Subtractive Renormalization of One-Pion-Exchange and Contact Interactions

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A recursive subtractive renormalization of the scattering equation is applied to the nucleon-nucleon $^1S_0$ channel with one-pion-exchange plus derivative contact interactions. This method can be easily extended to any derivative order of the singular interaction. Although we limit this work to the singlet partial wave, the method can be used as well in higher waves and coupled channels. The $^1S_0$ renormalization parameters are fitted to the data.

I. INTRODUCTION

The inspiring work of Weinberg [1] provided the basis for the effective field theory (EFT) of nuclear forces starting from the expansion of an effective chiral Lagrangian. In leading order, it gives the one-pion-exchange potential (OPEP) plus a Dirac-δ contact interaction. Many works where EFT methods were applied to the NN system have been developed by many groups with important results [2–11].

Recent works [12, 13] handle the OPEP plus derivative Dirac-δ interactions making use of dimensional and boundary conditions regularizations. The leading order interaction, renormalized with subtracted scattering equations, dominates the coupled $^3S_1 - ^3D_1$ channel while the singlet $^1S_0$ channel requires high order terms in the effective interaction [11].

The description of the $^1S_0$ singlet wave up to $p_{Lab} \sim$ 300 MeV/c demands an effective NN interaction with second order derivatives of the Dirac-δ, which in the relative momentum space reads

$$\langle \vec{p}' | V_{EFT} | \vec{p} \rangle = \langle \vec{p}' | V_\pi^s | \vec{p} \rangle \ + \ \sum_{i,j=0}^{1} \lambda_{ij} p'^{2i} p^{2j}, \qquad (1)$$

where the λ's are unregulated strengths and the matrix element of the one-pion-exchange potential is $\langle \vec{p}' | V_\pi^s | \vec{p} \rangle$. The effective bare potential of Eq. (1) generates integrals that diverge as much as p^5 in the scattering equation. Therefore, it is necessary at least three subtractions in the kernel of the Lippman-Schwinger (LS) equation, since each subtraction introduces a factor of p^{-2}. Differently from the recent works [12] and [13], we implement the alternative method of subtracted scattering equations [15] to handle the divergencies.

The one-subtraction scheme used in our previous work [11] was generalized in [15] to allow multiple subtractions, which makes possible to treat derivatives of the contact interaction in the effective two-body potential. The driving term of the n-subtracted LS equation is constructed recursively, so that the model is renormalized at each subtraction order, being the approach renormalization group invariant [15].

In this work we obtain the $^1S_0$ NN amplitude from the effective interaction Eq. (1), using a scattering equation with three subtractions. Within this framework, we perform a systematic analysis of the physical contribution coming from each order term in the recursive order-by-order renormalization method.

The presentation of this paper is as follows. In section II, we revise the recursive subtraction method to treat the scattering equation of ultraviolet singular potentials, and we describe in detail the renormalized scattering equation, also we briefly discuss the invariance of the scattering amplitude under the dislocation of the subtraction point. In section III, we present the results for the singlet nucleon-nucleon phase-shift and in section IV, we conclude.

II. SUBTRACTED T-MATRIX EQUATIONS

The bare effective potential of Eq. (1) is ultraviolet divergent, making the Lippman-Schwinger equation singular which requires regularization and renormalization to allow a sensible scattering amplitude. The use of the subtracted scattering equations which by construction are regularized and renormalized where the Green's functions appear subtracted at certain scales, which are convenient for introducing the physical inputs.

In the present case we consider a four-term singular interaction of Eq. (1), which after partial-wave decomposition to the singlet s-wave state, gives the bare potential in the form:

$$V_{EFT,s}(p',p) \ = \ \underbrace{V_{\pi,s}^{reg}(p',p) + \lambda_{00}}_{V_{\pi+\delta}} + \underbrace{\lambda_{01} p'^2 + \lambda_{10} p^2 + \lambda_{11} p'^2 p^2}_{V_{\delta'}}, \qquad (2)$$

and we introduce $V_{\pi+\delta}$ which corresponds to the regular part of OPEP plus a Dirac-delta interaction V_δ. For the $^1S_0$ state,the regular part is

$$V_{\pi,s}^{reg}(p',p) = -\frac{g_a^2}{32\pi f_\pi^2} \int_{-1}^{1} dx \frac{m_\pi^2}{p^2 + p'^2 - 2pp'x + m_\pi^2}, \qquad (3)$$

which has a finite scattering matrix, solution of the partial-

wave projected LS equation:

$$T_\pi(p',p;k^2) = V^{reg}_{\pi,s}(p',p) + \frac{2}{\pi}\int_0^\infty dq q^2 \frac{V^{reg}_{\pi,s}(p',q)}{k^2-q^2+i\varepsilon} T_\pi(q,p;k^2). \quad (4)$$

