UNIVERSITY OF NOVI SAD
FACULTY OF SCIENCES
DEPARTMENT OF MATHEMATICS AND INFORMATICS

Sanja Kostadinova

Some classes of integral transforms on distribution spaces and generalized asymptotics

-doctoral dissertation-

Сања Костадинова

Неке класе интегралних трансформација на простору дистрибуција и уопштена асимптотика

-докторска дисертација-

Novi Sad, 2014

Contents

Preface

The dissertation is organized in five chapters. Our original contributions can be found in Chapters 3, 4, and 5.

In Chapter 1, we collect some notions and notation to be used in the thesis. Also, we state some known definitions and theorems from topology and analysis. Here, we give description of some distribution spaces such as the space of tempered distributions, distributions of M-exponential type and Lyzorkin distributions. In Chapter 2, we define the quasiasymptotic behavior and S-asymptotic behavior of distributions and try to provide a more deeper background about this notions.

In Chapter 3, we introduce the short-time Fourier transform (STFT) and we study it in the context of the space $\mathcal{K}_1'(\mathbb{R}^n)$ of distributions of exponential type, the dual of the space of exponentially rapidly decreasing smooth functions $\mathcal{K}_1(\mathbb{R}^n)$. We obtain various characterizations of $\mathcal{K}_1'(\mathbb{R}^n)$ and related spaces via the short-time Fourier transform.
First, Section 3.2 deals with Abelian and Tauberian theorems for quasiasymptotics in terms of the STFT. Then, in Section 3.3 we shall present continuity theorems for the STFT and its adjoint on the test function space $\mathcal{K}_1(\mathbb{R}^n)$ and the topological tensor product $\mathcal{K}_1(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}(\mathbb{C}^n)$, where $\mathcal{U}(\mathbb{C}^n)$ is the space of entire rapidly decreasing functions in any horizontal band of $\mathbb{C}^n$. We then use such continuity results to develop a framework for the STFT on $\mathcal{K}_1'(\mathbb{R}^n)$. We also introduce the space $\mathcal{B}'_\omega(\mathbb{R}^n)$ of ω-bounded distributions and its subspace $\dot{\mathcal{B}}'_\omega(\mathbb{R}^n)$ with respect to an exponentially moderate weight ω; when $\omega = 1$, these spaces coincide with the well-known Schwartz spaces [89, p. 200] of bounded distributions $\mathcal{B}'(\mathbb{R}^n)$ and $\dot{\mathcal{B}}'(\mathbb{R}^n)$, which are of great importance in the study of convolution and growth properties of distributions. Notice that the distribution space $\mathcal{B}'(\mathbb{R}^n)$ also plays an important role in Tauberian theory; see, for instance, Beurling's theorem [12, p. 230] and the distributional Wiener Tauberian theorem from [67]. The spaces $\mathcal{B}'_\omega(\mathbb{R}^n)$ and $\dot{\mathcal{B}}'_\omega(\mathbb{R}^n)$ will be characterized in Section 3.4 in terms of the short-time Fourier transform and also in terms properties of the set of translates of their elements. Section 3.5 is devoted to the characterization of $\mathcal{K}_1'(\mathbb{R}^n)$ and related spaces via modulation spaces. The conclusive Section 3.6 deals with Tauberian theorems. Our Tauberian hypotheses are actually in terms of membership to suitable modulation spaces, this allows us to reinterpret the S-asymptotics in the weak* topology of modulation spaces.

