Electronic Journal of Differential Equations, Vol. 2003(2003), No. 109, pp. 1–25.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

EXISTENCE AND REGULARITY OF SOLUTIONS OF A PHASE FIELD MODEL FOR SOLIDIFICATION WITH CONVECTION OF PURE MATERIALS IN TWO DIMENSIONS

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Abstract. We study the existence and regularity of weak solutions of a phase field type model for pure material solidification in presence of natural convection. We assume that the non-stationary solidification process occurs in a two dimensional bounded domain. The governing equations of the model are the phase field equation coupled with a nonlinear heat equation and a modified Navier-Stokes equation. These equations include buoyancy forces modelled by Boussinesq approximation and a Carman-Koseny term to model the flow in mushy regions. Since these modified Navier-Stokes equations only hold in the non-solid regions, which are not known a priori, we have a free boundary-value problem.

1. Introduction

One of the first papers to consider phase field models applied to change of phases was written by Fix [17]. This article fostered many other studies in this subject; see for instance the sequence of papers [8], [5], [4], [7]. Caginalp and others took over the task of understanding the phase field approach, both in its mathematical aspects and in its relationship to the classical approach of using sharp interfaces to separate the phases, which gives rise to what is known by Stefan type problems. We remark that, for the derivation of the kinetic equation for the phase field, Caginalp and others used the free energy functional as a basis of the argument (see also Hoffman and Jiang [18]). An alternative derivation, suggested by Peronse and Fife [25, 26], uses an entropy functional which gives a kinetic equation for the phase field ensuring monotonic increase of the entropy in time. Peronse and Fife exhibit a specific choice of entropy density which essentially recovers the phase field model employed by Caginalp [8] by linearizing the heat flux. Thus, phase field models have a sound physical basis and provide simple and elegant descriptions of phase transition processes. Moreover, it is more versatile than the enthalpy method because it allows effects such as supercooling to be included. An important

2000 *Mathematics Subject Classification.* 76E06, 80A22, 82B26, 76D05.
Key words and phrases. Phase-field, phase transition, solidification, convection, Navier-Stokes equations.

Submitted September 14, 2001. Published November 3, 2003.
J. L. B. was partially supported by grant 300513/87-9 from CNPq, Brazil.

example of the utility of the phase field approach is its use for the numerical study of dendritic growth (see Caginalp [5] and Kobayshi [19]).

One point of importance is that many papers interested in the mathematical analysis of these models, whatever the approach used to model phase change, have neglected the possibility of flow occurring in non solidified portions of the material. In many practical situations, however, this assumption is not satisfactory because the existence of such motions may affect in important ways the outcome of the process of phase change. In fact, melt convection adds new length and time scales to the problem and results in morphologies that are potentially much different from those generated by purely diffusive heat and solute transport. Moreover, not only does convection influence the solidification pattern, but the evolving microstructure can also trigger unexpected and complicated phenomena. For instance, Coriell *et all* [11] and Davis [12] studied in detail the coupled convective and morphological instabilities at a growth front. This suggests that models that do not consider melt convection may have some limitations.

One must realize, however, that the inclusion of this possibility brings another very difficult aspect to an already difficult problem. For this, it is enough to observe that such a flow must occur only in an a priori unknown non-solid region, and thus one may be left with a rather difficult free boundary-value problem.

In recent years, some authors have considered convective effects; for instance: Cannon *et al* [9, 10], DiBenedetto and Friedman [15], DiBenedetto and O'Leary[14] and O'Leary [20], who addressed such questions by using weak formulations of the Stefan type approach. Blanc *et al.* [3], Pericleouns *et al.* [24] and Voller *et al.* [30, 31] considered convective effects in phase change problems by using the enthalpy approach to describe change of phases, together with modified Navier-Stokes equations to model the flow. In these works, the phases may be distinguished by the values of a variable corresponding to the solid fraction that is associated to the enthalpy; this same variable is used in a term that is added to the Navier-Stokes equations to cope with the influence of the mushy zones in the flow. Particular expressions for this term may be obtained by modelling such mushy zones as porous media. Beckermann *et al.* [2] and Diepers *et al.* [16] used the phase field methodology to obtain models including phase change and melt convection. By using numerical simulations they studied the influence of the convection in the melt on phenomena like dendritic growth and coarsening. An interesting discussion of the application of diffusive-interface methods (phase field being one of them) to fluid mechanics can be found in Anderson and McFadden [1].

In this paper we are interested in the mathematical analysis of a model problem having some of the main aspects that a reasonable model for a solidification process with convection should have. We will consider a rather simple situation of this sort in the hope to obtain a better understanding of the mathematical difficulties brought by the coupling of terms describing phase change and the terms describing convection. We also restrict the subject to the analysis of solidification of pure materials (the corresponding mathematical analysis for alloys will be considered elsewhere.)

As in [2] and [16], we employ a phase field methodology to model phase change; we also assume the solid phase to be rigid and stationary. However, differently from their models, convective effects will be included by using the ideas suggested by Blanc *et al* [3] and Voller *et al* [30]. Since the indicator of phase in these last papers is the solid fraction, we relate the two approaches by postulating a functional relationship between the solid fraction and the phase field. The governing equations of the model are the following: the phase field equation is as in Hoffman and Jiang [18]; it is coupled with equations for the temperature and velocity that are based on usual conservation principles. These last equations become respectively a nonlinear heat equation and modified a Navier-Stokes equations which include buoyancy forces modelled by Boussinesq approximation and a Carman-Koseny type term to model the flow in mushy regions. Since these modified Navier-Stokes equations only hold in a priori unknown non-solid regions, we actually have a free boundary value problem.

We remark that the phase field model that we consider here is rather simple and does not take care of several important transition events such as nucleation or spinodal decomposition. However, more complete and complex phase field models for phase change could be similarly considered. Our choice in this paper was just guided by mathematical simplicity. We do not present a detailed derivation of the model equations directly from physical background because this would be basically a repetition of the arguments of the references (they use just the usual balance of internal energy and linear momentum arguments arguments suitably adapted to the situation) and they would be lengthy in an already large paper. We give the idea of such adaptations for the balance of internal energy in Section 2; for the balance of linear momentum the argument is exactly as in Voller *et al* [31] (see also [30].) The details of the model problem can be found in Section 2, equations (2.1); the corresponding weak formulation can be found in Definition 3.5.

Our objective is to present a result on the existence and regularity of solutions of these model equations corresponding to a nonstationary phase change process in a bounded domain, which for technical reasons in this paper is assumed to be two dimensional.

Existence will be obtained by using a regularization technique similar to the one used by Blanc *et al* in [3]: an auxiliary positive parameter will be introduced in the equations in such way that the original free boundary value problem will be transformed in a more standard (penalized) one. We say that this is the regularized problem. By solving this, one hopes to recover the solution of the original problem as the parameter approaches zero. To accomplish such program, we will solve the regularized problem by using Leray-Schauder degree theory (see Section 8.3, p. 56 in Deimling [13]); and use results for certain modified Navier-Stokes equations presented in Vaz [29]. Then, by taking a sequence of values of the parameter approaching zero, we will correspondingly have a sequence of approximate solutions. By obtaining suitable uniform estimates for this sequence, we will then be able to take the limit along a subsequence and, by compactness arguments, to show that we have in fact a solution of the original problem. The stated regularity of this solution will be obtained by applying the L_p-theory of the parabolic linear equations together with bootstrapping arguments.

We should stress that the ideas presented in this mathematical analysis, in particular the penalization used for obtaining the approximate solutions, suggest a convenient discretization scheme for numerical simulations of phase change problems with melt convection. Such scheme would be similar to, but different from, the ones in [2] and [16]. Moreover, such methods would not rely on specifying a variable viscosity across the diffusive interface regions that tend to a large value in the rigid solid, as several other methods propose. This would be realistic only for certain classes of materials, and certainly difficult to specify for rigid solids. We also remark that the mathematical analysis corresponding to the models presented in [2] and [16] are presently under investigation.

This paper is organized as follows. In Section 2, we describe the mathematical model and its variables. In Section 3, we fix the notation and describe the the basic functional spaces to be used; we recall certain results and present auxiliary problems; we also state assumptions holding throughout the paper and define the concept of generalized solution. In Section 4, we consider the question of existence, uniqueness and regularity of solutions of the regularized problem. Section 5 is dedicated to the proof of the existence of a solution of the original free boundary value problem.

Finally, as it is usual in papers of this sort, C will denote a generic constant depending only on a priori known quantities.

2. Model Equations

The model problem presented here has aspects of the models studied in the works of Blanc [3], Caginalp [8] and Voller *et al* [30, 31]. As we said in the Introduction, the phase of the material will be described by using the phase field methodology, which in its simplest approach assumes that there is a scalar field $\varphi(x,t)$, the phase field, depending on the spatial variable x and time t and real values $\varphi_s < \varphi_\ell$ such that if $\varphi(x,t) \le \varphi_s$ then the material at point x at time t is in solid state; if $\varphi_\ell \le \varphi(x,t)$ then the material at point x at time t is in liquid state; if $\varphi_s < \varphi(x,t) < \varphi_\ell$ then, at time t the point x is in the mushy region. We follow Caginalp [8] and Hoffman and Jiang[18] and take the phase field equation as

$$\frac{\partial \varphi}{\partial t} - \alpha \Delta \varphi = a\varphi + b\varphi^2 - \varphi^3 + \theta,$$

where θ is the temperature; α is a (small) fixed positive constant, and a and b are known functions which regularity will be described later on.

We observe that the function $g(s) = as + bs^2 - s^3$ used at the right hand side of the above equation is the classical possibility coming from the classical double-well potential (see Hoffman and Jiang [18]). Other possibilities for the double-well potential can be found for instance in Caginalp [8] and Penrose [26].

To obtain a equation for the temperature, we observe that when there is phase change, the thermal energy has the following expression:

$$e = \theta + \frac{\ell}{2}(1 - f_s),$$

where θ and $\ell/2$ represent respectively the sensible heat (for simplicity of notation, we took the specific heat coefficient to be one) and latent heat. f_s is the solid fraction ($1 - f_s$ is the non-solid fraction), which for simplicity we assume to be a known function only of the phase field (obviously dependent on the material being considered.)

Then, the energy balance in pure material solidification process may be written (see Vaz [29]) as follows:

$$\frac{\partial \theta}{\partial t} - \Delta \theta + v.\nabla \theta = \frac{\ell}{2} \frac{\partial f_s}{\partial \varphi}(\varphi) \frac{\partial \varphi}{\partial t}$$

where v represent the velocity of the material.

We will assume that only non solid portions of the material can move, and this is done as an incompressible flow. Consequently, in non-solid regions Navier- Stokes type equations are required. According to Voller *et al* [30] and Blanc *et al* [3] these equations can be taken as

$$\frac{\partial v}{\partial t} - \nu \Delta v + (v.\nabla)v + \nabla p = G(f_s, v) + F(\theta)$$
$$\operatorname{div} v = 0$$

where v is velocity, p is pressure, ν is viscosity and $G(f_s, v)$ and $F(\theta)$ are source terms which will be defined below.

Assuming the Boussinesq treatment to be valid, natural convection effects can be accounted for by defining the buoyancy source term to be

$$F(\theta) = C\rho\mathbf{g}(\theta - \theta_r)$$

where ρ is the mean value of the density, $\mathbf{g}$ is the gravity, C is a constant and θ_r is a reference temperature. In order to simplify the calculations let us consider $F(\theta) = \vec{\sigma}\theta$.

