm{SO}^e_{p,q}=\frac{\mathrm{Spin}^e_{p,q}}{\mathbb{Z}_2}. \tag{C9}$$

The group homomorphism between $\mathrm{Spin}^e_{p,q}$ and $\mathrm{SO}^e(p,q)$ will be denoted by

$$\mathbf{L}:\mathrm{Spin}^e_{p,q}\to\mathrm{SO}^e_{p,q}. \tag{C10}$$

The following theorem that first appears in Porteous book[141] is very important. (In particular, when Theorem 49 is taken into account together with some of the coincidence between the complexifications of some low dimensions Clifford algebras it becomes clear that the construction of Dirac–Hestenes spinors [and its representation as in Eq. (D20)] for Minkowski vector space has no generalization for vector spaces of arbitrary dimensions and signatures.[109])

Theorem 60 (Porteous): *For* $p+q\leqslant 5$, $\mathrm{Spin}^e(p,q)=\{u\in\mathbb{R}_{p,q}|u\tilde{u}=1\}$.

Lie algebra of $\mathrm{Spin}^e_{1,3}$: It can be shown[109] that for each $u\in\mathrm{Spin}^e_{1,3}$ it holds $u=\pm e^F$, $F\in\Lambda^2\mathbb{R}^{1,3}\subset\mathbb{R}_{1,3}$ and F can be chosen in such a way to have a positive sign in Eq. (C8), except in the particular case $F^2=0$ when $u=-e^F$. From Eq. (C8) it follows immediately that the Lie algebra of $\mathrm{Spin}^e_{1,3}$ is generated by the bivectors $F\in\Lambda^2\mathbb{R}^{1,3}\subset\mathbb{R}_{1,3}$ through the commutator product. More details on the relations of Clifford algebras and the rotation groups may be found, e.g., in Refs. 7 and 170.

APPENDIX D: SPINOR REPRESENTATIONS OF $\mathbb{R}_{4,1}$, $\mathbb{R}^+_{4,1}$, AND $\mathbb{R}_{1,3}$

Let $b_0=\{E_0,E_1,E_2,E_3\}$ be an orthogonal basis of $\mathbb{R}^{1,3}\subset\mathbb{R}_{1,3}$, such that $E_\mu E_\nu+E_\nu E_\mu=2\eta_{\mu\nu}$, with $\eta_{\nu\mu}=\mathrm{diag}(+1,-1,-1,-1)$. Now, with the results of Appendix B we can verify without difficulties that the elements e, e', $e''\in\mathbb{R}_{1,3}$,

$$e=\tfrac{1}{2}(1+E_0), \tag{D1}$$

$$e'=\tfrac{1}{2}(1+E_3E_0), \tag{D2}$$

$$e''=\tfrac{1}{2}(1+E_1E_2E_3), \tag{D3}$$

are primitive idempotents of $\mathbb{R}_{1,3}$ The minimal left ideals, $I=\mathbb{R}_{1,3}e$, $I'=\mathbb{R}_{1,3}e'$, $I''=\mathbb{R}_{1,3}e''$ are *right* two dimension linear spaces over the quaternion field (e.g., $\mathbb{H}e=e\mathbb{H}=e\mathbb{R}_{1,3}e$). According to a definition given originally in Ref. 150 these ideals are algebraically equivalent. For example, $e'=ueu^{-1}$, with $u=(1+E_3)\notin\Gamma_{1,3}$.

Definition 61: The elements $\Phi\in\mathbb{R}_{1,3}\frac{1}{2}(1+E_0)$ are called mother *spinors*.

The above denomination has been given (with justice) by Lounesto.[109] It can be shown[67,68] that each Φ can be written

$$\Phi=\psi_1 e+\psi_2 E_3 E_1 e+\psi_3 E_3 E_0 e+\psi_4 E_1 E_0 e=\sum_i \psi_i s_i, \tag{D4}$$

$$s_1=e,\quad s_2=E_3E_1e,\quad s_3=E_3E_0e,\quad s_4=E_1E_0e \tag{D5}$$

and where the ψ_i are *formally* complex numbers, i.e., each $\psi_i=(a_i+b_iE_2E_1)$ with $a_i,\ b_i\in\mathbb{R}$ and the set $\{s_i, i=1,2,3,4\}$ is a basis in the mother spinors space.

