An estimate of the isovariant Borsuk-Ulam constant for a group of type B_2

Ikumitsu NAGASAKI [1)]

Abstract. The isovariant Borsuk-Ulam constant c_G of a compact Lie group G is defined to be the supremum of constants $c \in \mathbb{R}$ with the following property: If there exists a G-isovariant map $f : S(V) \to S(W)$, then

$$c(\dim V - \dim V^G) \leq \dim W - \dim W^G$$

holds. Several estimates of c_G are known. In our previous study, we provided a new estimate of c_G for G of type A_2. In this paper, we treat a group of type B_2 and provide a new estimate of c_G by using representation theory.

1. Introduction

Let G be a compact Lie group, and $S(V)$ and $S(W)$ the unit spheres of (orthogonal) G-representations V and W respectively. A G-map $f : S(V) \to S(W)$ is called *isovariant* if it preserves the isotropy groups.

Definition. The isovariant Borsuk-Ulam constant c_G of a compact Lie group G is defined to be the supremum of constants $c \in \mathbb{R}$ with the following property: If there exists a G-isovariant map $f : S(V) \to S(W)$, then

$$c(\dim V - \dim V^G) \leq \dim W - \dim W^G$$

holds.

The determination of c_G is an interesting and important problem for the study of the isovariant Borsuk-Ulam type theorem; however, it is difficult at present and therefore we shall provide estimates of c_G. For any connected compact Lie group, we have already provided some mild estimates in [3]; for example, $\frac{3}{4} \leq c_G \leq 1$ for G of type A_2 and $\frac{4}{5} \leq c_G \leq 1$ for G of type B_2. In [4], we provided a better estimate $\frac{26}{27} \leq c_G \leq 1$ for type A_2. The goal of this paper is to provide a better estimate for type B_2; namely, we shall show

[1)] Department of Mathematics, Kyoto Prefectural University of Medicine, 1-5 Shimogamohangi-cho, Sakyo-ku, Kyoto 606-0823, Japan. e-mail: nagasaki@koto.kpu-m.ac.jp

2010 *Mathematics Subject Classification.* Primary 57S15; Secondary 57S25, 55M20, 22E99.

Theorem 1.1. *Let G be of type B_2. Then* $\frac{13}{14} \le c_G \le 1$.

2. From representation theory

We collect necessary facts from representation theory in order to obtain the new estimate. Since representation theory of connected compact Lie groups is equivalent to that of complex Lie groups or algebraic groups, we may consider

$$G = B_2 = \{A \in \mathrm{SL}_5(\mathbb{C}) \,|\, {}^tAJA = J\} \cong \mathrm{SO}_5(\mathbb{C})$$

as a group of type B_2, where

$$J = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that a maximal compact subgroup of G is isomorphic to SO(5) and a maximal torus T of G can be taken as the group consisting of all elements

$$t = \begin{pmatrix} t_1 & & & & 0 \\ & t_2 & & & \\ & & 1 & & \\ & & & t_2^{-1} & \\ 0 & & & & t_1^{-1} \end{pmatrix}$$

for t_1, $t_2 \in \mathbb{C}^*$, which is isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$.

Irreducible representations of G are parametrized by the dominant weights. In type B_2, the set of dominant weights is given by

$$\Lambda^+ = \{\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2 \,|\, \lambda_1 \ge \lambda_2 \ge 0\}.$$

We denote by $V(\lambda)$ the irreducible representation corresponding to $\lambda \in \Lambda^+$. Consider elements of G:

$$a = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

belonging to $N = N_G(T)$. The subgroup W generated by a, b, c is isomorphic to the dihedral group D_4 of order 8 and N is isomorphic to $T \rtimes W$. Therefore the Weyl group $N_G(T)/T$ of G is isomorphic to W. Thus we identify W with the Weyl group of G.