The source term $G(f_s, v)$ is used to modify the Navier-Stokes equations in the mushy regions, and according to [30, 31], can be taken of form $G(f_s, v) = -k(f_s)v$. Usually the function $k(f_s)$ is taken as the Carman-Koseny expression (see again [30, 31]), which is

$$k(f_s) = \frac{f_s^2}{(1 - f_s)^3}.$$

As in Blanc *et al* [3], we will consider a more general situation including the previous one. We will assume that assuming that k is a nonnegative function in $C^0(-\infty, 1)$, $k = 0$ in $\mathbb{R}^-$ and $\lim_{\mathrm{y}\to 1} k(\mathrm{y}) = +\infty$, and in this case, we will refer to G as the Carman-Kosen type term.

To complete the description of the model problem, we must define the regions where the above equations are valid. By using the solid fraction, the following subsets of Q, denoted by Q_l, Q_m and Q_s and corresponding respectively to the liquid, mushy and solid regions, are defined as:

$$Q_l = \{(x,t) \in Q : f_s(\varphi(x,t)) = 0\}$$
$$Q_s = \{(x,t) \in Q : f_s(\varphi(x,t)) = 1\}$$
$$Q_m = \{(x,t) \in Q : 0 < f_s(\varphi(x,t)) < 1\}$$

In the following, $Q_{ml} = Q \setminus \bar{Q}_s$ will denote the non-solid part of Q. Moreover, for each time $t \in [0,T]$, we define $\Omega_s(t) = \{x \in \Omega : f_s(\varphi(x,t)) = 1\}$, $\Omega_{ml}(t) = \Omega \setminus \bar{\Omega}_s(t)$ and $S_{ml} = \{(x,t) \in \bar{Q} : x \in \partial\Omega_{ml}(t)\}$.

We must emphasize that this model is the free boundary problem since that Q_l, Q_m and Q_s are a priori unknown. Now, we can now summarize the formulation of the problem to be analyzed as:

$$\begin{gathered}
\frac{\partial \varphi}{\partial t} - \alpha\Delta\varphi = a\varphi + b\varphi^2 - \varphi^3 + \theta \quad \text{in } Q, \\
\frac{\partial \theta}{\partial t} - \Delta\theta + v.\nabla\theta = \frac{\ell}{2}\frac{\partial f_s}{\partial \varphi}(\varphi)\frac{\partial \varphi}{\partial t} \quad \text{in } Q, \\
\frac{\partial v}{\partial t} - \nu\Delta v + (v.\nabla)v + \nabla p + k(f_s(\varphi))v = \vec{\sigma}\theta \quad \text{in } Q_{ml}, \\
\operatorname{div} v = 0 \quad \text{in } Q_{ml}, \\
v = 0 \quad \text{in } \mathring{Q}_s,
\end{gathered} \tag{2.1}$$

subject to the boundary conditions

$$\begin{gathered}
\frac{\partial \varphi}{\partial n} = 0 \quad \text{on } S, \\
\theta = 0 \quad \text{on } S, \\
v = 0 \quad \text{on } S_{ml}.
\end{gathered} \tag{2.2}$$

and to the initial conditions

$$\begin{gathered}
\varphi(x,0) = \varphi_0(x) \quad \text{in } \Omega, \\
\theta(x,0) = \theta_0(x) \quad \text{in } \Omega, \\
v(x,0) = v_0(x) \quad \text{in } \Omega_{ml}(0),
\end{gathered} \tag{2.3}$$

where φ_0, θ_0 and v_0 are suitably given functions such that for compatibility v_0 is identically zero outside $\Omega_{ml}(0)$.

3. Preliminaries and Main Result

3.1. **Notation, functional spaces and auxiliary results.** Let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain with a sufficiently smooth boundary $\partial\Omega$ and $\mathrm{Q} = \Omega \times [0,T]$ the space-time cylinder with lateral surface $S = \partial\Omega \times [0,T]$. For $t \in [0,T]$, we denote $Q_t = \Omega \times [0,t]$.

We denote by $W_q^p(\Omega)$ the usual Sobolev space and $W_q^{2,1}(Q)$ the Banach space consisting of functions $u(x,t)$ in $L^q(Q)$ whose generalized derivatives $D_x u$, $D_x^2 u$, u_t are L^q-integrable ($q \geq 1$). The norm in $W_q^{2,1}(Q)$ is defined by

$$\|u\|_{q,Q}^{(2)} = \|u\|_{q,Q} + \|D_x u\|_{q,Q} + \left\|D_x^2 u\right\|_{q,Q} + \|u_t\|_{q,Q} \tag{3.1}$$

where D_x^s denotes any partial derivatives with respect to variables $x_1, x_2, ..., x_n$ of order s=1,2 and $\|\cdot\|_q$ the usual norm in the space $L^q(Q)$.

Moreover, $W_2^{1,0}(Q)$ is a Hilbert space for the scalar product

$$(u,v)_{W_2^{1,0}(Q)} = \int_Q uv + \nabla u.\nabla v \, dx\, dt$$

and $\overset{0}{W}{}_2^{1,1}(Q)$ is a Hilbert space for the scalar product

$$(u,v)_{W_2^{1,1}(Q)} = \int_Q uv + \nabla u.\nabla v + u_t v_t \, dx\, dt$$

whose functions vanish on S in the sense of traces.

We also denote by $V_2(Q)$ the Banach space consisting of function $u(x,t)$ in $W_2^{1,0}(Q)$ having the following finite norm

$$|u|_{V_2(Q)} = \operatorname*{ess\,sup}_{0\le t\le T} \|u(x,t)\|_{2,\Omega} + \|\nabla u(x,t)\|_{2,Q}\,. \tag{3.2}$$

Let $\overset{0}{V}_2(Q)$ be the Banach space consisting of those elements of $V_2(Q)$ that vanish on S in the sense of traces.

We now define spaces consisting of functions that are continuous in the sense of hölder. We say that a function $u(x,t)$ defined in $\overline{Q}$ is hölder continuous in x and t, respectively with exponents α and $\beta \in (0,1)$, if following quantities, called hölder constants, are finite:

$$\langle u\rangle_x^{(\alpha)} = \sup_{(x_1,t),(x_2,t)\,\in\overline{Q},\; x_1\ne x_2} \frac{|u(x_1,t)-u(x_2,t)|}{|x_1-x_2|^\alpha}$$

$$\langle u\rangle_t^{(\beta)} = \sup_{(x,t_1),(x,t_2)\,\in\overline{Q},\; t_1\ne t_2} \frac{|u(x,t_1)-u(x,t_2)|}{|t_1-t_2|^\beta}$$

Then, we define the hölder space $H^{\tau,\tau/2}(Q)$, with $0\le\tau<1$, (see Ladyzenskaja *et al* [21]), as the Banach space of functions u(x,t) that are continuous in $\overline{Q}$, having finite norm given by:

$$|u|_Q^{(\tau)} = \max_{\overline{Q}}|u| + \langle D_x u\rangle_x^{(\tau)} + \langle u\rangle_t^{(\tau/2)}. \tag{3.3}$$

For the functional spaces associated to the velocity field, we denote $\mathcal{D} = \{u \in \mathcal{C}^\infty(\Omega)^2 : \operatorname{supp} u \subset \Omega\}$ and $\mathcal{V} = \{u \in \mathcal{D} : \operatorname{div} u = 0\}$. The closure of $\mathcal{V}$ in $L^2(\Omega)^2$ is denoted by H and the closure of $\mathcal{V}$ in $\overset{0}{W}{}_2^1(\Omega)^2$ is denoted by V. These functional spaces appear in the mathematical theory of the Navier-Stokes equations; their properties can be found for instance in Temam [28].

The following two lemmas are particular case of Lemma 3.3 in Ladyzenskaja *et al* ([21]). They are stated here for ease of reference. The first lemma is immediate consequence of Lemma 3.3 in [21, p. 80], by taking there $l=1$, $n=2$ and $r=s=0$.

Lemma 3.1. *Let Ω and Q as in the beginning of this section. Then for any function $u\in W_q^{2,1}(Q)$ we also have $u\in L^p(Q)$, and it is valid the following inequality*

$$\|u\|_{p,Q} \le C\,\|u\|_{q,Q}^{(2)}\,, \tag{3.4}$$

provided that

$$p = \begin{cases} \infty & \text{if } \frac1q-\frac12<0\\ p\ge 1 & \text{if } \frac1q-\frac12=0\\ \left(\frac1q-\frac12\right)^{-1} & \text{if } \frac1q-\frac12>0\,. \end{cases}$$

The constant $C>0$ depends only on T, Ω, p and q.

The second lemma is immediately obtained from Lemma 3.3 in [21, p. 80], by taking there $l=1$, $n=2$, $r=s=0$ and $q=3$.

Lemma 3.2. *Let Ω and Q be as in the beginning of this section. Then for any function $u\in W_3^{2,1}(Q)$ we also have $u\in H^{2/3,1/3}(Q)$ satisfying the estimate*

$$|u|_Q^{(2/3)} \le C\,\|u\|_{3,Q}^{(2)}\;. \tag{3.5}$$

The constant $C>0$ depends only on T and Ω.

In the following we will consider two auxiliary problems, respectively related to the phase field and the velocity equations.

The first problem is

$$\begin{aligned} \frac{\partial \varphi}{\partial t} - \alpha \Delta \varphi = a\varphi + b\varphi^2 - \varphi^3 + g(x,t) \quad \text{in } Q, \\ \frac{\partial \varphi}{\partial \eta} = 0 \quad \text{on } S, \\ \varphi(x,0) = \varphi_0(x) \quad \text{in } \Omega. \end{aligned} \tag{3.6}$$

where α in a positive constant. This problem was treated by Hoffman and Jaing [18] when the initial date satisfies $\varphi_0 \in W^2_\infty(\Omega)$. Since we will need an existence result for $\varphi_0 \in W_q^{2-2/q}(\Omega) \cap W_2^{3/2-\delta}(\Omega)$, with $\delta \in (0,1)$, we restate the result of [18]. We remark that exactly the same proof presented in [18] holds in this situation (see also Vaz [29] for details, where some other specific results concerning (3.6) are proved.)

Proposition 3.3. *Let Ω and Q be as in the beginning of this section. Assume that $a(x,t)$ and $b(x,t)$ in $L^\infty(Q)$, $g \in L^q(Q)$, $\varphi_0 \in W_q^{2-2/q}(\Omega) \cap W_2^{3/2-\delta}(\Omega)$, where $q \geq 2$, $\delta \in (0,1)$ and $\frac{\partial \varphi_0}{\partial \eta} = 0$ in $\partial\Omega$. Then there exists an unique solution $\varphi \in W_q^{2,1}(Q)$ of problem (3.6), which satisfies the estimate*

$$\|\varphi\|^{(2)}_{q,Q} \leq C\left(\|\varphi_0\|_{W_q^{2-2/q}(\Omega)} + \|g\|_{q,Q}\right), \tag{3.7}$$

where C depends only on T, α, Ω, $\|a(x,t)\|_{\infty,Q}$ and on $\|b(x,t)\|_{\infty,Q}$.

The second auxiliary problem is

$$\begin{aligned} \frac{\partial v}{\partial t} - \nu\Delta v + (v.\nabla)v + \nabla p + k(x,t)v = f(x,t) \quad \text{in } Q, \\ \operatorname{div} v = 0 \quad \text{in } Q, \\ v = 0 \quad \text{on } S, \\ v(x,0) = v_0(x) \quad \text{in } \Omega. \end{aligned} \tag{3.8}$$

Proposition 3.4. *Let Ω and Q be as in the beginning of this section. Assume that $k(x,t) \in C^0(Q)$, $k(x,t) \geq 0$, $f(x,t) \in L^2(Q)^2$ and $v_0(x) \in H$. Then there exists an unique solution $v(x,t) \in L^2(0,T;V) \cap L^\infty(0,T;H)$ of problem (3.8) which satisfies the estimate*

$$\|v\|_{L^\infty(0,T,H)} + \|v\|_{L^2(0,T,V)} \leq C\big(\|v_0\|_H + \|f\|_{2,Q}\big) \tag{3.9}$$

Moreover, by interpolation results $v \in L^4(Q)^2$ and

$$\|v\|_{4,Q} \leq C\big(\|v_0\|_H + \|f\|_{2,Q}\big), \tag{3.10}$$

where C depends only on T and on Ω.