Chapter 4 is dedicated to the ridgelet and the Radon transform. We provide a thorough analysis of the ridgelet transform and its transpose, called here the ridgelet synthesis operator, on various test function spaces. The crucial continuity results for test function spaces are given in Section 4.3. In Section 4.4 we show that the ridgelet transform and the ridgelet synthesis operator can be extended as continuous mappings $\mathcal{R}_\psi : \mathcal{S}'_0(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{Y}^{n+1})$ and $\mathcal{R}^t_\psi : \mathcal{S}'(\mathbb{Y}^{n+1}) \to \mathcal{S}'_0(\mathbb{R}^n)$. We then use our results to develop a distributional framework for the ridgelet transform that is, we treat the ridgelet transform on $\mathcal{S}'_0(\mathbb{R}^n)$ via a duality approach. The ridgelet transform is intimately connected with the Radon and wavelet transforms. Helgason [33] proved range theorems for the Radon and dual Radon transform on the Lizorkin test function spaces $\mathcal{S}_0$. In Section 4.5 we apply our continuity theorems for the ridgelet transform to discuss the continuity of the Radon transform on these spaces and their duals. The Radon transform on Lizorkin spaces naturally extends the one considered by Hertle [34] on various distribution spaces. We use in Section 4.6 ideas from the theory of tensor products of topological vector spaces to study the relation between the distributional ridgelet, Radon, and wavelet transforms. We proof the desingularization formula, which essentially shows that the ridgelet transform of a Lizorkin distribution is smooth in the position and scale variables. In Section 4.7, we present a ridgelet transform characterization of the bounded subsets of $\mathcal{S}'_0(\mathbb{R}^n)$; we also show in this section that the Radon transform on $\mathcal{S}'_0(\mathbb{R}^n)$ is a topological isomorphism into its range. It is interesting to notice that the Radon transform may fail to have the latter property even on spaces of test functions; for instance, Hertle has shown [34] that the Radon transform on $\mathcal{D}(\mathbb{R}^n)$ is not an isomorphism of topological vector spaces into its range. We conclude Chapter 4 with some asymptotic results relating the quasiasymptotic behavior of distributions with the quasiasymptotics of its Radon and ridgelet transform.

The last chapter is devoted to the multiresolution analysis (MRA) of M-exponential distributions. We study in Section 5.2 the convergence of multiresolution expansions in various test function and distribution spaces. Section 5.3 treats the pointwise convergence of multiresolution expansions to the distributional point values of a distribution. Finally, Section 5.4 gives the asymptotic behavior of the sequence $\{q_j f(\mathbf{x}_0)\}_{j\in\mathbb{N}}$ as $j \to \infty$ when f has quasiasymptotic behavior at $\mathbf{x}_0$; we also provide there a characterization of the quasiasymptotic behavior in terms of multiresolution expansions and give an MRA sufficient condition for the existence of α-density points of positive measures.

Novi Sad, 2014 Sanja Kostadinova

Chapter 0

Introduction

The term generalized asymptotics refers to asymptotic analysis on spaces of generalized functions. Perhaps, the most developed approaches to generalized asymptotics are those of Vladimirov, Drozhzhinov and Zavialov [113], and of Estrada and Kanwal [17]. The work of Pilipović and his coworkers have great contribution in this field too, [69, 65, 67, 62, 105, 106, 110, 86, 87, 88]. A survey of definitions and results related to generalized asymptotics up to 1989 can be found in [65]. A more resent and complete account on the subject can be found in the book [73].
In general case, distributions do not have value at a point. One way to define the value at a point(if possible) is in sense of Łojasiewicz [50]. Natural generalization of this notion is the quasiasymptotic behavior of distributions. The introduction of the quasiasymptotic behavior of distributions was one of major steps toward the understanding of asymptotic properties of distributions. The concept is due to Zavialov [113]. The motivation for its introduction came from theoretical questions in quantum field theory. Roughly speaking, the idea is to study the asymptotic behavior at large or small scale of the dilates of a distribution.
The study of structural theorems in quasiasymptotic analysis has always had a privileged place in the theory [50, 66, 65, 113]. In general, the word structural theorem refers in distribution theory to the description of convergence properties of distributions in terms of ordinary convergence or uniform convergence of continuous functions. Vladimirov and collaborators gave the first general structural theorems in [113], and many authors dedicated efforts to extend the structural characterization and remove the support type restrictions [65]. In the work of Vindas [107, 104, 105, 106, 111, 18] there is a complete structural characterization for quasiasymptotics of Schwartz distributions (in one dimension).
The name Abelian (or direct) theorem usually refers to those results which obtain asymptotic information after performing an integral transformation to a (generalized) function. On the other hand, a Tauberian (or inverse) theorem is the converse to an Abelian result, subject to an additional assumption, the so called Tauberian hypothesis. In general, Tauberian theorems are much deeper and more difficult to show than Abelian ones. Tauberian theory is interesting by itself, but the study of Tauberian type results had been historically stimulated by their potential applications in diverse fields of mathematics. More historical details about Abelian and Tauberian theorems can be found in [47]. Tauberian theorems are an essential