We recall from the general result of Appendix C that $\mathrm{Pin}_{1,3}/\mathbb{Z}_2\simeq\mathrm{O}_{1,3}$, $\mathrm{Spin}_{1,3}/\mathbb{Z}_2\simeq\mathrm{SO}_{1,3}$, $\mathrm{Spin}^e_{1,3}/\mathbb{Z}_2\simeq\mathrm{SO}^e_{1,3}$, and $\mathrm{Spin}^e_{1,3}\simeq\mathrm{Sl}(2,\mathbb{C})$ is the universal covering group of $\mathcal{L}_+^\uparrow\equiv\mathrm{SO}^e_{1,3}$, the *special* (proper) *orthocronous* Lorentz group.

In order to determine the relation between $\mathbb{R}_{4,1}$ and $\mathbb{R}_{3,1}$ we proceed as follows: let $\{F_0,F_1,F_2,F_3,F_4\}$ be an orthonormal basis of $\mathbb{R}_{4,1}$ with

$$-F_0^2=F_1^2=F_2^2=F_3^2=F_4^2=1, F_AF_B=-F_BF_A(A\neq B;A,B=0,1,2,3,4).$$

Define the pseudoscalar

$$\mathbf{i}=F_0F_1F_2F_3F_4,\quad \mathbf{i}^2=-1,\quad \mathbf{i}F_A=F_A\mathbf{i},\quad A=0,1,2,3,4. \tag{D6}$$

Define

$$\mathcal{E}_\mu=F_\mu F_4. \tag{D7}$$

We can immediately verify that $\mathcal{E}_\mu\mathcal{E}_\nu+\mathcal{E}_\nu\mathcal{E}_\mu=2\eta_{\mu\nu}$. Taking into account that $\mathbb{R}_{1,3}\simeq\mathbb{R}^0_{4,1}$ we can explicitly exhibit here this isomorphism by considering the map $\mathsf{j}:\mathbb{R}_{1,3}\to\mathbb{R}_{4,1}$ generated by the linear extension of the map $\mathsf{j}^\#:\mathbb{R}^{1,3}\to\mathbb{R}_{4,1}$, $\mathsf{j}^\#(F_\mu)=\mathcal{E}_\mu=F_\mu F_4$, where $\mathcal{E}_\mu$ $(\mu=0,1,2,3)$ is an orthogonal basis of $\mathbb{R}^{1,3}$. Also $\mathsf{j}(1_{\mathbb{R}_{1,3}})=1_{\mathbb{R}^+_{4,1}}$, where $1_{\mathbb{R}_{1,3}}$ and $1_{\mathbb{R}^+_{4,1}}$ are the identity elements in $\mathbb{R}_{1,3}$ and $\mathbb{R}^+_{4,1}$. Now consider the primitive idempotent of $\mathbb{R}_{1,3}\simeq\mathbb{R}^0_{4,1}$,

$$e_{41}=\mathsf{j}(e)=\tfrac{1}{2}(1+\mathcal{E}_0) \tag{D8}$$

and the minimal left ideal $I_{4,1}=\mathbb{R}_{4,1}e_{41}$.

In what follows we use (when convenient) for minimal idempotents and the minimal ideals generated by them, the labels involving the notion of spinorial frames discussed in Sec. II. Let then, Ξ_0 be a fiducial spinorial frame. The elements [in what follows we use (when convenient) for minimal idempotents and the minimal ideals generated by them, the labels involving the notion of spin frames discussed in Sec. II] $Z_{\Xi_0}\in I_{4,1}$ can be written analogously to $\Phi\in\mathbb{R}_{1,3}\frac{1}{2}(1+E_0)$ as

$$Z_{\Xi_0}=\sum z_i\overline{s_i}, \tag{D9}$$

where

$$\overline{s_1}=e_{41},\quad \overline{s_2}=\mathcal{E}_1\mathcal{E}_3e_{41},\quad \overline{s_3}=\mathcal{E}_3\mathcal{E}_0e_{41},\quad \overline{s_4}=\mathcal{E}_1\mathcal{E}_0e_{41} \tag{D10}$$

and where

$$z_i = a_i + \mathcal{E}_2\mathcal{E}_1 b_i$$

are formally complex numbers, a_i, $b_i \in \mathbb{R}$.

Consider now the element $f_{\Xi_0} \in \mathbb{R}_{4,1}$,

$$f_{\Xi_0} = e_{41}\tfrac{1}{2}(1+\mathbf{i}\mathcal{E}_1\mathcal{E}_2) = \tfrac{1}{2}(1+\mathcal{E}_0)\tfrac{1}{2}(1+\mathbf{i}\mathcal{E}_1\mathcal{E}_2), \tag{D11}$$

with $\mathbf{i}$ defined as in Eq. (D6).