The proof of Proposition 3.4 is done by using the same arguments used in the classical theory of weak solutions of the Navier-Stokes equations. As in this classical situation, the fact that the domain is two dimensional is important to obtain uniqueness of solutions (see Temam [28], p.282, for instance.)

3.2. **Technical Hypotheses and Generalized Solution.** For the rest of this article we will be using the following technical hypotheses:

(H1) $\Omega \subset \mathbb{R}^2$ is an open and bounded domain with sufficiently smooth boundary $\partial\Omega$; T is a finite positive number; $Q = \Omega \times (0, T)$.

(H2) $a(x,t), b(x,t)$ are given functions in $L^\infty(Q)$; $f_s \in C_b^{1,1}(\mathbb{R})$, $0 \le f_s(z) \le 1$ for all $z \in \mathbb{R}$; $k(y) \in C^0(-\infty, 1)$, $k(0) = 0$, $k(y) = 0$ in $\mathbb{R}^-$, $k(y)$ is nonnegative and $\lim_{y\to 1} k(y) = +\infty$.

(H3) $v_0 \in$ H; $\theta_0 \in W_2^1(\Omega)$, $\theta_0 = 0$ on $\partial\Omega$; $\varphi_0 \in W_3^{4/3}(\Omega) \cap W_2^{3/2+\delta}(\Omega)$, for some $\delta \in (0,1)$, $\frac{\partial\varphi_0}{\partial\eta} = 0$ on $\partial\Omega$.

Now we explain in what sense we will understand a solution of (2.1), (2.2), (2.3).

Definition 3.5. By a **generalized solution** of the problem (2.1), (2.2), (2.3), we mean a triple of functions (φ, θ, v) such that $\varphi \in V_2(Q)$, $\theta \in \overset{0}{V}_2(Q)$ and $v \in L^2(0,T;V) \cap L^\infty(0,T;H)$. Moreover, being

$$\begin{gathered}
Q_s = \{(x,t) \in Q : f_s(\varphi(x,t)) = 1\}, \\
\Omega_s(t) = \{x \in \Omega : f_s(\varphi(x,t)) = 1\}, \\
Q_{ml} = Q \setminus \bar{Q}_s, \\
\Omega_{ml}(0) = \Omega \setminus \bar{\Omega}_s(0)
\end{gathered}$$

we have $v = 0$ a.e in $\overset{\circ}{Q}_s$, and φ, θ and v satisfy the integral relations

$$\begin{aligned}
&-\int_Q \varphi\, \beta_t \, dx\, dt + \alpha \int_Q \nabla\varphi\nabla\beta \, dx\, dt \\
&= \int_Q \left(a + b\varphi - \varphi^2\right) \varphi\, \beta dx\, dt + \int_Q \theta\beta \, dx\, dt + \int_\Omega \varphi_0(x)\beta(x,0)dx \,,
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
&-\int_Q \theta\, \xi_t dx\, dt + \int_Q \nabla\theta\nabla\xi dx\, dt + \int_Q v.\nabla\theta\, \xi \, dx\, dt \\
&= \int_Q \frac{\ell}{2} \frac{\partial f_s}{\partial\varphi}(\varphi)\, \varphi_t\, \xi \, dx\, dt + \int_\Omega \theta_0(x)\xi(x,0)dx \,,
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
&-\int_{Q_{ml}} v\, \phi_t \, dx\, dt + \nu \int_{Q_{ml}} \nabla v\nabla\phi \, dx\, dt \\
&+ \int_{Q_{ml}} (v.\nabla)v\, \phi \, dx\, dt + \int_{Q_{ml}} k(f_s(\varphi))v\, \phi \, dx\, dt \\
&= \int_{Q_{ml}} \vec{\sigma}\theta\phi \, dx\, dt + \int_{\Omega_{ml}(0)} v_0(x)\phi(x,0)dx \,,
\end{aligned} \tag{3.13}$$

for all β in $W_2^{1,1}(Q)$ such that $\beta(x,T) = 0$; for all ξ in $\overset{0}{W}{}_2^{1,1}(Q)$ such that $\xi(x,T) = 0$, and for all $\phi \in C([0,T]; W_2^1(\Omega_{ml}(t)))$ such that $\phi(.,T) = 0$, $\operatorname{div}\phi(.,t) = 0$ for all $t \in [0,T]$ and $\operatorname{supp}\phi(x,t) \subset Q_{ml} \cup \Omega_{ml}(0)$.

Note that due to our technical hypotheses and choice of functional spaces, all of the integrals in Definition 3.5 are well defined.

3.3. **Existence of Generalized Solutions.** The purpose of this paper is to prove the following result

Theorem 3.6. *Under the hypotheses (H1), (H2), (H3), there is a generalized solution of the problem (2.1), (2.2), (2.3) in the sense of the Definition 3.5. Moreover, when* $\varphi_0 \in W_q^{2-2/q}(\Omega) \cap W_2^{3/2+\delta}(\Omega)$ *for some* $\delta \in (0,1)$ *and* $q \geq 3$, *and* $\theta_0 \in W_p^{2-2/p}(\Omega)$ *with* $3 \leq p < 4$, *then such solution satisfies* $\varphi \in W_q^{2,1}(Q) \cap L^\infty(Q)$, $\theta \in W_p^{2,1}(Q) \cap L^\infty(Q)$, $v \in L^2(0,T;V) \cap L^\infty(0,T;H)$.

The proof of the previous result is long and will be done in the following sections. Here we want just to sketch it: existence of a solution of problem (2.1), (2.2), (2.3) will proved by using a regularization technique already used by Blanc *et al* in [3]. The purpose this regularization is to deal with the Navier-Stokes equations in whole domain instead of unknown regions. Thus, the problem will be adequately regularized with the help of a positive parameter, and the existence of solutions for this regularized problem will obtained by using the Leray-Schauder degree theory (see Theorem 4.1). Then, as this parameter approaches zero, a sequence of regularized solutions is obtained. With the help of suitable estimates and compactness arguments, a limit of a subsequence is then proved to exist and to be a solution of problem (2.1), (2.2), (2.3).

We also remark that the phase field equation admits classic solution when φ_0 is sufficiently smooth. In fact, its right hand side term satisfies $a\varphi + b\varphi^2 - \varphi^3 + \theta \in L^\infty(Q)$, and, in particular when $\varphi_0 \in W_q^{2-2/q}(\Omega) \cap W_2^{3/2+\delta}(\Omega)$, with $q \geq 2$, we obtain a strong solution with the equation satisfied in the a.e-sense. The boundary and initial conditions are also satisfied in the pontual sense because $\varphi \in C^1(Q)$. When $\theta_0 \in W_p^{2-2/p}(\Omega)$, with $3 \leq p < 4$, the same sort of arguments applies and the solution is strong with $\theta \in C^0(Q)$; the temperature equation and the boundary and initial conditions are valid in point wise sense. Unfortunately, we are not able to improve the regularity of the corresponding solution even if the initial velocity is very regular. Thus, we only generalized solutions are obtained for the velocity equation.

4. Regularized Problem

In this section we regularize problem (2.1), (2.2), (2.3) by changing the term $k(f_s(\varphi))v$ in the velocity equation. We will obtain a result of the existence, uniqueness and regularity for this associated regularized problem.

Theorem 4.1. *Fix* $\varepsilon \in (0,1]$. *Under the hypotheses* (H_1), (H_2), (H_3), *there exists an unique solution* $(\varphi_\varepsilon, v_\varepsilon, \theta_\varepsilon) \in W_3^{2,1}(Q) \times (L^2(0,T;V) \cap L^\infty(0,T;H)) \times W_2^{2,1}(Q) \subset L^6(Q) \times L^2(0,T;H) \times L^3(Q)$ *of the following problem:*

$$\begin{gathered}
\frac{\partial \varphi_\varepsilon}{\partial t} - \alpha \Delta \varphi_\varepsilon = a\varphi_\varepsilon + b\varphi_\varepsilon^2 - \varphi_\varepsilon^3 + \theta_\varepsilon, \\
\frac{\partial v_\varepsilon}{\partial t} - \nu \Delta v_\varepsilon + (v_\varepsilon.\nabla) v_\varepsilon + \nabla p_\varepsilon + k(f_s(\varphi_\varepsilon) - \varepsilon) v_\varepsilon = \vec{\sigma}\theta_\varepsilon, \\
\operatorname{div} v_\varepsilon = 0, \\
\frac{\partial \theta_\varepsilon}{\partial t} - \Delta \theta_\varepsilon + v_\varepsilon.\nabla \theta_\varepsilon = \frac{\ell}{2} \frac{\partial f_s}{\partial \varphi}(\varphi_\varepsilon) \frac{\partial \varphi_\varepsilon}{\partial t}
\end{gathered} \tag{4.1}$$

in Q;

$$\frac{\partial \varphi_\varepsilon}{\partial n} = 0, \quad \theta_\varepsilon = 0, \quad v_\varepsilon = 0 \tag{4.2}$$

on S; and

$$\varphi_\varepsilon(x,0)=\varphi_0(x),\quad \theta_\varepsilon(x,0)=\theta_0(x),\quad v_\varepsilon(x,0)=v_0(x) \tag{4.3}$$

in Ω. Moreover, as ε varies in $[0,1]$, such solutions $(\varphi_\varepsilon, v_\varepsilon, \theta_\varepsilon)$ are uniformly bounded with respect to ε in $W_3^{2,1}(Q)\times(L^2(0,T;V)\cap L^\infty(0,T;H))\times W_2^{2,1}(Q)$.

This theorem will be proven the end of this section, after some preparation and auxiliary lemmas. The solvability of problem (4.1), (4.2), (4.3) will be proved by applying the Leray-Schauder degree theory (see Deimling [13]) as in Morosanu and Motreanu [23]. For this, we will reformulate the problem as $T(1,\varphi,v,\theta)=(\varphi,v,\theta)$, where $T(\lambda,\cdot)$ is a compact homotopy depending on a parameter $\lambda\in[0,1]$ to be described shortly.

Basic tools in our argument are L_p-theory of parabolic equations and Theorems 3.3 and 3.4 in Section 3. Moreover, we emphasize that the regularity of solution of Navier-Stokes and phase field equations plays an essential role in this proof. Such connection is strictly related with a selection of the order of the equations in quasilinear problem, mainly in deriving *a priori estimates* for possible solutions. Moreover, since that the phase field has smooth solution (classical solution), the regularity of Navier-Stokes equations becomes very important but this regularity is governed by the additional Carman-Koseny type term $k(f_s(\varphi))v$ that one not permits one to obtain uniform estimate in some different as $L^2(0,T;\mathrm{V})\cap L^\infty(0,T;H)$.

For simplicity of notation, we omit the subscript ε in the rest of this section.

Definition 4.2. Define the homotopy $T:[0,1]\times L^6(Q)\times L^2(0,T;H)\times L^3(Q)\to L^6(Q)\times L^2(0,T;H)\times L^3(Q)$ as

$$T(\lambda,\phi,u,\omega)=(\varphi,v,\theta) \tag{4.4}$$

where (φ,v,θ) is the unique solution of the quasilinear problem:

$$\begin{gathered}
\frac{\partial\varphi}{\partial t}-\alpha\Delta\varphi=a\varphi+b\varphi^2-\varphi^3+\lambda\omega\,,\\
\frac{\partial v}{\partial t}-\nu\Delta v+(v.\nabla)v+\nabla p+k(f_s(\varphi)-\varepsilon)v=\lambda\vec{\sigma}\omega\,,\\
\operatorname{div} v=0\,,\\
\frac{\partial\theta}{\partial t}-\Delta\theta+v.\nabla\theta=\lambda\frac{\ell}{2}\frac{\partial f_s}{\partial\varphi}(\varphi)\frac{\partial\varphi}{\partial t}
\end{gathered} \tag{4.5}$$

in Q;

$$\frac{\partial\varphi}{\partial n}=0\,,\quad \theta=0\,,\quad v=0 \tag{4.6}$$

on S; and

$$\varphi(x,0)=\varphi_0(x)\,,\quad \theta(x,0)=\theta_0(x)\,,\quad v(x,0)=v_0(x) \tag{4.7}$$

in Ω.