In our normalization conventions, the scattering amplitude in the angular momentum basis is $T(k,k,;k^2) = -\frac{1}{kcot\delta - i\,k}$, and reminding that the low-energy parameters are defined by the effective range expansion $kcot\delta = -\frac{1}{a} + \frac{1}{2} r_0 k^2 + \cdots$, with a the scattering length and r_0 the effective range.

A. Renormalized T-matrix for $V_{\pi+\delta}$

The T-matrix of the one-pion-exchange plus the Dirac-delta potential [11] is the input of the subtracted scattering equations for the complete singular potential of Eq. (2). Here, it is obtained using Distorted Wave Theory [16] (see Ref. ([12]):

$$T_{\pi+\delta}(E) = T_\pi(E) + [\Omega^-(E)]^\dagger T_\delta(E) \Omega^+(E) \,, \quad (5)$$

where $\Omega^+(E) = 1 + G_0^{(+)}(E) T_\pi(E)$. The singular part of the T-matrix is:

$$T_\delta(E) = V_\delta + V_\delta G_\pi^{(+)}(E) T_\delta(E) \,, \quad (6)$$

where

$$G_\pi^{(+)}(E) = G_0^{(+)}(E) + G_0^{(+)}(E) T_\pi(E) G_0^{(+)}(E) \,, \quad (7)$$

is the Green's function obtained from the regular part of the pion-exchange interaction.

Eq. (6) is ill-defined due to ultraviolet divergences in the momentum integration. In our method, this scattering equation is reformulated in a subtracted form, which allows to get a finite scattering amplitude with only one subtraction [11]. Therefore, we apply the method of subtracted equations to Eq. (6) subtracting the interacting Green's function, Eq. (7), at a certain energy scale $-\mu^2$ and the driving term $T_\delta(-\mu^2)$ of the subtracted equation has now a finite value. Then, we have:

$$T_\delta(E) = T_\delta(-\mu^2) + T_\delta(-\mu^2)\left[G_\pi^{(+)}(E) - G_\pi(-\mu^2)\right] T_\delta(E). \quad (8)$$

We observe that Eq. (8) has the same operator form as the original one-fold subtracted T-matrix equation [11] with the only difference being that the interacting Green's function appears in the place of the free one.

The renormalized strength of the interaction is the value of $T_\delta(E)$ at the subtraction point, $T_\delta(-\mu^2) = \lambda_{\mathcal{R}00}$, which allows to solve Eq. (8) resulting in a finite scattering amplitude for the OPEP plus a Dirac-delta:

$$T_{\pi+\delta}(p',p;-\mu^2) = T_\pi(p',p;-\mu^2) + \omega(p';-\mu^2)\,\lambda_{\mathcal{R}00}\,\omega(p;-\mu^2), \quad (9)$$

where

$$\omega(p;-\mu^2) = 1 + \frac{2}{\pi}\int_0^\infty dq\, q^2 \frac{T_\pi(p,q;-\mu^2)}{-\mu^2-q^2} \,, \quad (10)$$

with Eq. (4) giving the T-matrix for the regular part of the one-pion-exchange potential in the $^1S_0$ channel.