The conjugate action of W on T is given by

$$a\begin{pmatrix} t_1 & & & & 0 \\ & t_2 & & & \\ & & 1 & & \\ & & & t_2^{-1} & \\ 0 & & & & t_1^{-1} \end{pmatrix}a^{-1} = \begin{pmatrix} t_1^{-1} & & & & 0 \\ & t_2 & & & \\ & & 1 & & \\ & & & t_2^{-1} & \\ 0 & & & & t_1 \end{pmatrix},$$

$$b\begin{pmatrix} t_1 & & & & 0 \\ & t_2 & & & \\ & & 1 & & \\ & & & t_2^{-1} & \\ 0 & & & & t_1^{-1} \end{pmatrix}b^{-1} = \begin{pmatrix} t_1 & & & & 0 \\ & t_2^{-1} & & & \\ & & 1 & & \\ & & & t_2 & \\ 0 & & & & t_1^{-1} \end{pmatrix},$$

$$c\begin{pmatrix} t_1 & & & & 0 \\ & t_2 & & & \\ & & 1 & & \\ & & & t_2^{-1} & \\ 0 & & & & t_1^{-1} \end{pmatrix}c^{-1} = \begin{pmatrix} t_2 & & & & 0 \\ & t_1 & & & \\ & & 1 & & \\ & & & t_1^{-1} & \\ 0 & & & & t_2^{-1} \end{pmatrix}.$$

There are five conjugacy classes of elements of W given by

$$(1), \quad (a) = (b), \quad (c) = (abc), \quad (ab), \quad (ac) = (bc).$$

The subgroups and the conjugacy classes of subgroups of W are described in figures 1 and 2.

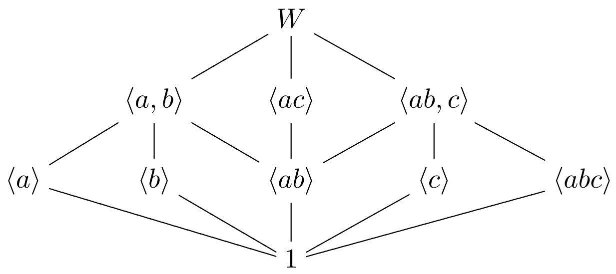

FIGURE 1. The subgroup lattice of W

There are five irreducible representations of W, say $\mathbb{C}$, $\mathbb{C}_{\mathrm{sgn}}$, $\mathbb{C}_{-}$, $\mathbb{C}_{\mathrm{sgn}-}$ and U, whose characters are described in table 1.

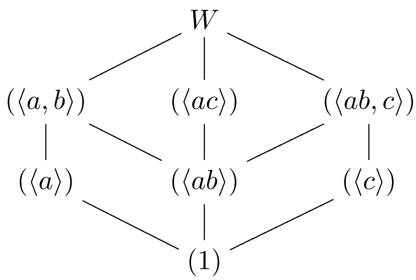

Figure 2. The lattice of conjugacy classes

	(1)	(a)	(ab)	(ac)	(c)
$\mathbb{C}$	1	1	1	1	1
$\mathbb{C}_{\mathrm{sgn}}$	1	-1	1	-1	1
$\mathbb{C}_{-}$	1	1	1	-1	-1
$\mathbb{C}_{\mathrm{sgn}-}$	1	-1	1	1	-1
U	2	0	-2	0	0

Table 1. The character table of W

From the character table, we see the following.

Proposition 2.1. *There are the following isomorphisms as W-representations.*

(I) (a) $\mathbb{C}[W/W] \cong \mathbb{C}$.
(b) $\mathbb{C}[W/\langle a\rangle] \cong \mathbb{C}[W/\langle b\rangle] \cong \mathbb{C} \oplus \mathbb{C}_{-} \oplus U$.
(c) $\mathbb{C}[W/\langle a, b\rangle] \cong \mathbb{C} \oplus \mathbb{C}_{-}$.
(d) $\mathbb{C}[W/\langle c\rangle] \cong \mathbb{C}[W/\langle abc\rangle] \cong \mathbb{C} \oplus \mathbb{C}_{\mathrm{sgn}} \oplus \mathbb{C}_{-} \oplus U$.
(e) $\mathbb{C}[W] \cong \mathbb{C} \oplus \mathbb{C}_{\mathrm{sgn}} \oplus \mathbb{C}_{-} \oplus \mathbb{C}_{\mathrm{sgn}-} \oplus 2U$.