We observe that the homotopy $T(\lambda,\cdot)$ is well defined. In fact, for fixed $\lambda\in[0,1]$, by using Proposition 3.3 and Lemma 3.1, we conclude that first equation of problem (4.5), (4.6), (4.7) has a unique solution $\varphi\in W_3^{2,1}(Q)\cap L^\infty(Q)$. Once φ is known, Proposition 3.4 implies that the modified Navier-Stokes equations has an unique solution $v\in L^2(0,T;V)\cap L^\infty(0,T;H)$. By usual interpolation, it results that $v\in L^4(Q)^2$. Now that φ and v are known, the L_p-theory of parabolic equations , that also is valid for Neumann boundary condition (see Ladyzenskaja *et al* [21, p.351]), Lemma 3.1 and the facts that $\frac{\partial\varphi}{\partial t}\in L^3(Q)$, $v\in L^4(Q)^2$, and $f_s\in C_b^{1,1}(\mathbb{R})$

imply that there is a unique solution $\theta \in W_3^{2,1}(Q) \cap L^\infty(Q)$ for the third equation of (4.5).

Lemma 4.3. *Under assumptions (H1), (H2), (H3), the mapping* $T : [0,1]\times L^6(Q)\times L^2(0,T;H) \times L^3(Q) \to L^6(Q) \times L^2(0,T;H) \times L^3(Q)$ *is a compact mapping, i.e, it is continuous and maps bounded sets into relatively compact sets.*

Proof. Let us check the continuity of $T(\lambda, .)$. For this, let $\lambda_n \to \lambda$ in [0,1] and $(\phi_n, u_n, \omega_n) \to (\phi, u, \omega)$ in $L^6(Q)\times L^2(0,T;\mathrm{H})\times L^3(Q)$. Denoting $T(\lambda_n, \phi_n, u_n, \omega_n) = (\varphi_n, v_n, \theta_n)$, from (4.4), we write

$$\frac{\partial \varphi_n}{\partial t} - \alpha\Delta\varphi_n = a\varphi_n + b\varphi_n^2 - \varphi_n^3 + \lambda_n\omega_n\,, \tag{4.8}$$

$$\frac{\partial v_n}{\partial t} - \nu\Delta v_n + (v_n.\nabla)v_n + \nabla p_n + k(f_s(\varphi_n) - \varepsilon)v_n = \lambda_n\vec{\sigma}\omega_n\,, \tag{4.9}$$

$$\operatorname{div} v_n = 0\,, \tag{4.10}$$

$$\frac{\partial \theta_n}{\partial t} - \Delta\theta_n + v_n.\nabla\theta_n = \lambda_n\frac{\ell}{2}\frac{\partial f_s}{\partial\varphi}(\varphi_n)\frac{\partial \varphi_n}{\partial t} \tag{4.11}$$

in Q;

$$\frac{\partial \varphi_n}{\partial \eta} = 0\,, \quad v_n = 0\,, \quad \theta_n = 0\,, \tag{4.12}$$

on S; and

$$\varphi_n(x,0) = \varphi_0(x)\,, \quad v_n(x,0) = v_0(x)\,, \quad \theta_n(x,0) = \theta_0(x) \tag{4.13}$$

in Ω. By applying Proposition 3.3 with $\omega_n \in L^2(Q)$, we obtain the following estimate for the phase-field equation (4.8)

$$\|\varphi_n\|_{2,Q}^{(2)} \le C\big(|\lambda_n|\,\|\omega_n\|_{2,Q} + \|\varphi_0\|_{W_2^1(\Omega)}\big) \tag{4.14}$$

Now, by applying Proposition 3.4, we obtain the following estimates for the velocity equation (4.9)

$$\|v_n\|_{L^\infty(0,T,H)} + \|v_n\|_{L^2(0,T,V)} \le C\big(\|v_0\|_H + |\lambda_n|\|\omega_n\|_{2,Q}\big), \tag{4.15}$$

which by usual interpolation implies

$$\|v_n\|_{4,Q} \le C\big(\|v_0\|_H + |\lambda_n|\|\omega_n\|_{2,Q}\big)\,. \tag{4.16}$$

For (4.11), the L_p-theory of the parabolic equation (see Ladyzenskaja [21, p. 351]) with the facts that $\frac{\partial\varphi_n}{\partial t} \in L^2(Q)$, $\frac{\partial f_s}{\partial\varphi}(\varphi_n) \in L^\infty(Q)$, $v_n \in L^4(Q)^2$ and $\theta_0 \in W_2^1(\Omega)$ provides the estimate

$$\|\theta_n\|_{2,Q}^{(2)} \le C\Big(\|v_n\|_{4,Q}\|\theta_0\|_{W_2^1(\Omega)} + |\lambda_n|\Big\|\frac{\partial\varphi_n}{\partial t}\Big\|_{2,Q} + \|\theta_0\|_{W_2^1(\Omega)}\Big)\,. \tag{4.17}$$

Since the sequences (ω_n) and (λ_n) are respectively bounded in $L^2(Q)$ and [0,1], from (4.14) and (4.15) we obtain for all n that

$$\|\varphi_n\|_{2,Q}^{(2)} \le C \quad \text{and} \quad \|v_n\|_{L^\infty(0,T,H)} + \|v_n\|_{L^2(0,T,V)} \le C\,. \tag{4.18}$$

Consequently, from (4.17) we have for all n

$$\|\theta_n\|_{2,Q}^{(2)} \le C\,. \tag{4.19}$$

From (4.18) and (4.19), it follows $\{T(\lambda_n, \phi_n, u_n, \omega_n)\} = \{(\varphi_n, v_n, \theta_n)\}$ is uniformly bounded sequence with respect to n in the functional space $W_2^{2,1}(Q)\times(L^2(0,T;V)\cap L^\infty(0,T;H)) \times W_2^{2,1}(Q)$. Moreover, we observe that for fixed $\varepsilon \in (0,1]$, from the properties of $k(y)$ (see the conditions stated in (H_2)), there is a finite positive constant C depending only on ε such that $\sup\{k(y-\varepsilon)\} \le C$. By using this and our previous estimates as in Lions [22, p. 71], we conclude that for all n,

$$\|(v_n)_t\|_{L^2(0,T;V')} \le C(\varepsilon). \tag{4.20}$$

Thus, the previous estimates, and the Aubin-Lions Lemma (see Temam [28] or Lions [22]), allow us to select a subsequence, which we denote $\{T(\lambda_k, \phi_k, u_k, \omega_k)\} = \{(\varphi_k, v_k, \theta_k)\}$ such that

$$\varphi_k \rightharpoonup \varphi \text{in } W_2^{2,1}(Q) \tag{4.21}$$

$$v_k \rightharpoonup v \quad \text{in } L^2(0,T;V) \tag{4.22}$$

$$v_k \overset{*}{\rightharpoonup} v \quad \text{in } L^\infty(0,T;H) \tag{4.23}$$

$$\theta_k \rightharpoonup \theta \quad \text{in } W_2^{2,1}(Q) \tag{4.24}$$

$$(v_k)_t \rightharpoonup v_t \quad \text{in } L^2(0,T;V') \tag{4.25}$$

$$\varphi_k \to \varphi \quad \text{in } L^6(Q) \tag{4.26}$$

$$v_k \to v \quad \text{in } L^2(0,T;H) \tag{4.27}$$

$$\theta_k \to \theta \quad \text{in } L^3(Q) \tag{4.28}$$

Now, let us verify that $T(\lambda, \phi, u, \omega) = (\varphi, v, \theta)$, in other words, that (φ, v, θ) is solution of (4.5), (4.6),(4.7). For this, we are going to pass to the limit with respect to the above subsequence in equations (4.8)-(4.11) together with the conditions (4.12)-(4.13).

Let us prove that the equations are satisfied in the sense distribution. For this, fix in the sequel $g \in C_c^\infty(Q)$, and let us describe the process of taking the limit only for those terms of the equations that are neither trivial nor standard. We observe that by using (4.26) and $\lambda_k \to \lambda$, we obtain

$$\int_Q \lambda_k(a\varphi_k + b\varphi_k^2 - \varphi_k^3) g \,dx\,dt \to \int_Q \lambda(a\varphi + b\varphi^2 - \varphi^3) g \,dx\,dt \qquad \forall\, g \tag{4.29}$$

Thus, passing to the limit in phase field equation (4.8), using the convergence (4.21), (4.26) and (4.29), we obtain the first equation in (4.5).

To verify the convergence

$$\int_Q k(f_s(\varphi_k) - \varepsilon) v_k \, g \,dx\,dt \to \int_Q k(f_s(\varphi) - \varepsilon) v \, g \,dx\,dt \tag{4.30}$$

we use (4.27), the fact that for fixed $\varepsilon \in (0,1]$, $k(f_s(\cdot) - \varepsilon$ is bounded, and following argument. Consider $h_k = |k(f_s(\varphi_k) - \varepsilon) - k(f_s(\varphi) - \varepsilon)|^2$. Since $k(f_s(\cdot) - \varepsilon)$ is continuous and (4.26) is valid, passing to a subsequence if necessary, we know that $h_k \to 0$ almost everywhere in Q. Moreover, $|h_k| \le C\,\|f_s(\varphi)\|_\infty^2$ a.e and therefore $h_k \to 0$ in $L^1(Q)$ by Lebesgue dominated convergence theorem. Thus, $k(f_s(\varphi_k) - \varepsilon) \to k(f_s(\varphi) - \varepsilon)$ in $L^2(Q)$, what together with (4.22) implies (4.30). By passing to the limit in velocity equation (4.9), using the convergence (4.22), (4.27) and (4.30) we obtain the second equation in (4.5).

Now, we use (4.21), (4.26), $\lambda_k \to \lambda$ and arguments similar to the ones previously with $\frac{\partial f_s}{\partial \varphi}(\varphi_k)$ in place of f_s to obtain

$$\int_Q \lambda_k \frac{\partial f_s}{\partial \varphi}(\varphi_k) \frac{\partial \varphi_k}{\partial t} \, g \,dx\,dt \to \int_Q \lambda \frac{\partial f_s}{\partial \varphi}(\varphi) \frac{\partial \varphi}{\partial t} \, g \,dx\,dt \tag{4.31}$$

By passing to the limit in temperature equation (4.11), using the convergence (4.24), (4.28) and (4.31), we obtain the third equation in (4.5)

The required boundary conditions are included in the definitions of the functional spaces where (φ, v, θ) is in. Also, with the estimates we have obtained, it is standard to prove that φ, v and θ satisfy the required initial conditions. Hence, (φ, v, θ) is solution of (4.8)-(4.13).

Note that for any given subsequence of $\{T(\lambda_n, \varphi_n, v_n, \theta_n)\}$, the above arguments can be applied to conclude that this subsequence admits another subsequence converging to a solution of (4.8)-(4.13). Since (ϕ, u, ω) is also fixed and the solution of this last problem is unique, we conclude that $\{T(\lambda_n, \varphi_n, v_n, \theta_n)\}$ is a sequence with the property that any one of its subsequences has by its turn a subsequence converging to a limit that is independent of the chosen subsequence. Hence, $\{T(\lambda_n, \varphi_n, v_n, \theta_n)\}$ converges to this limit, and the continuity of T is proved.