B. Recursively subtracted T-matrix equations

The singular potential of Eq. (2) requires a three-fold subtracted T-matrix equation to allow a finite scattering amplitude from its solution. The n-fold subtracted equation in operator form is written as [15]:

$$T(E) = V^{(n)}(-\mu^2;E) + V^{(n)}(-\mu^2;E) G_n^{(+)}(E;-\mu^2) T(E), \quad (11)$$

where

$$V^{(n)}(-\mu^2;E) \equiv \left[1 - (-\mu^2 - E)^{n-1} V^{(n-1)}(-\mu^2;E) G_0^3(-\mu^2)\right]^{-1} \times V^{(n-1)}(-\mu^2;E) \,, \quad (12)$$

with the n-fold subtracted Green's function defined by

$$G_n^{(+)}(E;-\mu^2) \equiv \left[(-\mu^2-E) G_0(-\mu^2)\right]^n G_0^{(+)}(E), \quad (13)$$

and $G_0^{(+)}(E) = [E - H_0 + i\varepsilon]^{-1}$ with $H_0 = p^2$ in the two-nucleon rest-frame.

The driving term $V^{(3)}(-\mu^2)$ is constructed recursively starting from $V^{(1)}(-\mu^2) = T_{\pi+\delta}(-\mu^2)$ given by Eq. (9). Then, the higher order singular terms of the potential Eq.(2) are introduced in the driving term of the three-fold subtracted equation as we are going to show in detail below.

The driving term of the three-fold subtracted scattering equation is obtained numerically by solving recursively the integral equations for the matrix elements $V^{(n)}_{\pi+\delta}(p',p\,;\,-\mu^2;k^2)$ which are explicitly given by:

$$V^{(n)}_{\pi+\delta}(p',p;-\mu^2;k^2) = V^{(n-1)}_{\pi+\delta}(p',p;-\mu^2;k^2) + \frac{2}{\pi}\int_0^\infty dq q^2 \left(\frac{\mu^2+k^2}{\mu^2+q^2}\right)^{n-1} \frac{V^{(n-1)}_{\pi+\delta}(p',q;-\mu^2;k^2)}{-\mu^2-q^2} \times \times V^{(n)}_{\pi+\delta}(q,p;-\mu^2;k^2), \quad (14)$$

with $k = \sqrt{E}$. Then, the higher singular terms of the full effective interaction of Eq. (2) are introduced directly in the matrix element of $V^{(3)}_{\pi+\delta+\delta'}(-\mu^2)$ in the form:

$$V^{(3)}_{\pi+\delta+\delta'}(p',p;-\mu^2;k^2) = V^{(3)}_{\pi+\delta}(p',p;-\mu^2;k^2) + \lambda_{\mathcal{R}10}(p'^2+p^2) + \lambda_{\mathcal{R}11} p'^2 p^2 \,. \quad (15)$$

The driving term of the three-fold subtracted scattering equation gets contribution from the derivatives of the Dirac-delta interaction, that have $\lambda_{\mathcal{R}ij}$ as the renormalized strengths of the corresponding singular part of the potential.

Finally, within our method we solve numerically the three-

fold subtracted LS equation given by:

$$T(p',p;k^2) = V^{(3)}_{\pi+\delta+\delta'}(p',p;-\mu^2;k^2) + \frac{2}{\pi}\int_0^\infty dq q^2 \left(\frac{\mu^2+k^2}{\mu^2+q^2}\right)^3 \frac{V^{(3)}_{\pi+\delta+\delta'}(p',q;-\mu^2;k^2)}{k^2-q^2+i\varepsilon} T(q,p;k^2)\,, \quad (16)$$

where in all steps to get and solve the above equation the ultraviolet divergences were removed in favor of the renormalized strengths of the singular interaction at the subtraction energy $-\mu^2$. Indeed, the arbitrary subtraction point can be moved without affecting the physics at the expense of changing the driving term of the subtracted LS-equation.

C. Renormalization group invariance

The physics of the nucleon-nucleon scattering should not depend on the arbitrary subtraction energy scale, $-\mu^2$. One could work with any convenient value of μ^2. However, the detailed form of the driving term in Eq. (15) which defines the scattering amplitude depends on the prescription used to define the renormalized theory. The renormalization group method guide us how to modify this prescription while keeping unchanged the predictions of the theory [17].