(II) (a) $\mathbb{C}[W/W] \otimes \mathbb{C}_{\mathrm{sgn}} \cong \mathbb{C}_{\mathrm{sgn}}$.
(b) $\mathbb{C}[W/\langle a\rangle] \otimes \mathbb{C}_{\mathrm{sgn}} \cong \mathbb{C}_{\mathrm{sgn}} \oplus \mathbb{C}_{\mathrm{sgn}-} \oplus U$.
(c) $\mathbb{C}[W/\langle a, b\rangle] \otimes \mathbb{C}_{\mathrm{sgn}} \cong \mathbb{C}_{\mathrm{sgn}} \oplus \mathbb{C}_{\mathrm{sgn}-}$.
(d) $\mathbb{C}[W/\langle c\rangle] \otimes \mathbb{C}_{\mathrm{sgn}} \cong \mathbb{C}[W/\langle c\rangle]$.
(e) $\mathbb{C}[W] \otimes \mathbb{C}_{\mathrm{sgn}} \cong \mathbb{C}[W]$.

Proof. For any subgroup H, the character χ of a permutation representation $\mathbb{C}[W/H]$ is given by

$$\chi(g) = |(W/H)^{\langle g\rangle}|.$$

All isomorphisms are verified by comparing the characters of both sides. □

By Proposition 2.1, we have

Proposition 2.2. *Let M be one of W-representations in Proposition 2.1. Then the W-fixed space $M^W = 0$ for cases (II) (a), (b) and (c), and $M^W = \mathbb{C}$ otherwise.*

3. Computation by a method of Ariki-Matsuzawa-Terada

Let $G = B_2$, $N = N_G(T)$ and $W = N/T$ as before. We determine $\dim V^N = \dim(V^T)^W$ for irreducible representation V of G using a method of Ariki-Matsuzawa-Terada [1]. Let $V = V(\lambda)$ be the irreducible G-representation with the highest weight $\lambda = (\lambda_1, \lambda_2) \in \Lambda^+$. Let $S^k = S^k(\mathbb{C}^5)$ be the k-th symmetric power of the natural G-representation on $\mathbb{C}^5$. By a result of Weyl [6, p.228, Theorem 7.9A], $V(\lambda)$ is described in the representation ring $R(G)$ as follows.

Lemma 3.1.

$$\begin{aligned} V(\lambda) &= \begin{vmatrix} S^{\lambda_1} - S^{\lambda_1-2} & S^{\lambda_1+1} - S^{\lambda_1-3} \\ S^{\lambda_2-1} - S^{\lambda_2-3} & S^{\lambda_2} - S^{\lambda_2-4} \end{vmatrix} \\ &= S^{\lambda_1} \otimes S^{\lambda_2} + S^{\lambda_1-2} \otimes S^{\lambda_2-4} + S^{\lambda_1+1} \otimes S^{\lambda_2-3} + S^{\lambda_1-3} \otimes S^{\lambda_2-1} \\ &\quad - (S^{\lambda_1-2} \otimes S^{\lambda_2} + S^{\lambda_1} \otimes S^{\lambda_2-4} + S^{\lambda_1+1} \otimes S^{\lambda_2-1} + S^{\lambda_1-3} \otimes S^{\lambda_2-3}) \in R(G). \end{aligned}$$

In this notation, if $k < 0$, then S^k is understood to be 0.