tool of the theory of probability and statistics, number theory, the theory of generalized functions and many others. In the work of [82, 83, 84, 70, 71, 69, 65, 113] they are applied in the study of the asymptotics of integral transforms such as the Laplace, Stieltjes and wavelet transform on distributions. In this dissertation we use Abelian and Tauberian ideas for asymptotic analysis of the short-time Fourier transform, Radon, and ridgelet transforms, and multiresolution approximations. Remarkably, many of our Tauberian theorems turn out to be full characterizations of the asymptotic properties of a distribution.

Time-frequency has its origin in the early development in quantum mechanics by H. Weyl, E. Wigner and J. von Neuman around 1930. D. Gabor in 1946 set the foundation of information theory and signal analysis. At the end of 20th century time-frequency analysis had been establish as a independent mathematical field by the work of Guido Janssen. Because the growth of time-frequency analysis is connected with the rise of wavelet theory, both theories grew in parallel. Their mutual interaction is beautifully summarized in Ingrid Daubechies's textbook[10]. The Fourier transform is probably the most widely applied signal processing tool in science and engineering. It reveals the frequency composition of a time series by transforming it from the time domain into the frequency domain. However, it does not reveal how the signals frequency contents vary with time. Because the temporal structure of the signal is not revealed, the merit of the Fourier transform is limited; specifically, it is not suited for analyzing nonstationary signals. On the other hand, as signals encountered in manufacturing are generally nonstationary in nature (e.g., subtle, time-localized changes caused by structural defects are typically seen in vibration signals measured from rotary machines), a new signal processing technique that is able to handle the nonstationarity of a signal is needed. A straightforward solution to overcoming the limitations of the Fourier transform is to introduce an analysis window of certain length that glides through the signal along the time axis to perform a time-localized Fourier transform. Such a concept led to the short-time Fourier transform (STFT), introduced by Dennis Gabor. The most cited textbook where one can find a full treatment on STFT is [26].

The short-time Fourier transform (STFT) is a very effective device in the study of function spaces. The investigation of major test function spaces and their duals through time-frequency representations has attracted much attention. For example, the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ and the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ were studied in [29] (cf. [26]). Characterizations of Gelfand-Shilov spaces and ultradistribution spaces by means of the short-time Fourier transform and modulation spaces are also known [30, 59, 102]. We study in this dissertation the short-time Fourier transform in the context of the space $\mathcal{K}_1'(\mathbb{R}^n)$ of distributions of exponential type, the dual of the space of exponentially rapidly decreasing smooth functions $\mathcal{K}_1(\mathbb{R}^n)$. We will obtain various characterizations of $\mathcal{K}_1'(\mathbb{R}^n)$ and related spaces via the short-time Fourier transform. The space $\mathcal{K}_1'(\mathbb{R}^n)$ was introduced by Silva [90] and Hasumi [31] in connection with the so-called space of Silva tempered ultradistributions $\mathcal{U}'(\mathbb{C}^n)$. Let us mention that $\mathcal{K}_1'(\mathbb{R}^n)$ and $\mathcal{U}'(\mathbb{R}^n)$ were also studied by Morimoto through the theory of ultra-hyperfunctions [58] (cf. [60]). We refer to [19, 38, 91, 121] for some applications of the Silva spaces. Also, we present Abelian and Tauberian theorems for the short-time Fourier transform of tempered distributions and we prove new Tauberian theorems where the exponential asymptotics of functions and distributions can be obtained from those of the short-time Fourier transform.