Since $f_{\Xi_0}\mathbb{R}_{4,1}f_{\Xi_0} = \mathbb{C}f_{\Xi_0} = f_{\Xi_0}\mathbb{C}$ it follows that f_{Ξ_0} is a primitive idempotent of $\mathbb{R}_{4,1}$. We can easily show that each $\Phi_{\Xi_0} \in I_{\Xi_0} = \mathbb{R}_{4,1}f_{\Xi_0}$ can be written

$$\Psi_{\Xi_0} = \sum_i \psi_i f_i, \quad \psi_i \in \mathbb{C},$$
$$f_1 = f_{\Xi_0}, \quad f_2 = -\mathcal{E}_1\mathcal{E}_3 f_{\Xi_0}, \quad f_3 = \mathcal{E}_3\mathcal{E}_0 f_{\Xi_0}, \quad f_4 = \mathcal{E}_1\mathcal{E}_0 f_{\Xi_0} \tag{D12}$$

with the methods described in Refs. 67 and 68 we find the following representation in $\mathbb{C}(4)$ for the generators $\mathcal{E}_\mu$ of $\mathbb{R}_{4,1} \simeq \mathbb{R}_{1,3}$:

$$\mathcal{E}_0 \mapsto \underline{\gamma}_0 = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix} \leftrightarrow \mathcal{E}_i \mapsto \underline{\gamma}_i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}, \tag{D13}$$

where $\mathbf{1}_2$ is the unit 2×2 matrix and σ_i $(i=1,2,3)$ are the standard Pauli matrices. We immediately recognize the $\underline{\gamma}$-matrices in Eq. (D13) as the standard ones appearing, e.g., in Ref. 13.

The matrix representation of $\Psi_{\Xi_0} \in I_{\Xi_0}$ will be denoted by the same letter without the index, i.e., $\Psi_{\Xi_0} \mapsto \mathbf{\Psi} \in \mathbb{C}(4)f$, where

$$f = \tfrac{1}{2}(1 + i\,\underline{\gamma}_1\underline{\gamma}_2) \quad i = \sqrt{-1}. \tag{D14}$$

We have

$$\mathbf{\Psi} = \begin{pmatrix} \psi_1 & 0 & 0 & 0 \\ \psi_2 & 0 & 0 & 0 \\ \psi_3 & 0 & 0 & 0 \\ \psi_4 & 0 & 0 & 0 \end{pmatrix}, \quad \psi_i \in \mathbb{C}. \tag{D15}$$

Equations (D13), (D14), and (D15) are sufficient to prove that there are bijections between the elements of the ideals $\mathbb{R}_{1,3}\frac{1}{2}(1+E_0)$, $\mathbb{R}_{4,1}\frac{1}{2}(1+\mathcal{E}_0)$, and $\mathbb{R}_{4,1}\frac{1}{2}(1+\mathcal{E}_0)\frac{1}{2}(1+\mathbf{i}\mathcal{E}_1\mathcal{E}_2)$.

We can easily find that the following relation exist between $\Psi_{\Xi_0} \in \mathbb{R}_{4,1}f_{\Xi_0}$ and $Z_{\Xi_0} \in \mathbb{R}_{4,1}\frac{1}{2}(1+\mathcal{E}_0)$, $\Xi_0 = (u_0, \Sigma_0)$ being a spinorial frame (see Sec. I)

$$\Psi_{\Xi_0} = Z_{\Xi_0}\tfrac{1}{2}(1+\mathbf{i}\mathcal{E}_1\mathcal{E}_2). \tag{D16}$$

Decomposing Z_{Ξ_0} into even and odd parts relative to the $\mathbf{Z}_2$-graduation of $\mathbb{R}^0_{4,1} \simeq \mathbb{R}_{1,3}$, $Z_{\Xi_0} = Z^0_{\Xi_0} + Z^1_{\Xi_0}$ we obtain $Z^0_{\Xi_0} = Z^1_{\Xi_0}\mathcal{E}_0$ which clearly shows that all information of Z_{Ξ_0} is contained in $Z^0_{\Xi_0}$. Then,

$$\Psi_{\Xi_0} = Z^0_{\Xi_0}\tfrac{1}{2}(1+\mathcal{E}_0)\tfrac{1}{2}(1+\mathbf{i}\mathcal{E}_1\mathcal{E}_2). \tag{D17}$$