The same arguments prove that mapping T is a compact mapping. In fact, if $\{(\phi_n, u_n, \omega_n)\}$ is any bounded sequence in $L^6(Q) \times (L^2(0,T;V) \cap L^\infty(0,T;H)) \times L^3(Q)$, the above arguments can be applied to obtain exactly the same sort of estimates for $T(\lambda_n, \phi_n, u_n, \omega_n)$. These imply that $\{(\varphi_n, v_n, \theta_n)\}$ is relatively compact in $L^6(Q) \times L^2(0,T;H) \times L^3(Q)$, and thus there exists a subsequence of $T(\lambda_n, \phi_n, u_n, \omega_n)$ converging in $L^6(Q) \times L^2(0,T;H) \times L^3(Q)$. Therefore, the compactness is proved. ☐

The next lemma give us an uniform estimate for any possible fix point of $T(\lambda, \cdot)$.

Lemma 4.4. *Under assumptions (H1), (H2), (H3), there exists a positive number ρ, depending only on the given data of the problem and in particular independent of $\lambda \in [0,1]$, with the property any fix point of $T(\lambda, .)$ is in the interior of the ball of radius ρ in $L^6(Q) \times L^2(0,T;H) \times L^3(Q)$. That is,*

$$T(\lambda, \varphi, v, \theta) = (\varphi, v, \theta) \Rightarrow \|(\varphi, v, \theta)\| < \rho, \tag{4.32}$$

where $\|\cdot\|$ denotes the norm in $L^6(Q) \times L^2(0,T;H) \times L^3(Q)$.

Proof. Using (4.4), the condition $T(\lambda, \varphi, v, \theta) = (\varphi, v, \theta)$ is equivalent to

$$\frac{\partial \varphi}{\partial t} - \alpha \Delta \varphi = a\varphi + b\varphi^2 - \varphi^3 + \lambda\theta, \tag{4.33}$$

$$\frac{\partial v}{\partial t} - \nu \Delta v + (v.\nabla)v + \nabla p + k(f_s(\varphi) - \varepsilon)v = \lambda \vec{\sigma}\theta, \tag{4.34}$$

$$\operatorname{div} v = 0, \tag{4.35}$$

$$\frac{\partial \theta}{\partial t} - \Delta\theta + v.\nabla\theta = \lambda \frac{\ell}{2} \frac{\partial f_s}{\partial \varphi}(\varphi) \frac{\partial \varphi}{\partial t} \tag{4.36}$$

in Q;

$$\frac{\partial \varphi}{\partial \eta} = 0, \quad \theta = 0, \quad v = 0 \tag{4.37}$$

on S;

$$\varphi(x,0) = \varphi_0(x), \quad \theta(x,0) = \theta_0(x), \quad v(x,0) = v_0(x) \tag{4.38}$$

in Ω. To obtain estimates for (φ, v, θ), we start by multiplying the first equation (4.33) by φ. After integrating of the result over Q_t ($t \in (0,T]$), using Fubini's theorem, Green's formula and Young's inequality, we get

$$\begin{aligned} &\int_\Omega \varphi^2 dx + \int_0^t \int_\Omega |\nabla\varphi|^2 dx\,dt + \frac{\lambda}{2}\int_0^t \int_\Omega \varphi^4 dx\,dt \\ &\le C\Big(\|\varphi_0\|_{2,\Omega}^2 + \int_0^T \int_\Omega |\theta|^2 dx\,dt + \int_0^T \int_\Omega |\varphi|^2 dx\,dt\Big), \end{aligned} \tag{4.39}$$

where C depends on α and $\max_{(x,t)\in Q}\Big(a(x,t)+b(x,t)s-\frac{1}{2}s^2\Big)$. Applying Gronwall's inequality in (4.39), we get

$$\int_0^T \int_\Omega |\varphi|^2 dx\,dt \le C\Big(\|\varphi_0\|_{2,\Omega}^2 + \int_0^T \int_\Omega |\theta|^2 dx\,dt\Big). \tag{4.40}$$

Thus, combining (4.39) and (4.40), we conclude

$$\int_0^T \int_\Omega |\nabla\varphi|^2 dx\,dt \le C\Big(\|\varphi_0\|_{2,\Omega}^2 + \int_0^T \int_\Omega |\theta\|^2 dx\,dt\Big) \tag{4.41}$$

$$\frac{\lambda}{2}\int_0^T \int_\Omega \varphi^4 dx\,dt \le C\Big(\|\varphi_0\|_{2,\Omega}^2 + \int_0^T \int_\Omega |\theta|^2 dx\,dt\Big) \tag{4.42}$$

Now, we multiply equation (4.36) by θ and integrate over Q_t. Then we use the fact that $\frac{\partial f_s}{\partial\varphi}\in L^\infty(\mathbb{R})$, (4.35), Green's formula and also Poincaré's and Young's inequalities to obtain

$$\int_\Omega \theta^2 dx + \int_0^t \int_\Omega |\nabla\theta|^2\,dx\,dt \le C\Big(\|\theta_0\|_{2,\Omega}^2 + \int_0^T \int_\Omega |\varphi_t|^2\,dx\,dt\Big), \tag{4.43}$$

where C depends on Ω, ℓ and $\|\frac{\partial f_s}{\partial\varphi}\|_\infty$.

Multiplying the first equation (4.33) by $\frac{\partial\varphi}{\partial t}$, integrating over Q_t, using Green's formula and Young's inequality, we obtain

$$\begin{aligned} &\int_0^t \int_\Omega \big(\frac{\partial\varphi}{\partial t}\big)^2 dx\,dt + \int_0^t \int_\Omega |\nabla\varphi|^2 dx\,dt \\ &\le C\Big(\|\nabla\varphi_0\|_{2,\Omega}^2 + \int_0^T \int_\Omega |\varphi|^2 dx\,dt + \int_0^T \int_\Omega |\theta|^2 dx\,dt\Big), \end{aligned} \tag{4.44}$$

where C depends on α and $\max_{(x,t)\in Q}\big(a(x,t)+b(x,t)s-s^2\big)$. Using (4.40) in (4.44) and applying the resulting estimate in (4.43), we get

$$\int_\Omega \theta^2 dx + \int_0^t \int_\Omega |\nabla\theta|^2\,dx\,dt \le C\Big(\|\theta_0\|_{2,\Omega}^2 + \|\varphi_0\|_{W_2^1(\Omega)}^2 + \int_0^T \int_\Omega |\theta|^2\,dx\,dt\Big) \tag{4.45}$$

Applying Gronwall inequality in (4.45), we obtain

$$\|\theta\|_{2,Q} \le C\Big(\|\theta_0\|_{2,\Omega} + \|\varphi_0\|_{W_2^1(\Omega)}\Big), \tag{4.46}$$

and, consequently, $\|\nabla\theta\|_{2,Q} \le C\big(\|\theta_0\|_{2,\Omega} + \|\varphi_0\|_{W_2^1(\Omega)}\big)$. Moreover, by interpolation results (see Ladyzenskaja [21, p. 74]), we have

$$\|\theta\|_{4,Q} \le M\left(\|\theta_0\|_{2,\Omega} + \|\varphi_0\|_{W_2^1(\Omega)}\right). \tag{4.47}$$

Using (4.46) in (4.40), (4.41) and (4.42), we conclude that

$$\|\varphi\|_{2,Q} \le C\left(\|\theta_0\|_{2,\Omega} + \|\varphi_0\|_{W_2^1(\Omega)}\right), \tag{4.48}$$

$$\|\nabla\varphi\|_{2,Q} \le C\left(\|\theta_0\|_{2,\Omega} + \|\varphi_0\|_{W_2^1(\Omega)}\right), \tag{4.49}$$

$$\lambda\,\|\varphi\|_{4,Q} \le C\left(\|\theta_0\|_{2,\Omega} + \|\varphi_0\|_{W_2^1(\Omega)}\right). \tag{4.50}$$

Using (4.46) and (4.48) in (4.44), we have

$$\|\frac{\partial\varphi}{\partial t}\|_{2,Q} \le C\left(\|\theta_0\|_{2,\Omega} + \|\varphi_0\|_{W_2^1(\Omega)}\right). \tag{4.51}$$

Multiplying the first equation (4.33) equation by $-\Delta\varphi$ integrating over Q_t, using Green formula and Young inequalities, we obtain

$$\begin{aligned} &\int_\Omega |\nabla\varphi|^2 dx + \int_0^t\int_\Omega |\Delta\varphi|^2 dx\,dt + 3\lambda\int_0^t\int_\Omega \varphi^2|\nabla\varphi|^2 dx\,dt \\ &\le C\Big(\|\nabla\varphi_0\|_{2,\Omega}^2 + \int_0^T\int_\Omega |\theta|^2 dx\,dt + \int_0^T\int_\Omega |\varphi|^2 dx\,dt + \lambda\int_0^T\int_\Omega |\varphi|^4 dx\,dt\Big), \end{aligned} \tag{4.52}$$

where C depends on Ω, α, $\|a\|_{\infty,Q}$, $\|b\|_{\infty,Q}$ and $\|\frac{\partial f_s}{\partial\varphi}\|_\infty$. Using (4.46), (4.48) and (4.50) in (4.52), we obtain

$$\|\Delta\varphi\|_{2,Q} \le C\left(\|\theta_0\|_{2,\Omega} + \|\varphi_0\|_{W_2^1(\Omega)}\right) \tag{4.53}$$

Combining estimates (4.48), (4.49), (4.51) and (4.53), using the imbedding in Lemma 3.1, we have

$$\|\varphi\|_{p,Q} \le C\,\|\varphi\|_{2,Q}^{(2)} \le C\left(\|\theta_0\|_{2,\Omega} + \|\varphi_0\|_{W_2^1(\Omega)}\right) \quad (p \ge 6). \tag{4.54}$$

Now, by multiplying the second equation (4.34) by v, integrating over Q_t, using Green's formula, and Poncaré's and Young's inequalities, we get

$$\begin{aligned} &\frac{1}{2}\int_\Omega v^2 dx + \frac{\nu}{2}\int_0^t\int_\Omega |\nabla v|^2\,dx\,dt + \int_0^t\int_\Omega k\left(f_s(\varphi) - \varepsilon\right) v^2 dx\,dt \\ &\le C\Big(\|v_0\|_{\mathrm{H}} + \int_0^T\int_\Omega |\theta|^2\,dx\,dt\Big). \end{aligned} \tag{4.55}$$

Combining (4.46) and (4.55), using that $k(f_s(\varphi) - \epsilon) \ge 0$, we conclude that

$$\|v\|_{L^\infty(0,T;\mathrm{H})} + \|v\|_{L^2(0,T;\mathrm{V})} \le C\left(\|v_0\|_{\mathrm{H}} + \|\theta_0\|_{2,\Omega} + \|\varphi_0\|_{W_1^2(\Omega)}\right).$$

Finally, by the interpolation result given in Theorem (3.4),

$$\|v\|_{4,Q} \le C\left(\|v_0\|_{\mathrm{H}} + \|\theta_0\|_{2,\Omega} + \|\varphi_0\|_{W_1^2(\Omega)}\right). \tag{4.56}$$

□

The next lemma tell us that there is an unique fix point in the special case $\lambda = 0$.