Another subject of research in this dissertation is the ridgelet and the Radon transforms. The ridge function terminology was introduced in the 1970s by Logan and Shepp. In recent years, ridge functions (and ridgelets) have appeared often in the literature of approximation theory, statistics, and signal analysis. In [4, 5] Candès introduced and studied the continuous ridgelet transform. He developed a harmonic analysis groundwork for this transform and showed that it is possible to obtain constructive and stable approximations of functions by ridgelets. One of the motivations for the introduction of the "X-let" transforms, such as the ridgelet or curvelet transforms, comes from the search of optimal representations of signals in high-dimensions. Wavelets are very good in detecting point singularities in the sense that wavelet coefficients near the discontinuity are significantly higher than those at the smooth region, but they have several difficulties in localizing edges of higher dimension [6]. We can construct two-dimensional wavelets by simply taking the tensor product and compute wavelet coefficients. However, these edges, while separating smooth regions, are themselves smooth curves. As a result, a direct applications of 2D wavelets will not be able to localize coefficients near the edges as a 1D wavelet transform does. The ridgelet transform is more sensitive to higher dimensional discontinuities, as it essentially projects a hyperplane singularity into a point singularity (this is done with the Radon transform) and then takes a one-dimensional wavelet transform.
The ridgelet transform of distributions must be more carefully handled than the wavelet transform. While the wavelet transform of a distribution can be defined by direct evaluation of the distribution at the wavelets, this procedure fails for the ridgelet transform because the ridgelets do not belong to the Schwartz class $\mathcal{S}(\mathbb{R}^n)$. The larger distribution space where the direct approach works is $\mathcal{D}'_{L^1}(\mathbb{R}^n)$. Actually, in earlier works by other authors [79], the continuous ridgelet transform was not properly extended to distributional spaces. In this dissertation (cf. Chapter 3) we provide a thorough analysis of the ridgelet transform and its transpose, called here the ridgelet synthesis operator, on various test function spaces. Our main results are continuity theorems on such function spaces (cf. Section 4.3). We then use our results to develop a distributional framework for the ridgelet transform. It should be noticed that Roopkumar has proposed a different definition for the ridgelet transform of distributions [78, 79]; however, his work contains several major errors (see Remark 4.3.1 in Chapter 4). This motivated us in this doctoral work to develop a correct theoretical framework for treating the ridgelet transform of distributions.
The ridgelet transform is intimately connected with the Radon and wavelet transforms. The Radon transform was first introduce by Johann Radon (l887-1956) in a paper from 1917. Today, the Radon transform is widely known by working scientists in medicine, engineering, physical science, optics and holographic interferometry, geophysics, radio astronomy and mathematics. Helgason [33] proved

range theorems for the Radon and dual Radon transform on the Lizorkin test function spaces $\mathcal{S}_0$. We apply our continuity theorems for the ridgelet transform to discuss the continuity of the Radon transform on these spaces and their duals. The Radon transform on Lizorkin spaces naturally extends the one considered by Hertle [34] on various distribution spaces. We use in Section 4.6 ideas from the theory of tensor products of topological vector spaces to study the relation between the distributional ridgelet, Radon, and wavelet transforms. Moreover, we give a desingularization formula, which essentially shows that the ridgelet transform of a Lizorkin distribution is smooth in the position and scale variables. Finally, we present a ridgelet transform characterization of the bounded subsets of $\mathcal{S}_0'(\mathbb{R}^n)$ and we prove some Abelian and Tauberian theorems for the ridgelet transform.
We point out that the wavelet transform has shown usefulness to study pointwise scaling properties of distributions [37, 55, 71, 87, 95, 110]. One can then expect that the ridgelet transform of distributions might provide a tool for studying higher dimensional scaling notions, such as those introduced by Łojasiewicz in [49].

From a historical point of view, the first reference to wavelets goes back to the early twentieth century by Alfred Haar. His research on orthogonal systems of functions led to the development of a set of rectangular basis functions. Later, an entire wavelet family, the Haar wavelet, was named on the basis of this set of functions, and it is also the simplest wavelet family developed till this date.
Several mathematicians, such as John Littlewood, Richard Paley, Elias M. Stein, and Norman H. Ricker have great contribution to what is today known as wavelet analysis. A major advancement in the field was attributed to Jean Morlet who developed and implemented the technique of scaling and shifting of the analysis window functions in analyzing acoustic echoes while working for an oil company in the mid 1970s. When Morlet first used the STFT to analyze these echoes, he found that keeping the width of the window function fixed did not work. As a solution to the problem, he experimented with keeping the frequency of the window function constant while changing the width of the window by stretching or squeezing the window function. The resulting waveforms of varying widths were called by Morlet the "wavelet", and this marked the beginning of the era of wavelet research.
The notion of multiresolution analysis (MRA) was introduced by Mallat and Meyer as a natural approach to the construction of orthogonal wavelets [53, 56]. Approximation properties of multiresolution expansions in function and distribution spaces have been extensively investigated, see e.g. [56]. The problem of pointwise convergence of multiresolution expansions is very important from a computational point of view and has also been studied by many authors. In [39] (see also [40]), Kelly, Kon, and Raphael showed that the multiresolution expansion of a function $f \in L^p(\mathbb{R}^n)$ $(1 \leq p \leq \infty)$ converges almost everywhere; in fact, at every Lebesgue point of f. Related pointwise convergence questions have been investigated by Tao [100] and Zayed [120].