The invariance of the theory under the changes of the renormalization prescriptions defines a rule to modify consistently $V^{(n)}$ in Eq.(11) to keep unchanged the T-matrix. Using Eq. (11) and $\frac{\partial T(E)}{\partial \mu^2} = 0$, one can derive the non-relativistic Callan-Symanzik (NRCS) differential equation [15, 18]:

$$\frac{\partial V^{(n)}(-\mu^2;E)}{\partial \mu^2} = -V^{(n)}(-\mu^2;E)\frac{\partial G_n^{(+)}(E;-\mu^2)}{\partial \mu^2} V^{(n)}(-\mu^2;E)\,. \quad (17)$$

The solution of the above equation gives the driving term at any subtraction point with the boundary condition given by Eq. (15). The NRCS equation concretizes the invariance of the renormalized T-matrix under dislocation of the subtraction point. From that one immediately realizes that the dependence on the subtraction point appearing in the driven term of the subtracted scattering equation is highly nontrivial, although the physical results of the model are kept unchanged.

III. NUMERICAL RESULTS

The free parameters of our calculation are the renormalized strengths $\lambda_{\mathcal{R}00}$, $\lambda_{\mathcal{R}10}$, $\lambda_{\mathcal{R}11}$ and the subtraction point μ. To simplify the fitting procedure we fix $\lambda_{\mathcal{R}10}$, $\lambda_{\mathcal{R}11}$ and μ, and adjust $\lambda_{\mathcal{R}00}$ to reproduce the singlet scattering length $a_s = -23.739$ fm. The three parameters left are adjusted to reproduce the Nijgemen data [14] up to the center of mass momentum of k =300 MeV/c. From the fitting procedure we got $\mu\lambda_{\mathcal{R}00} = -0.1465$, $\mu^3\lambda_{\mathcal{R}01} = 4.7124$ and $\mu^5\lambda_{\mathcal{R}11} = 5.0265$, with $\mu = 214$ MeV. The resulting effective range is $r_{0,s} = 2.73$ fm compared with the value of 2.68 fm from Ref. [14].

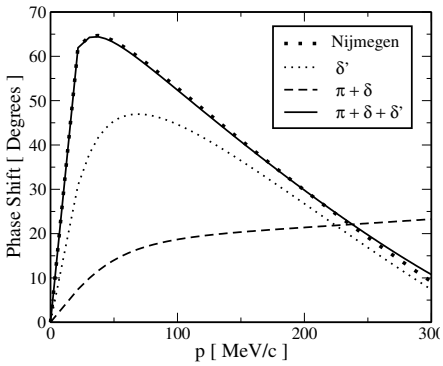

FIG. 1: $^1S_0$ phase shift as a function of the c.m. momentum calculated for $V_{\pi+\delta}$, $V_{\delta'}$ and $V_{\pi+\delta+\delta'}$. The dots are the Nijmegen data [14].

In the figure, we show our study for the different contributions of the effective potential as well as the full calculation. We performed calculations for the $^1S_0$ phase shifts obtained with $V_{\pi+\delta}$, $V_{\delta'}$ and $V_{\pi+\delta+\delta'}$, with the parameters fixed at the values we found by the fitting procedure. Below $p \sim 20$ MeV/c the calculation with only $V_{\pi+\delta}$ underestimates the data, while for $V_{\delta'}$ the phase shifts are better described. The dominant contribution in this channel comes from $V_{\delta'}$, while the pion appears at low energies providing the long range part of the NN interaction.

It is interesting to study how the observables depend on the scale μ. The dependence of the singlet $^1S_0$ phase shifts on the subtraction point μ was studied in Ref. [19].

IV. CONCLUDING REMARKS

We show how to apply the method of subtracted scattering equations [15] to calculate the NN singlet phase-shift, when the matrix elements of the effective interaction diverges in the ultraviolet region as p^2. In this case three subtractions are required to renormalize the model. Then, the integrand of the original scattering equation is automatically regularized by the subtractions of the propagator at some energy scale. The Born term in our calculation is the T-matrix at the subtraction energy scale differently from the usual potential term of the standard LS equation. For regular potentials the subtracted equations is fully equivalent to the LS formalism. If one desires the subtraction scale can be moved without modifying the calculated observables as long as new driven term of the subtracted scattering equation comes from the solution of the Non-Relativistic Callan-Symanzik (CS) equation. In our calculation, the boundary condition of the CS equation is determined by the renormalized coupling constants and by the T-matrix of the OPE and Dirac-δ potentials at the specified value of the subtraction point. And finally, we would like to add that the method of subtracted scattering equation is in principle suitable to treat also higher order singular terms of

the effective interaction.

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