Lemma 4.5. *Under assumptions (H1), (H2), (H3), there exists an unique solution of the problem* $T(0,\varphi,v,\theta) = (\varphi,v,\theta)$ *(*T *defined in* (4.4)*).*

Proof. The equation $T(0,\varphi,v,\theta)=(\varphi,v,\theta)$ is equivalent to the nonlinear system

$$\begin{gathered}\frac{\partial\varphi}{\partial t}-\alpha\Delta\varphi=a\varphi+b\varphi^2-\varphi^3,\\ \frac{\partial v}{\partial t}-\nu\Delta v+(v.\nabla)v+\nabla p+k(f_s(\varphi)-\varepsilon)v=0,\\ \operatorname{div} v=0,\\ \frac{\partial\theta}{\partial t}-\Delta\theta+v.\nabla\theta=0\end{gathered}$$

in Q;

$$\frac{\partial\varphi}{\partial\eta}=0,\quad \theta=0,\quad v=0$$

on S;

$$\varphi(x,0)=\varphi_0(x),\quad \theta(x,0)=\theta_0(x), v(x,0)=v_0(x)$$

in Ω. For these equations, Proposition 3.3 ensures the existence and uniqueness of φ; then Proposition 3.4 gives the existence and uniqueness v. The L_p-theory of the linear parabolic equations ensures then the existence and uniqueness of θ. $\square$

Proof of Theorem 4.1. According to Lemma 4.4, there exists a number ρ satisfying (4.32). Let us consider the open ball

$$B_\rho=\left\{(\varphi,v,\theta)\in L^6(Q)\times L^2(0,T;H)\times L^3(Q):\|(\varphi,v,\theta)\|<\rho\right\},$$

where $\|\cdot\|$ is the norm in the space $L^6(Q)\times L^2(0,T;H)\times L^3(Q)$. Lemma 4.3 ensures that the mapping $T:[0,1]\times L^6(Q)\times L^2(0,T;\mathrm{H})\times L^3(Q)\to L^6(Q)\times L^2(0,T;\mathrm{H})\times L^3(Q)$ is a homotopy of compact transformations on the closed ball $\overline{B}_\rho$ and Lemma 4.4 implies that

$$T(\lambda,\varphi,v,\theta)\neq(\varphi,v,\theta)\quad \forall(\varphi,v,\theta)\in\partial B_\rho,\ \forall\lambda\in[0,1]$$

The foregoing properties allow us consider the Leray-Schauder degree $D(Id-T(\lambda,\cdot),B_\rho,0)$, $\forall\lambda\in[0,1]$ (see Deimling [13]). The homotopy invariance of Leray-Schauder degree shows that the equality below holds

$$D(Id-T(0,\cdot),B_\rho,0)=D(Id-T(1,\cdot),B_\rho,0) \tag{4.57}$$

Moreover, the Lemma 4.5 ensures that the problem $T(0,\varphi,v,\theta)=(\varphi,v,\theta)$ has a unique solution in $L^6(Q)\times L^2(0,T;\mathrm{H})\times L^3(Q)$. Hence we can choose a sufficiently large $\rho>0$ such that the ball B_ρ contains this solution, it turns out that $D(Id-T(0,\cdot),B_\rho,0)=1$. Then relation (4.57) ensures that the equation $T(1,\varphi,v,\theta)-(\varphi,v,\theta)=0$ has a solution $(\varphi,v,\theta)\in B_\rho\subset L^6(Q)\times L^2(0,T;\mathrm{H})\times L^3(Q)$. By (4.4) with $\lambda=1$, this is just a solution of the problem (4.1)-(4.2)-(4.3).

The uniqueness and regularity of problem (4.1), (4.2), (4.3) are consequence of the application of the Propositions 3.3 and 3.4 and L_p-regularity theory for linear parabolic equations. To prove uniqueness let φ_i, v_i and θ_i with i = 1, 2 be two solutions of problem (4.1), (4.2), (4.3), with corresponding pressures p_i (for simplicity of exposition, we omit the subscript ε). We first observe that by using the previously obtained estimates and arguments similar to the ones used to prove that T_λ is well defined (Definition 4.2), we conclude that $\varphi_i\in W_3^{2,1}(Q)\cap L^\infty(Q)$, $v_i\in L^2(0,T;V)\cap L^\infty(0,T;H)$ and $\theta\in W_2^{2,1}(Q)\cap L^p(Q)$ (for any finite $p\geq1$).

Let $\tilde\varphi = \varphi_1 - \varphi_2$, $\tilde v = v_1 - v_2$, $\tilde\theta = \theta_1 - \theta_2$ and $\tilde p = p_1 - p_2$. These functions satisfy the following conditions:

$$\frac{\partial\tilde\varphi}{\partial t} - \alpha\Delta\tilde\varphi = [a(x,t) + b(x,t)(\varphi_1+\varphi_2) - (\varphi_1^2 + \varphi_1\varphi_2 + \varphi_2^2)]\tilde\varphi + \tilde\theta, \tag{4.58}$$

$$\begin{aligned}&\frac{\partial\tilde v}{\partial t} - \nu\Delta\tilde v + (v_1.\nabla)\tilde v + \nabla\tilde p + k(f_s(\varphi_1)-\varepsilon)\,\tilde v\\ &= \vec\sigma\tilde\theta - (\tilde v.\nabla)v_2 + \{k(f_s(\varphi_1)-\varepsilon) - k(f_s(\varphi_2)-\varepsilon)\}\tilde v,\end{aligned} \tag{4.59}$$

$$\operatorname{div}\tilde v = 0, \tag{4.60}$$

$$\frac{\partial\tilde\theta}{\partial t} - \Delta\tilde\theta + v_1.\nabla\tilde\theta = \frac{\ell}{2}\frac{\partial f_s}{\partial\varphi}(\varphi_1)\frac{\partial\tilde\varphi}{\partial t} - (\tilde v.\nabla)\theta_2 + \frac{\ell}{2}\left(\frac{\partial f_s}{\partial\varphi}(\varphi_1) - \frac{\partial f_s}{\partial\varphi}(\varphi_2)\right)\frac{\partial\varphi_2}{\partial t}; \tag{4.61}$$

$$\frac{\partial\tilde\varphi}{\partial\eta} = 0, \quad \tilde\theta = 0, \quad \tilde v = 0 \tag{4.62}$$

on S; and

$$\tilde\varphi(x,0) = \tilde\theta(x,0) = 0, \quad \tilde v(x,0) = 0 \tag{4.63}$$

in Ω. Multiplying equation (4.58) by $\tilde\varphi$ and integrating on Ω, after usual integration by parts, using the fact that $a(\cdot)$, $b(\cdot)$, φ_1, $\varphi_2 \in L^\infty(Q)$ and Holder's inequality, we obtain

$$\frac{d}{dt}\|\tilde\varphi(t)\|_{2,\Omega}^2 + 2\alpha\|\nabla\tilde\varphi(t)\|_{2,\Omega}^2 \le C_1\big[\|\tilde\varphi(t)\|_{2,\Omega}^2 + \|\tilde\theta(t)\|_{2,\Omega}^2\big]. \tag{4.64}$$

Multiply (4.58) by $\frac{\partial\tilde\varphi}{\partial t}$ and integrate on Ω. Proceeding as before, we obtain

$$\|\frac{\partial\tilde\varphi}{\partial t}(t)\|_{2,\Omega}^2 + \frac{\alpha}{2}\frac{d}{dt}\|\nabla\tilde\varphi(t)\|_{2,\Omega}^2 \le C_2\big[\|\tilde\varphi(t)\|_{2,\Omega}^2 + \|\tilde\theta(t)\|_{2,\Omega}^2\big]. \tag{4.65}$$

Multiply (4.59) by $\tilde v$ and proceed as usual with the help of the facts that $\operatorname{div} v_1 = 0$, $k(f_s(\varphi_1) - \epsilon \ge 0$ and Holder's inequality to obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\tilde v(t)\|_{2,\Omega}^2 + \nu\|\nabla\tilde v(t)\|_{2,\Omega}^2\\ &\le C\big[\|\tilde\theta(t)\|_{2,\Omega}^2 + \|\tilde v(t)\|_{2,\Omega}^2 + \int_\Omega(\tilde v(t).\nabla)v_2(t)\tilde v(t)\\ &\quad + \int_\Omega[k(f_s(\varphi_1(t)) - \epsilon) - k(f_s(\varphi_2(t)) - \epsilon)]|\tilde v(t)|^2\big]\end{aligned} \tag{4.66}$$

The integral terms on the right hand side of the previous inequality can be estimated as follows.

$$\begin{aligned}|\int_\Omega(\tilde v(t).\nabla)v_2(t)\tilde v(t)| &\le C\|\nabla v_2(t)\|_{2,\Omega}\|\tilde v(t)\|_{4,\Omega}^2\\ &\le C\|\nabla v_2(t)\|_{2,\Omega}\|\tilde v(t)\|_{2,\Omega}\|\nabla\tilde v(t)\|_{2,\Omega}\\ &\le C_\nu\|\nabla v_2(t)\|_{2,\Omega}^2\|\tilde v(t)\|_{2,\Omega}^2 + \frac{\nu}{4}\|\nabla\tilde v(t)\|_{2,\Omega}^2\end{aligned}$$

Next, by using the facts that $k(\cdot)$ is a Lipschitz function on $(-\infty, 1-\epsilon)$ and $f_s(\cdot)$ is a L^∞-function, we obtain

$$\begin{aligned}
&|\int_\Omega [k(f_s(\varphi_1(t)) - \epsilon) - k(f_s(\varphi_2(t)) - \epsilon)]|\tilde{v}(t)|^2 dx| \\
&\leq C_\epsilon \int_\Omega |[f_s(\varphi_1(t)) - \epsilon)] - [f_s(\varphi_1(t)) - \epsilon]||\tilde{v}(t)|^2 dx \\
&= C_\epsilon \int_\Omega |f_s(\varphi_1(t)) - f_s(\varphi_1(t))||\tilde{v}(t)|^2 dx \leq C\|\tilde{v}(t)\|^2_{2,\Omega}.
\end{aligned}$$

Using the last two estimates in (4.66), we obtain

$$\frac{d}{dt}\|\tilde{v}(t)\|^2_{2,\Omega} + \frac{3}{2}\nu\|\nabla\tilde{v}(t)\|^2_{2,\Omega} \leq C_3\|\tilde{\theta}(t)\|^2_{2,\Omega} + C_4(1 + \|\nabla v_2(t)\|^2_{2,\Omega})\|\tilde{v}(t)\|^2_{2,\Omega} \quad (4.67)$$

We proceed by multiplying equation (4.61) by $\tilde{\theta}$, integranting on Ω. After integration by parts and the use of the facts that div $\mathrm{v}_1 = 0$, $\frac{\partial f_s}{\partial\varphi} \in L^\infty(R)$, with the help of Holder's inequality, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\tilde{\theta}(t)\|^2_{2,\Omega} + \|\nabla\tilde{\theta}(t)\|^2_{2,\Omega} \\
&\leq C(\|\tilde{\theta}(t)\|^2_{2,\Omega} + \|\frac{\partial\tilde{\varphi}}{\partial t}(t)\|^2_{2,\Omega}) \qquad (4.68)\\
&\quad + \int_\Omega (\tilde{v}(t).\nabla)\theta_2\,\tilde{\theta}(t)dx + \int_\Omega \frac{\ell}{2}(\frac{\partial f_s}{\partial\varphi}(\varphi_1(t)) - \frac{\partial f_s}{\partial\varphi}(\varphi_1(t)))\frac{\partial\varphi_2}{\partial t}(t)\,\tilde{\theta}(t)\,dx
\end{aligned}$$