A basis $\mathcal{B}$ of $S^{\mu_1} \otimes S^{\mu_2}$ is given by the elements

$$e_1^{v_{11}} e_2^{v_{12}} e_3^{\alpha_1} e_4^{u_{12}} e_5^{u_{11}} \otimes e_1^{v_{21}} e_2^{v_{22}} e_3^{\alpha_2} e_4^{u_{22}} e_5^{u_{21}}$$

with $v_{i1} + v_{i2} + \alpha_i + u_{i2} + u_{i1} = \mu_i$ $(i = 1, 2)$. For simplicity, we indicate these elements by

$$(v_1, v_2, \alpha, u_2, u_1) = \begin{pmatrix} v_{11} & v_{12} & \alpha_1 & u_{12} & u_{11} \\ v_{21} & v_{22} & \alpha_2 & u_{22} & u_{21} \end{pmatrix} \in M_{2,5}(\mathbb{Z}_{\geq 0}).$$

The action of T on $S^{\mu_1} \otimes S^{\mu_2}$ is given by

$$\begin{aligned} &t \cdot e_1^{v_{11}} e_2^{v_{12}} e_3^{\alpha_1} e_4^{u_{12}} e_5^{u_{11}} \otimes e_1^{v_{21}} e_2^{v_{22}} e_3^{\alpha_2} e_4^{u_{22}} e_5^{u_{21}} \\ &= t_1^{v_{11}-u_{11}+v_{21}-u_{21}} t_2^{v_{12}-u_{12}+v_{22}-u_{22}} e_1^{v_{11}} e_2^{v_{12}} e_3^{\alpha_1} e_4^{u_{12}} e_5^{u_{11}} \otimes e_1^{v_{21}} e_2^{v_{22}} e_3^{\alpha_2} e_4^{u_{22}} e_5^{u_{21}}. \end{aligned}$$

Thus $(S^{\mu_1} \otimes S^{\mu_2})^T$ has a basis $\mathcal{B}^T$ consisting elements of

$$e_1^{v_{11}} e_2^{v_{12}} e_3^{\alpha_1} e_4^{u_{12}} e_5^{u_{11}} \otimes e_1^{v_{21}} e_2^{v_{22}} e_3^{\alpha_2} e_4^{u_{22}} e_5^{u_{21}}$$

satisfying the condition

$$(*)\begin{cases}|v_i| := v_{1i} + v_{2i} = |u_i| := u_{1i} + u_{2i} \quad (i=1,2)\\ v_{i1} + v_{i2} + \alpha_i + u_{i2} + u_{i1} = \mu_i \quad (i=1,2).\end{cases}$$

$\mathcal{B}^T$ is identified with $S = S(\mu_1, \mu_2)$ which consists of all elements

$$\boldsymbol{v} = (v_1, v_2, \alpha, u_2, u_1) = \begin{pmatrix} v_{11} & v_{12} & \alpha_1 & u_{12} & u_{11} \\ v_{21} & v_{22} & \alpha_2 & u_{22} & u_{21}\end{pmatrix} \in M_{2,5}(\mathbb{Z}_{\geq 0})$$

satisfying the condition $(*)$.

Let a', b', c' be the following elements of the symmetric group S_Γ over the set $\Gamma = \{\pm 1, \pm 2\}$:

$$a' = \begin{pmatrix} 1 & 2 & -2 & -1 \\ -1 & 2 & -2 & 1\end{pmatrix}, \ b' = \begin{pmatrix} 1 & 2 & -2 & -1 \\ 1 & -2 & 2 & -1\end{pmatrix}, \ c' = \begin{pmatrix} 1 & 2 & -2 & -1 \\ 2 & 1 & -1 & -2\end{pmatrix}.$$

Let $W' = \langle a', b', c'\rangle$, which is isomorphic to W. The W'-action on S is given by:

$$\begin{aligned} a' \cdot (v_1, v_2, \alpha, u_2, u_1) &= (u_1, v_2, \alpha, u_2, v_1),\\ b' \cdot (v_1, v_2, \alpha, u_2, u_1) &= (v_1, u_2, \alpha, v_2, u_1),\\ c' \cdot (v_1, v_2, \alpha, u_2, u_1) &= (v_2, v_1, \alpha, u_1, u_2).\end{aligned}$$

Note that these actions are identified with the actions of a, b, c on $\mathcal{B}^T$ respectively. Hence we obtain a complete system $\overline{S} = \overline{S}(\mu_1, \mu_2)$ of representatives of S/W':

$$\overline{S} = \{v = ((v_1, v_2, \alpha, u_2, u_1) \in S \,|\, v_1 \geq u_1,\ v_2 \geq u_2,\ (v_1, u_1) \geq (v_2, u_2)\},$$

where

$$(v_1, u_1) \geq (v_2, u_2) \iff v_1 > v_2, \text{ or } v_1 = v_2 \text{ and } u_1 \geq u_2.$$