Walter was the first to study the pointwise convergence of multiresolution expansions for tempered distributions. Under mild conditions, he proved [114] (cf. [117]) in dimension 1 that the multiresolution expansion of a tempered distribu-

tion is convergent at every point where $f \in \mathcal{S}'(\mathbb{R})$ possesses a distributional point value. As noted, the notion of distributional point value for generalized functions was introduced by Łojasiewicz [50, 49]. Not only is this concept applicable to distributions that might not even be locally integrable, but also includes the Lebesgue points of locally integrable functions as particular instances. Interestingly, the distributional point values of tempered distributions can be characterized by the pointwise Fourier inversion formula in a very precise fashion [104, 109], but in contrast to multiresolution expansions, one should employ summability methods in the case of Fourier transforms and Fourier series. The problem of pointwise summability of distribution expansions with respect to various orthogonal systems had been considered by Walter in [115].
The result of Walter on pointwise convergence of multiresolution expansions was generalized by Sohn and Pahk [97] to distributions of superexponential growth, that is, elements of $\mathcal{K}'_M(\mathbb{R})$. The important case $M(\mathbf{x}) = |\mathbf{x}|^p$, $p > 1$, of $\mathcal{K}'_M(\mathbb{R}^n)$ was introduced by Sznajder and Zieleźny in connection with solvability questions for convolution equations [99]. We extend in this dissertation the results from [114, 97] to the multidimensional case. Actually, our results improve those from [114, 97], even in the one-dimensional case, because our hypotheses on the order of distributional point values are much weaker. We provide in Chapter 5 of this work pointwise convergence results for multiresolution expansions of tempered distributions and tempered measures as well as distributions and measures of superexponential growth.

In [70], Pilipović, Takači, and Teofanov studied the quasiasymptotic properties of a tempered distribution f in terms of its multiresolution expansion $\{q_j f\}$ with respect to an r-regular MRA. A similar study was carried out by Sohn [95] for distributions of exponential type. In these works it was claimed that $q_j f$ has the same quasiasymptotic properties as f. Unfortunately, such results turn out to be false. Here, we revisit the problem and provide an appropriate characterization of the quasiasymptotic behavior in terms of multiresolution expansions. As an application, we give an MRA criterion for the determination of (symmetric) α-density points of measures. Finally, for other studies about the rich interplay between wavelet analysis and quasiasymptotics, we refer to [70, 72, 86, 87, 95, 110].

Notions

Ω	open subset of $\mathbb{R}^n$
$C^m(\Omega)$, $m \in \mathbb{N}_0$	space of complex valued functions over Ω with continuous derivatives up to order m
$C^\infty(\Omega)$	space of infinite differentiate functions over Ω (smooth functions over Ω)
$C_0^m(\Omega)$, $m \in \mathbb{N}_0$	subspace of $C^m(\Omega)$ whose elements have compact support in Ω
$C_0^\infty(\Omega)$	subspace of $C^\infty(\Omega)$ whose elements have compact support of Ω
$C_0^m(K)$, $m \in \mathbb{N}_0$	subspace of $C^m(\Omega)$ whose elements have compact support contained in some compact set $K \subset \Omega$
$C_0^\infty(K)$	subspace of $C^\infty(\Omega)$ whose elements have compact support contained in some compact set $K \subset \Omega$
$L^1(\mathbb{R}^n)$	space of absolute integrable functions over $\mathbb{R}^n$
$L^1_{loc}(\mathbb{R}^n)$	space of locally integrable functions over $\mathbb{R}^n$
$L^2(\mathbb{R}^n)$	space of square integrable functions over $\mathbb{R}^n$
$\mathcal{D}(\Omega)$	the locally convex space $C_0^\infty(\Omega)$ (space of test functions)
$\mathcal{D}'(\Omega)$	space of continuous linear functionals over $\mathcal{D}(\Omega)$ (space of distributions)
$\mathcal{E}(\Omega)$	$= C^\infty(\Omega)$
$\mathcal{E}'(\Omega)$	space of continuous linear functionals over $\mathcal{E}(\Omega)$ (space of distributions with compact support in Ω)