The last two integrals terms in the above inequality can be estimated as follows:

$$\begin{aligned}
|\int_\Omega (\tilde{v}(t).\nabla)\theta_2\tilde{\theta}(t)dx| &= |\int_\Omega \mathrm{div}\ (\tilde{\mathrm{v}}(\mathrm{t})\theta_2)\ \tilde{\theta}(\mathrm{t})\mathrm{dx}| \\
&= |\int_\Omega (\tilde{v}(t)\theta_2)\ \nabla\tilde{\theta}(t)dx| \\
&\leq \|\tilde{v}(t)\|_{4,\Omega}\|\theta_2(t)\|_{4,\Omega}\|\nabla\tilde{\theta}(t)\|_{2,\Omega} \\
&\leq 4\|\tilde{v}(t)\|^2_{4,\Omega}\|\theta_2(t)\|^2_{4,\Omega} + \frac{1}{4}\|\nabla\tilde{\theta}(t)\|^2_{2,\Omega} \\
&\leq C\|\tilde{v}(t)\|_{2,\Omega}\|\nabla\tilde{v}(t)\|_{2,\Omega}\|\theta_2(t)\|^2_{4,\Omega} + \frac{1}{4}\|\nabla\tilde{\theta}(t)\|^2_{2,\Omega} \\
&\leq C_\nu\|\theta_2(t)\|^4_{4,\Omega}\|\tilde{v}(t)\|^2_{2,\Omega} + \frac{\nu}{2}\|\nabla\tilde{v}(t)\|^2_{2,\Omega} + \frac{1}{4}\|\nabla\tilde{\theta}(t)\|^2_{2,\Omega}.
\end{aligned}$$

Using the fact that $\frac{\partial f_s}{\partial\varphi}$ is a Lipschitz function, we obtain

$$\begin{aligned}
&|\int_\Omega \frac{\ell}{2}(\frac{\partial f_s}{\partial\varphi}(\varphi_1(t)) - \frac{\partial f_s}{\partial\varphi}(\varphi_1(t)))\frac{\partial\varphi_2}{\partial t}(t)\,\tilde{\theta}(t)\,dx| \\
&\leq C\int_\Omega |\tilde{\varphi}(t)|\,|\frac{\partial\varphi_2}{\partial t}(t)|\,|\tilde{\theta}(t)|\,dx \\
&\leq C\|\tilde{\varphi}(t)\|_{2,\Omega}\|\frac{\partial\varphi_2}{\partial t}(t)\|_{3,\Omega}\|\tilde{\theta}(t)\|_{6,\Omega} \\
&\leq C\|\tilde{\varphi}(t)\|_{2,\Omega}\|\frac{\partial\varphi_2}{\partial t}(t)\|_{3,\Omega}\|\nabla\tilde{\theta}(t)\|_{2,\Omega} \\
&\leq C\|\frac{\partial\varphi_2}{\partial t}(t)\|^2_{3,\Omega}\|\tilde{\varphi}(t)\|^2_{2,\Omega} + \frac{1}{4}\|\nabla\tilde{\theta}(t)\|^2_{2,\Omega}.
\end{aligned}$$

Using the last two inequalities in (4.68),

$$\begin{aligned}&\frac{d}{dt}\|\tilde\theta(t)\|^2_{2,\Omega}+\|\nabla\tilde\theta(t)\|^2_{2,\Omega}\\&\le C_5(\|\tilde\theta(t)\|^2_{2,\Omega}+\|\frac{\partial\tilde\varphi}{\partial t}(t)\|^2_{2,\Omega})+C_6\|\theta_2(t)\|^4_{4,\Omega}\|\tilde v(t)\|^2_{2,\Omega}+C_7\|\frac{\partial\varphi_2}{\partial t}(t)\|^2_{3,\Omega}\|\tilde\varphi\|^2_{2,\Omega}.\end{aligned}\tag{4.69}$$

Now, we multiply (4.65) by $2C_5$ and add the result to (4.64), (4.67) and (4.69). After some simplifications, we obtain

$$\begin{aligned}&\frac{d}{dt}\|\tilde\varphi(t)\|^2_{2,\Omega}+\frac{d}{dt}\|\tilde v(t)\|^2_{2,\Omega}+\frac{d}{dt}\|\tilde\theta(t)\|^2_{2,\Omega}+\alpha C_5\frac{d}{dt}\|\nabla\tilde\varphi(t)\|^2_{2,\Omega}\\&\le C_8(1+\|\frac{\partial\varphi_2(t)}{\partial t}\|^2_{3,\Omega})\|\tilde\varphi(t)\|_{2,\Omega}\\&\quad+C_9(1+\|\nabla v_2(t)\|^2_{2,\Omega}+\|\theta_2(t))\|^4_{4,\Omega})\|\tilde v(t)\|_{2,\Omega}+C_{10}\|\tilde\theta\|^2_{2,\Omega}.\end{aligned}$$

By denoting $z(t)=\|\tilde\varphi(t)\|^2_{2,\Omega}+\|\tilde v(t)\|^2_{2,\Omega}+\|\tilde\theta(t)\|^2_{2,\Omega}+\alpha C_5\|\nabla\tilde\varphi(t)\|^2_{2,\Omega}$, the last inequality implies

$$\frac{d}{dt}z(t)\le C\big[1+\|\frac{\partial\varphi_2}{\partial t}(t)\|^2_{3,\Omega})+\|\nabla v_2(t)\|^2_{2,\Omega}+\|\theta_2(t))\|^4_{4,\Omega}\big]z(t).$$

This inequality implies that for $t\in[0,T]$,

$$0\le z(t)\le z(0)\exp\big\{C(T)[1+\|\frac{\partial\varphi_2}{\partial t}(t)\|^2_{3,Q})+\|v_2(t)\|^2_{L^2(0,T;V)}+\|\theta_2(t))\|^4_{4,Q}]\big\}.$$

Since $\|\frac{\partial\varphi_2}{\partial t}(t)\|^2_{L^3(Q)})+\|v_2(t)\|^2_{L^2(0,T;V)}+\|\theta_2(t))\|^4_{L^4(Q)}$ is finite, due to the known regularity of the involved functions, and $z(0)=0$, we conclude that $z(t)\equiv0$, and therefore $\tilde\varphi\equiv0$, $\tilde v\equiv0$, $\tilde\theta\equiv0$, which imply the uniqueness of the solutions.

Next, we show that the solutions $(\varphi_\varepsilon,v_\varepsilon,\theta_\varepsilon)\in L^6(Q)\times L^2(0,T;\mathrm{H})\times L^3(Q)$ of the problem (4.1), (4.2), (4.3) are uniformly bounded with respect to ε in the space $W^{2,1}_3(Q)\times L^2(0,T;V)\cap L^\infty(0,T;H)\times W^{2,1}_2(Q)$. For this, note first that $\theta_\varepsilon\in L^3(Q)$; the L_p-theory of parabolic linear equation combined with Theorem 3.3 and Lemma 3.1 allow us to conclude that there exists an unique $\varphi_\varepsilon\in W^{2,1}_3(Q)\cap L^\infty(Q)$ such that

$$\|\varphi_\varepsilon\|_{\infty,Q}\le C\,\|\varphi_\varepsilon\|^{(2)}_{3,Q}\le C\Big(\big\|(a+b\varphi_\varepsilon-\varphi_\varepsilon^2)\varphi_\varepsilon\big\|_{3,Q}+\|\theta_\varepsilon\|_{3,Q}+\|\varphi_{0\varepsilon}\|_{W^{4/3}_3(\Omega)}\Big).\tag{4.70}$$

Since $\max_{(x,t)\in Q}\big(a(x,t)+b(x,t)s-s^2\big)$ is finite, from (4.70), we have

$$\|\varphi_\varepsilon\|_{\infty,Q}\le C\,\|\varphi_\varepsilon\|^{(2)}_{3,Q}\le C\Big(\,\|\varphi_\varepsilon\|_{6,Q}+\|\theta_\varepsilon\|_{3,Q}+\|\varphi_{0\varepsilon}\|_{W^{4/3}_3(\Omega)}\Big).\tag{4.71}$$

Combining (4.47), (4.54) and (4.71) and using usual Sobolev imbeddings, we conclude that

$$\|\varphi_\varepsilon\|^{(2)}_{3,Q}\le C\left(\|\theta_{0\varepsilon}\|_{2,\Omega}+\|\varphi_{0\varepsilon}\|_{W^{4/3}_3(\Omega)}\right).\tag{4.72}$$

Moreover, Lemma 3.2 gives us that $\varphi_\varepsilon\in H^{2/3,1/3}(Q)$ such that

$$|\varphi_\varepsilon|^{(2/3)}_Q\le C\,\|\varphi_\varepsilon\|^{(2)}_{3,Q}\le C\left(\|\theta_{0\varepsilon}\|_{2,\Omega}+\|\varphi_{0\varepsilon}\|_{W^{4/3}_3(\Omega)}\right).\tag{4.73}$$

We consider then the equation for the temperature. By applying the L_p-theory of parabolic linear equations (see Ladyzenskaja [21]) together with the facts that

$\frac{\partial\varphi_\varepsilon}{\partial t} \in L^2(Q)$, $f_s \in C_b^{1,1}(\mathbb{R})$ and $v_\varepsilon \in L^4(Q)^2$, we have that there exists an unique $\theta_\varepsilon \in W_2^{2,1}(Q) \cap L^p(Q)\,(p \geq 2)$ such that

$$\|\theta_\varepsilon\|_{2,Q}^{(2)} \leq C\Big(\|v_\varepsilon\|_{4,Q}\,\|\theta_{0\varepsilon}\|_{W_2^1(\Omega)} + \Big\|\frac{\partial f_s}{\partial\varphi}\Big\|_{\infty,Q}\Big\|\frac{\varphi_\varepsilon}{\partial t}\Big\|_{2,Q} + \|\theta_{0\varepsilon}\|_{W_2^1(\Omega)}\Big), \quad (4.74)$$

where the estimates $\|v_\varepsilon\|_{4,Q}$ and $\|\frac{\varphi_\varepsilon}{\partial t}\|_{2,Q}$ are given by (4.56) and (4.51), respectively.

Combining (4.54), (4.56) and (4.74), we obtain

$$\|\theta_\varepsilon\|_{2,Q}^{(2)} \leq C\left(\|v_0\|_{\mathrm{H}} + \|\varphi_0\|_{W_3^{4/3}(\Omega)} + \|\theta_0\|_{W_2^1(\Omega)}\right) \quad (4.75)$$

Therefore, the solutions $(\varphi_\varepsilon, v_\varepsilon, \theta_\varepsilon)$ of problem (4.1), (4.2), (4.3) are uniformly bounded with respect to ε in the space $W_3^{2,1}(Q) \times (L^2(0,T;V) \cap L^\infty(0,T;H)) \times W_2^{2,1}(Q)$, and this completes the proof of Theorem 4.1. □

5. Proof of Theorem 3.6

In this section we use the results of Theorem 4.1, the L_p-theory of parabolic equations, the imbedding of Lemma 3.2 and compactness arguments to prove a result on existence and regularity of solution for problem (2.1), (2.2), (2.3). This will be obtained by passing to the limit in the regularized problem (4.1), (4.2), (4.3) as ε approaches zero. Due to the estimates we present, the convergence of almost all the terms in the equations of the regularized problem will be standard ones, except for the regularized velocity equation that will require a local argument. The stated regularity of the solutions will be obtained by using bootstrapping arguments. Unfortunately, due to the additional Carman-Koseny type term in the velocity equation, we cannot improve the regularity of weak solution of Navier-Stokes equations.

Passing to the Limit. As a consequence of Theorem 4.1, for $\varepsilon \in (0,1]$, any solution $(\varphi_\varepsilon, v_\varepsilon, \theta_\varepsilon) \in L^6(Q) \times L^2(0,T;\mathrm{H}) \times L^3(Q)$ of problem (4.1), (4.2), (4.3) is uniformly bounded with respect to ε in the space $W_3^{2,1}(Q) \times (L^2(0,T;V) \cap L^\infty(0,T;H)) \times W_2^{2,1}(Q)$.