Also set $S' = S'(\mu_1, \mu_2) = \{v \in S \,|\, W'_v = 1, \langle c'\rangle, \langle a'b'c'\rangle\}$ and $\overline{S}' = \overline{S}'(\mu_1, \mu_2) = S'/W'$. By an easy argument, we have

Lemma 3.2. *The isotropy subgroup W'_v at $v \in \overline{S}$ is one of following:*

$$W', \quad \langle a'\rangle, \quad \langle b'\rangle, \quad \langle a', b'\rangle, \quad \langle c'\rangle, \quad \langle a'b'c'\rangle, \quad 1.$$

The result of [1, Theorem B.2] shows that

$$(S^{\mu_1} \otimes S^{\mu_2})^T \cong \begin{cases} \oplus_{v \in \overline{S}} \mathbb{C}[W'/W'_v] & (\mu_1 + \mu_2 \equiv 0\ (2))\\ \oplus_{v \in \overline{S}} \mathbb{C}[W'/W'_v] \otimes \mathbb{C}_{\mathrm{sgn}} & (\mu_1 + \mu_2 \equiv 1\ (2))\end{cases}$$

Thus, combining this and Propositions 2.1 and 2.2, we have

Proposition 3.3.

$$(S^{\mu_1} \otimes S^{\mu_2})^N = ((S^{\mu_1} \otimes S^{\mu_2})^T)^W \cong \begin{cases} \oplus_{v \in \overline{S}} \mathbb{C} & (\mu_1 + \mu_2 \equiv 0 \ (2)) \\ \oplus_{v \in \overline{S'}} \mathbb{C} & (\mu_1 + \mu_2 \equiv 1 \ (2)), \end{cases}$$

and hence

$$\dim (S^{\mu_1} \otimes S^{\mu_2})^N = \begin{cases} |\overline{S}| & (\mu_1 + \mu_2 \equiv 0 \ (2)) \\ |\overline{S'}| & (\mu_1 + \mu_2 \equiv 1 \ (2)). \end{cases}$$

We give a few examples.

Example 3.4. If $\lambda = (1, 0)$, then $V(\lambda) = S^1 \otimes S^0$. In this case, $S' = \emptyset$; hence $\dim V(\lambda)^N = 0$.

Example 3.5. If $\lambda = (2, 0)$, then $V = V(\lambda) = S^2 \otimes S^0 - S^0 \otimes S^0$. In this case, $\overline{S}(2, 0)$ consists of elements $\begin{pmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, and $\overline{S}(0, 0)$ consists of an element $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. Hence we have $\dim V^N = 2 - 1 = 1$.

Example 3.6. If $\lambda = (1, 1)$, then $V = V(\lambda) = S^1 \otimes S^1 - S^2 \otimes S^0$. In this case, $\overline{S}(1, 1)$ consists of elements $\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$, and $\overline{S}(2, 0)$ consists of two elements as in the previous example. Hence $\dim V^N = 2 - 2 = 0$.

Let $S_0(\mu_1, \mu_2)$ for $\mu_1 \geq 0$ and $\mu_2 \geq 0$ be the set of elements of $(v_1, v_2, u_2, u_1) \in M_{2,4}(\mathbb{Z}_{\geq 0})$ satisfying the condition $(*)$ with $\alpha = 0$. For $0 \leq d_i \leq \mu_i$, there is an injection $S_0(d_1, d_2) \to S(\mu_1, \mu_2)$ defined by

$$\begin{pmatrix} v_{11} & v_{12} & u_{12} & u_{11} \\ v_{21} & v_{22} & u_{22} & u_{21} \end{pmatrix} \mapsto \begin{pmatrix} v_{11} & v_{12} & \mu_1 - d_1 & u_{12} & u_{11} \\ v_{21} & v_{22} & \mu_2 - d_2 & u_{22} & u_{21} \end{pmatrix}.$$