Now, if we take into account[150] that $\mathbb{R}^0_{4,1}\frac{1}{2}(1+\mathcal{E}_0)=\mathbb{R}^{00}_{4,1}\frac{1}{2}(1+\mathcal{E}_0)$ where the symbol $\mathbb{R}^{00}_{4,1}$ means $\mathbb{R}^{00}_{4,1}\simeq\mathbb{R}^0_{1,3}\simeq\mathbb{R}_{3,0}$ we see that each $Z_{\Xi_0}\in\mathbb{R}_{4,1}\frac{1}{2}(1+\mathcal{E}_0)$ can be written

$$Z_{\Xi_0}=\psi_{\Xi_0}\tfrac{1}{2}(1+\mathcal{E}_0),\quad \psi_{\Xi_0}\in\mathbb{R}^{00}_{4,1}\simeq\mathbb{R}^0_{1,3}. \tag{D18}$$

Then setting $Z^0_{\Xi_0}=\psi_{\Xi_0}/2$, Eq. (D18) can be written

$$\Psi_{\Xi_0}=\psi_{\Xi_0}\tfrac{1}{2}(1+\mathcal{E}_0)\tfrac{1}{2}(1+\mathbf{i}\mathcal{E}_1\mathcal{E}_2)=Z^0_{\Xi_0}\tfrac{1}{2}(1+\mathbf{i}\mathcal{E}_1\mathcal{E}_2). \tag{D19}$$

The matrix representation of Z_{Ξ_0} and ψ_{Ξ_0} in $\mathbb{C}(4)$ (denoted by the same letter in boldface without index) in the spin basis given by Eq. (D12) are

$$\boldsymbol{\Psi}=\begin{pmatrix}\psi_1 & -\psi_2^* & \psi_3 & \psi_4^*\\ \psi_2 & \psi_1^* & \psi_4 & -\psi_3^*\\ \psi_3 & \psi_4^* & \psi_1 & -\psi_2^*\\ \psi_4 & -\psi_3^* & \psi_2 & \psi_1^*\end{pmatrix},\quad \mathbf{Z}=\begin{pmatrix}\psi_1 & -\psi_2^* & 0 & 0\\ \psi_2 & \psi_1^* & 0 & 0\\ \psi_3 & \psi_4^* & 0 & 0\\ \psi_4 & -\psi_3^* & 0 & 0\end{pmatrix}. \tag{D20}$$

APPENDIX E: WHAT IS A COVARIANT DIRAC SPINOR (CDS)

As we already know $f_{\Xi_0}=\frac{1}{2}(1+\mathcal{E}_0)\frac{1}{2}(1+\mathbf{i}\mathcal{E}_1\mathcal{E}_2)$ [Eq. (D12)] is a primitive idempotent of $\mathbb{R}_{4,1}\simeq\mathbb{C}(4)$. If $u\in\mathrm{Spin}(1,3)\subset\mathrm{Spin}(4,1)$ then all ideals $I_{\Xi_u}=I_{\Xi_0}u^{-1}$ are geometrically equivalent to I_{Ξ_0}. Now, let $\mathbf{s}(\Xi_u)=\{\mathfrak{E}_0,\mathfrak{E}_1,\mathfrak{E}_2,\mathfrak{E}_3\}$ and $\mathbf{s}(\Xi_{u'})=\{\mathfrak{E}'_0,\mathfrak{E}'_1,\mathfrak{E}'_2,\mathfrak{E}'_3\}$ with $\mathbf{s}(\Xi_u)=u^{-1}\mathbf{s}(\Xi_0)u$, $\mathbf{s}(\Xi_{u'})=u'^{-1}\mathbf{s}(\Xi_0)u'$ be two arbitrary basis for $\mathbb{R}^{1,3}\subset\mathbb{R}_{4,1}$. From Eq. (D13) we can write

$$I_{\Xi_u}\ni\Psi_{\Xi_u}=\sum\psi_i f_i,\quad\text{and}\quad I_{\Xi'_u}\ni\Psi_{\Xi_{u'}}=\sum\psi'_i f'_i, \tag{E1}$$

where

$$f_1=f_{\Xi_u},\quad f_2=-\mathfrak{E}_1\mathfrak{E}_3 f_{\Xi_u},\quad f_3=\mathfrak{E}_3\mathfrak{E}_0 f_{\Xi_u},\quad f_4=\mathfrak{E}_1\mathfrak{E}_0 f_{\Xi_u}$$

and

$$f'_1=f_{\Xi_{u'}},\quad f'_2=-\mathfrak{E}'_1\mathfrak{E}'_3 f_{\Xi_{u'}},\quad f'_3=\mathfrak{E}'_3\mathfrak{E}'_0 f_{\Xi_{u'}},\quad f_4=\mathfrak{E}'_1\mathfrak{E}'_0 f_{\Xi_{u'}}.$$