With the help of Aubin-Lions Lemma (see Temam [28], Lions [22] or Corollary 4, p. 85, in Simon [27]), there exists $(\varphi, v, \theta) \in L^6(Q) \times L^2(0,T;H) \times L^3(Q)$ and a subsequence, which for simplicity of notation is still indexed by ε, such that as $\varepsilon \to 0$

$$\begin{gathered}
\varphi_\varepsilon \to \varphi \quad \text{in } L^q(Q)\,(q \geq 6)\\
\nabla\varphi_\varepsilon \to \nabla\varphi \quad \text{in } L^3(Q)^2\\
\varphi_\varepsilon \rightharpoonup \varphi \quad \text{in } W_3^{2,1}(Q)\\
\theta_\varepsilon \to \theta \quad \text{in } L^p(Q)\,(p \geq 2)\\
\nabla\theta_\varepsilon \to \nabla\theta \quad \text{in } L^2(Q)^2\\
\theta_\varepsilon \rightharpoonup \theta \quad \text{in } W_2^{2,1}(Q)\\
v_\varepsilon \rightharpoonup v \quad \text{in } L^2(0,T;\mathrm{V})\\
v_\varepsilon \overset{*}{\rightharpoonup} v \quad \text{in } L^\infty(0,T;\mathrm{H})
\end{gathered}$$

Moreover, by Lemma 3.2, $\varphi_\varepsilon \in H^{2/3,1/3}(Q)$ and for all $\varepsilon \in [0,1]$ we have $|\varphi_\varepsilon|_Q^{(2/3)} \leq C\,\|\varphi_\varepsilon\|_3^{(2)}$. In particular, $\sup_{\overline{Q}}|\varphi_\varepsilon(x,t)| \leq C$, and $\langle\varphi_\varepsilon\rangle_t^{(1/3)} \leq C$. Thus, $\{\varphi_\varepsilon\}$ is uniformly bounded and equicontinuous family in $\overline{Q}$. By Arzela-Ascoli's Theorem it follows that there exists a subsequence, that we denote, for simplicity, again by $\{\varphi_\varepsilon\}$ such that $\varphi_\varepsilon \to \varphi$ uniformly in $\overline{Q}$.

We check now that $(\varphi, v, \theta) \in L^6(Q) \times L^2(0,T;H) \times L^3(Q)$ is a generalized solution of problem (2.1), (2.2), (2.3). We start by taking Q_s and Q_{ml} as in Definition 3.5 with the just obtained function φ.

Now, we have to prove that $v = 0$ in $\overset{\circ}{Q}_s$. For this, we will use an argument already used by Blanc *et al* [3]: we take K a compact subset in $\overset{\circ}{Q}_s$ and observe that $f_s \in C_b^{1,1}(\mathbb{R})$, $f_s(\varphi(x,t)) = 1$ in a neighborhood of K. Since $\varphi_\varepsilon \to \varphi$ uniformly in $\overline{Q}$, we conclude that there is a small positive ε_K such that

$$f_s(\varphi_\varepsilon(x,t)) = 1 \quad \text{in} \quad K$$

whenever $\varepsilon \in (0, \varepsilon_K)$. By multiplying the regularized velocity equation of problem (4.1)-(4.2)-(4.3) by v_ε, integrating over K, using Green's formula and Young's inequality, we obtain

$$k(1-\varepsilon) \|v_\varepsilon\|_{2,K} \leq C$$

with C a positive constant independent of $\varepsilon \in (0, \varepsilon_K)$. As ε approaches zero, $k(1-\varepsilon)$ blows up and compels $\|v_\varepsilon\|_{L^2(K)^2}$ to converge to 0. Therefore, $v_{\varepsilon|_K} \to 0$ in $L^2(K)$, and consequently $v = 0$ in K. Since K was an arbitrary compact set of $\overset{\circ}{Q}_s$, we conclude that $v = 0$ in $\overset{\circ}{Q}_s$.

Now we have to show that the triple of functions (φ, θ, v) satisfies equations (3.11), (3.12) and (3.13). We start by proving that (3.13) is satisfied. For this, we multiply the second equation in (4.1) by a test function $\phi \in C([0,T]; W_2^1(\Omega_{ml}(t)))$ such that $\operatorname{div} \phi(.,t) = 0$ for all $t \in [0,T]$, $\operatorname{supp} \phi(x,t) \subset Q_{ml} \cup \Omega_{ml}(0)$ and $\phi(.,T) = 0$ and integrate over Q. After some usual integrations by parts using (4.2), (4.3) and observing the properties of ϕ, we obtain

$$\begin{aligned}
&- \int_{Q_{ml}} v_\varepsilon \, \phi_t \, dx \, dt + \nu \int_{Q_{ml}} \nabla v_\varepsilon \nabla \phi \, dx \, dt + \int_{Q_{ml}} (v_\varepsilon . \nabla) v_\varepsilon \, \phi \, dx \, dt \\
&+ \int_{Q_{ml}} k(f_s(\varphi_\varepsilon) - \varepsilon) v_\varepsilon \, \phi \, dx \, dt \\
&= \int_{Q_{ml}} \vec{\sigma} \theta_\varepsilon \phi \, dx \, dt + \int_{\Omega_{ml}(0)} v_0(x) \phi(x,0) dx
\end{aligned} \tag{5.1}$$

The stated convergence for φ_ε, θ_ε and v_ε are enough to conclude the convergence of the first and second terms of the left hand side and also of the first term of the right hand side of equation (5.1). For the convergence of the third and fourth term of the left hand side, however, we need to be more careful. We first observe that

$$k(f_s(\varphi_\varepsilon) - \varepsilon) \to k(f_s(\varphi)) \quad \text{in } C^0(K_{ml}) \tag{5.2}$$

for any fixed compact $K_{ml} \subset Q_{ml} \cup \Omega_{ml}(0)$. In fact, in such K_{ml}, $k(f_s(\varphi_\varepsilon(x,t)) - \varepsilon)$ and $k(f_s(\varphi(x,t)))$ are bounded continuous functions, and, since $f_s(\varphi_\varepsilon) - \varepsilon$ converges to $f_s(\varphi)$ in $C^0(K_{ml})$, we obtain the stated result. In particular, this result holds for K_{ml} taken as *supp* ϕ, and this guarantees the convergence of the last term in the left hand side of the last equation.

For the convergence of the third term of the left hand side it is necessary to improve the convergence of v_ε. For this, we first observe that Q_{ml} is an open set and can be covered by a countable number of open cylinders $\Omega_i \times (a_i, b_i)$, such that for each $i = 1, \dots, \infty$, we have $\overline{\Omega}_i \subset \Omega$ and $[a_i, b_i] \subset (0, T)$. Thus, for each $i = 1, \dots, \infty$, we can take the compact set $\overline{\Omega}_i \times [a_i, b_i]$ as K_{ml} in (5.2) and conclude that there is $\varepsilon_i \in (0, 1]$ and $C_i > 0$ independent of $\varepsilon \in (0, \varepsilon_i]$ such that for such ε we have

$$\|k(f_s(\varphi_\varepsilon) - \varepsilon)\|_{L^\infty(\overline{\Omega}_i \times [a_i, b_i])} \leq C_i.$$

This and our previous estimates allow us to work with the second equation in (4.1) restricted to $\Omega_i \times (a_i, b_i)$ to obtain that there is $C_i > 0$ independent of $\varepsilon \in (0, \varepsilon_i]$ such that for such ε we have

$$\|\frac{\partial v_\varepsilon}{\partial t}\|_{L^2(a_i, b_i; V'(\Omega_i))} \leq C_i,$$

where $V'(\Omega_i)$ is the topological dual of the Banach space

$$V(\Omega_i) = \{u \in \overset{0}{W}{}^1_2(\Omega_i)^2; \operatorname{div} u = 0\},$$

considered with the norm of $\overset{0}{W}{}^1_2(\Omega_i)^2$.

Also, our previous estimates tell us in particular that $\{v_\varepsilon\}$ for is uniformly bounded with respect to $\varepsilon \in (0, \varepsilon_i]$ in $L^2(a_i, b_i; W(\Omega_i))$, where $W(\Omega_i) = \{u \in W^1_2(\Omega_i)^2; \operatorname{div} u = 0\}$ is a Banach space with the $W^1_2(\Omega_i)^2$-norm.

Consider the Banach space

$$H(\Omega_i) = \{u \in L^2(\Omega_i)^2; \operatorname{div} u = 0, \text{ and null normal trace}\}$$

with the $L^2(\Omega_i)^2$-norm (see Temam [28] for properties of this and the previous Banach spaces). We observe that $W(\Omega_i) \subset H(\Omega_i) \subset V'(\Omega_i)$, and the first imbedding is compact, we can use Corolary 4, p. 85, in Simon [27] to conclude that there is a subsequence of $\{v_\varepsilon\}$ converging to v in $L^2(a_i, b_i; H(\Omega_i))$. In particular, this implies that along such subsequence $v_\varepsilon \to v \quad \text{in} \quad L^2(\Omega_i \times (a_i, b_i))$.

Proceeding as above for each $i = 1, \dots, \infty$, with the help of the usual diagonal argument, we obtain a subsequence such that

$$v_\varepsilon \to v \quad \text{in} \quad L^2_{\text{loc}}(Q_{ml}).$$

Thus, along such subsequence, we can pass to the limit as $\varepsilon \to 0$ in (5.1) by proceeding exactly as in the case of the classical Navier-Stokes equations and conclude that (3.13) is satisfied.

To obtain the other equations in Definition 3.5, we multiply the first and third equations of (4.1) respectively by $\beta \in W^{1,1}_2(Q)$ with $\beta(.\,, T) = 0$ and $\xi \in \overset{0}{W}{}^{1,1}_2(Q)$ with $\xi(., T) = 0$, and proceed as before. Using arguments similar to the ones in (4.29) and (4.31), we conclude that

$$\int_Q \left(a\varphi_\varepsilon + b\varphi_\varepsilon - \varphi_\varepsilon^3\right) \beta \, dx \, dt \to \int_Q \left(a\varphi + b\varphi - \varphi^3\right) \beta \, dx \, dt,$$
$$\int_Q \frac{\partial f_s}{\partial \varphi}(\varphi_\varepsilon) \frac{\partial \varphi_\varepsilon}{\partial t} \xi \, dx \, dt \to \int_Q \frac{\partial f_s}{\partial \varphi}(\varphi) \frac{\partial \varphi}{\partial t} \xi \, dx \, dt,$$

as $\varepsilon \to 0$. With these results, it is easy to to pass to the limit as $\varepsilon \to 0$ and conclude that equations (3.11) and (3.12) are also satisfied.

Regularity of the Solution. Now we have to examine the regularity of (φ, θ, v). For this, we remark that by interpolation (see Ladyzenskaja [21] p. 74), $\theta \in L^4(Q)$. Thus, applying Proposition 3.3 with $\theta \in L^3(Q)$, we conclude that $\varphi \in W_3^{2,1}(Q) \cap L^\infty(Q)$. Also, Proposition 3.4 give us that $v \in L^4(Q)^2$.

Applying the L_p-theory of parabolic equations together with the facts that $f_s \in C_b^{1,1}(\mathbb{R})$, $v \in L^4(Q)^2$, $\frac{\partial \varphi}{\partial t} \in L_2(Q)$ and Lemma the result of 3.1, we conclude that $\theta \in W_2^{2,1}(Q) \cap L^p(Q)$ $(p \geq 2)$. Therefore, by using a bootstrapping argument with $\theta \in L^q(Q)$ where $q \geq 3$ and smoothness of the data φ_0 we conclude that $\varphi \in W_q^{2,1}(Q) \cap L^\infty(Q)$.

Applying again the L_p-theory of parabolic equations with $f_s \in C_b^{1,1}(\mathbb{R})$, $v \in L^4(Q)^2$, $\frac{\partial \varphi}{\partial t} \in L^p(Q)$, with $2 \leq p < 4$, recalling the given smoothness of θ_0 and the result of Lemma 3.1, we conclude that $\theta \in W_p^{2,1}(Q) \cap L^\infty(Q)$, with $2 \leq p < 4$. This completes the proof of Theorem 3.6.

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