Then it is easily seen that

Lemma 3.7.

$$S(\mu_1, \mu_2) \cong \coprod_{\substack{0 \leq d_1 \leq \mu_1 \\ 0 \leq d_2 \leq \mu_2}} S_0(d_1, d_2),$$

and

$$S'(\mu_1, \mu_2) \cong \coprod_{\substack{0 \leq d_1 \leq \mu_1 \\ 0 \leq d_2 \leq \mu_2}} S'_0(d_1, d_2)$$

as W'-sets, where $S'_0(d_1, d_2)$ be the subset of $S_0(d_1, d_2)$ consisting of the elements satisfying:

(1) $v_1 \neq u_1$, $v_2 \neq u_2$, *and* $v_1 \neq v_2$ *or* $u_1 \neq u_2$, *or*
(2) $v_1 \neq v_2$, $u_2 \neq u_1$, $v_1 \neq u_1$, *or*
(3) $v_1 \neq u_2$, $v_2 \neq u_1$, $v_1 \neq v_2$, *where*

$(v_1, v_2, u_2, u_1) \in M_{2,4}(\mathbb{Z}_{\geq 0})$ *satisfying the condition* $(*)$ *with* $\alpha = 0$.

Definition. We set

$$n_0(d_1, d_2) = |S_0(d_1, d_2)/W'|$$

$$n_0'(d_1, d_2) = |S_0'(d_1, d_2)/W'|$$

From condition $(*)$ with $\alpha = 0$, it is easy to verify that

Lemma 3.8. $v \in S_0(d_1, d_2) \iff$

$$\begin{cases} v_{11} = -d_2 + u_{11} + v_{21} + u_{22} + 2u_{21} \\ v_{12} = -\frac{1}{2}(d_1 + d_2) - u_{11} - v_{22} - u_{21} \\ u_{12} = \frac{1}{2}(d_1 + d_2) - u_{11} - u_{22} - u_{21} \\ v_{21} = d_2 - v_{22} - u_{22} - u_{21} \\ 0 \leq u_{ij} \leq d_i, \ 0 \leq v_{ij} \leq d_i. \end{cases}$$

In particular, we see

Corollary 3.9. *If* $d_1 + d_2$ *is odd, then* $n_0(d_1, d_2) = 0$ *and* $n_0'(d_1, d_2) = 0$.

As a finial result, we obtain the following formula.

Theorem 3.10. *With the notation above,*

(1) *If* $\lambda_1 + \lambda_2$ *is even, then* $\dim V(\lambda)^N =$

$$n_0(\lambda_1, \lambda_2) + n_0(\lambda_1 - 1, \lambda_2 - 3) - n_0(\lambda_1 + 1, \lambda_2 - 1) - n_0(\lambda_1 - 2, \lambda_2 - 2).$$

(2) *If* $\lambda_1 + \lambda_2$ *is odd, then* $\dim V(\lambda)^N =$

$$n_0'(\lambda_1 - 1, \lambda_2) + n_0'(\lambda_1, \lambda_2 - 3) - n_0'(\lambda_1 + 1, \lambda_2 - 2) - n_0'(\lambda_1 - 2, \lambda_2 - 1).$$

Proof. Suppose that $\lambda_1 + \lambda_2$ is even. By Proposition 3.3, Lemma 3.7 and Corollary 3.9, we have

$$\dim(S^{\mu_1} \otimes S^{\mu_2})^N = \sum_{\substack{0 \leq d_1 \leq \mu_1 \\ 0 \leq d_2 \leq \mu_2 \\ d_1 + d_2 \equiv 0 \ (2)}} n_0(d_1, d_2).$$

By Lemma 3.1, we obtain statement (1). Statement (2) is similar. □

4. **Proof of Theorem 1.1**

From representation theory of $G = B_2$, recall the dimension formula

$$\dim V(\lambda) = \frac{2}{3}(\lambda_1 - \lambda_2 + 1)(\lambda_1 + \lambda_2 + 2)(\lambda_1 + 3/2)(\lambda_2 + 1/2),$$

see for example [2]. Suppose that H is a closed subgroup of G with $c_H = 1$. Set

$$r_G(H) := \sup\{\dim V(\lambda)^H / \dim V(\lambda) \mid \lambda \in \Lambda^+ \smallsetminus \{0\}\}.$$

It follows from a similar argument as in [3, 4] that

$$1 - r_G(H) \leq c_G \leq 1.$$

Since N is solvable, it follows from [5] that $c_N = 1$. Therefore, it is suffices to show the following.