Since $\Psi_{\Xi_{u'}}=\Psi_{\Xi_u}(u'^{-1}u)^{-1}$, we get

$$\Psi_{\Xi_{u'}}=\sum_i\psi_i(u'^{-1}u)^{-1}f'_i=\sum_{i,k}S_{ik}[(u^{-1}u')]\psi_i f_k=\sum_k\psi'_k f_k.$$

Then

$$\psi'_k=\sum_i S_{ik}(u^{-1}u')\psi_i, \tag{E2}$$

where $S_{ik}(u^{-1}u')$ are the matrix components of the representation in $\mathbb{C}(4)$ of $(u^{-1}u')\in\mathrm{Spin}^e_{1,3}$. As proved in Refs. 67 and 68 the matrices $S(u)$ correspond to the representation $D^{(1/2,0)}\oplus D^{(0,1/2)}$ of $\mathrm{SL}(2,\mathbb{C})\simeq\mathrm{Spin}^e_{1,3}$.

We remark that all the elements of the set $\{I_{\Xi_u}\}$ of the ideals geometrically equivalent to I_{Ξ_0} under the action of $u \in \mathrm{Spin}^e_{1,3} \subset \mathrm{Spin}^e_{4,1}$ have the same image $I = \mathbb{C}(4)f$ where f is given by Eq. (D11), i.e.,

$$f = \frac{1}{2}(1+\underline{\gamma}_0)(1+i\underline{\gamma}_1\underline{\gamma}_2), \quad i=\sqrt{-1}, \tag{E3}$$

where $\underline{\gamma}_\mu$, $\mu = 0,1,2,3$ are the Dirac matrices given by Eq. (D14).

Then, if

$$\begin{gathered} \gamma: \mathbb{R}_{4,1} \to \mathbb{C}(4) \equiv \mathrm{End}(\mathbb{C}(4)f), \\ x \mapsto \gamma(x): \mathbb{C}(4)f \to \mathbb{C}(4)f \end{gathered} \tag{E4}$$

it follows that

$$\gamma(\mathfrak{E}_\mu) = \gamma(\mathfrak{E}'_\mu), \quad \gamma(f_\mu) = \gamma(f'_\mu) \tag{E5}$$

for all $\{\mathfrak{E}_\mu\}$, $\{\mathfrak{E}'_\mu\}$ such that $\mathfrak{E}'_\mu = (u'^{-1}u)\mathfrak{E}_\mu(u'^{-1}u)^{-1}$. Observe that *all information* concerning the geometrical images of the spinorial frames Ξ_u, $\Xi_{u'}, \ldots$, under $\mathbf{s}$ disappear in the matrix representation of the ideals I_{Ξ_u}, $I_{\Xi_{u'}}, \ldots$, in $\mathbb{C}(4)$ since all these ideals are mapped in the same ideal $I = \mathbb{C}(4)f$.

With the above remark and taking into account the definition of algebraic spinors given in Sec. II C and Eq. (E2) we are lead to the following.

Definition 62: A covariant Dirac spinor (CDS) for $\mathbb{R}^{1,3}$ *is an equivalence class of pairs* $(\Xi^m_u, \boldsymbol{\Psi})$, *where* Ξ^m_u *is a matrix spinorial frame associated to the spinorial frame* Ξ_u *through the* $S(u^{-1}) \in D^{(1/2,0)} \oplus D^{(0,1/2)}$ *representation of* $\mathrm{Spin}^e_{1,3}$, $u \in \mathrm{Spin}^e_{1,3}$. *We say that* $\Psi, \Psi' \in \mathbb{C}(4)f$ *are equivalent and write*

$$(\Xi^m_u, \Psi) \sim (\Xi^m_{u'}, \Psi') \tag{E6}$$

if and only if

$$\Psi' = S(u'^{-1}u)\Psi, \quad u\mathbf{s}\Xi_u u^{-1} = u'\mathbf{s}(\Xi_{u'})u'^{-1}. \tag{E7}$$

Remark 63: The definition of CDS just given agrees with that given in Ref. 40 except for the irrelevant fact that there, as well as in the majority of Physics textbook's, authors use as the space of representatives of a CDS a complex four-dimensional space $\mathbb{C}^4$ *instead of* $I = \mathbb{C}(4)f$.

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