Proposition 4.1. *$r_G(N) = \frac{1}{14}$ for $N = N_G(T)$.*

To show this, we first note the following.

Lemma 4.2. *If $\lambda_1 \geq 11$, then $\frac{\dim V(\lambda)^N}{\dim V(\lambda)} < \frac{1}{14}$.*

Proof. By [3, Lemma 5.5], we have

$$\frac{\dim V(\lambda)^N}{\dim V(\lambda)} \leq \frac{\dim V(\lambda)^T}{\dim V(\lambda)} \leq \frac{\lambda_1 + \lambda_2/2}{K_\lambda + \lambda_1 + \lambda_2/2},$$

where $K_\lambda = \lambda_1^2 + \lambda_2^2 + 3\lambda_1 + \lambda_2$. Then the inequality

$$\frac{\lambda_1 + \lambda_2/2}{K_\lambda + \lambda_1 + \lambda_2/2} < \frac{1}{14}$$

is equivalent to

$$(\lambda_1 - 5)^2 + (\lambda_2 - \frac{11}{4})^2 > \frac{521}{16}.$$

It is easy to verify that this inequality holds when $\lambda_1 \geq 11$. □

Proof of Proposition 4.1. By Lemma 4.2, we may assume that $0 \leq \lambda_2 \leq \lambda_1 \leq 10$ ($(\lambda_1, \lambda_2) \neq (0, 0)$). We compute $\dim V(\lambda)^N / \dim V(\lambda)$ by Theorem 3.10 and the dimension formula. By computation by Mathematica, we obtain the values of

$$\dim V(\lambda)^N / \dim V(\lambda)$$

in Table 2 below, and thus we conclude that $r_G(N) = \dfrac{1}{14}$. □

		λ_1										
		0	1	2	3	4	5	6	7	8	9	10
λ_2	0	–	0	$\frac{1}{14}$	0	$\frac{2}{55}$	0	$\frac{1}{70}$	0	$\frac{1}{95}$	0	$\frac{3}{506}$
	1		0	0	0	0	$\frac{1}{260}$	0	$\frac{1}{595}$	0	$\frac{1}{567}$	0
	2			$\frac{1}{35}$	$\frac{1}{105}$	$\frac{1}{110}$	$\frac{1}{390}$	$\frac{3}{625}$	$\frac{2}{935}$	$\frac{2}{665}$	$\frac{1}{910}$	$\frac{1}{483}$
	3				0	0	$\frac{1}{455}$	$\frac{1}{770}$	$\frac{1}{595}$	$\frac{1}{1729}$	$\frac{3}{2401}$	$\frac{1}{1610}$
	4					$\frac{2}{165}$	$\frac{1}{429}$	$\frac{1}{270}$	$\frac{1}{663}$	$\frac{1}{399}$	$\frac{1}{945}$	$\frac{1}{644}$
	5						0	$\frac{1}{715}$	$\frac{1}{1309}$	$\frac{1}{1045}$	$\frac{3}{3080}$	$\frac{3}{4301}$
	6							$\frac{2}{455}$	$\frac{2}{1105}$	$\frac{1}{494}$	$\frac{3}{3094}$	$\frac{2}{1495}$
	7								0	$\frac{1}{1615}$	$\frac{2}{2835}$	$\frac{3}{4370}$
	8									$\frac{1}{323}$	$\frac{2}{2261}$	$\frac{1}{782}$
	9										0	$\frac{2}{3059}$
	10											$\frac{3}{1771}$

TABLE 2

References

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