$\mathcal{S}(\mathbb{R})$ space of rapidly decreasing smooth functions over $\mathbb{R}$

$\mathcal{S}'(\mathbb{R})$ the space of tempered distributions (distributions with slow growth)

$\mathcal{S}_0(\mathbb{R}^n)$ subspace of $\mathcal{S}(\mathbb{R}^n)$ consisting of all those functions having all their moments vanishing (space of highly localized functions in time and frequency space)

$\mathcal{S}_0'(\mathbb{R}^n)$ Lizorkin distributions (dual of $\mathcal{S}_0(\mathbb{R}^n)$)

$\mathcal{S}(\mathbb{H})$ space of highly localized functions over the upper half plane $\mathbb{H} = \{(b,a) : b \in \mathbb{R},\, a > 0\}$

$\mathcal{S}_{r,l}(\mathbb{R}^n)$ completion of $\mathcal{D}(\mathbb{R}^n)$ with norm

$$\rho_{r,\,l}(\varphi) := \sup_{|\alpha|\le r,\,\mathbf{x}\in\mathbb{R}^n} (1+|\mathbf{x}|)^l |\varphi^{(\alpha)}(\mathbf{x})|, \quad r, l \in \mathbb{N}$$

$\mathcal{S}_r(\mathbb{R}^n)$ projective limit of $\mathcal{S}_{r,l}(\mathbb{R}^n)$ when $l \to \infty$

$\mathcal{K}_M(\mathbb{R}^n)$ space of all those smooth functions $\varphi \in C^\infty(\mathbb{R}^n)$ for which

$$\nu_{r,l}(\varphi) := \sup_{|\alpha|\le r,\,\mathbf{x}\in\mathbb{R}^n} e^{M(l\mathbf{x})} |\varphi^{(\alpha)}(\mathbf{x})| < \infty, \quad r, l \in \mathbb{N}$$

$\mathcal{K}_{M,r,l}(\mathbb{R}^n)$ $= \{\varphi \in C^r(\mathbb{R}^n) : \lim_{|\mathbf{x}|\to\infty} e^{M(l\mathbf{x})}\varphi^{(\alpha)}(\mathbf{x}) = 0,\ |\alpha| \le r\}$

$\mathcal{K}_{M,r}(\mathbb{R}^n)$ projective limit of $\mathcal{K}_{M,r,l}(\mathbb{R}^n)$ when $l \to \infty$

$\mathcal{K}_1(\mathbb{R}^n)$ the space of functions $\varphi \in C^\infty(\mathbb{R}^n)$ such that

$$\nu_k(\varphi) = \sup_{\mathbf{x}\in\mathbb{R},\alpha\le k} e^{k|\mathbf{x}|}|\varphi^{(\alpha)}(\mathbf{x})| < \infty,\ k \in \mathbb{N}_0$$

$\mathcal{K}_1'(\mathbb{R}^n)$ the dual of $\mathcal{K}_1(\mathbb{R}^n)$ (space of exponential distributions)

$\mathcal{D}_{L^1}(\mathbb{R}^n)$ the space of functions $\varphi \in C^\infty(\mathbb{R}^n)$ that together with their derivatives belong in $L^1(\mathbb{R}^n)$

$\mathcal{D}_{L^1}'(\mathbb{R})$ Schwartz space of integrable distributions

$\mathcal{D}_{L^p}(\mathbb{R}^n)$	the space of smooth functions with all their derivatives belonging to $L^p(\mathbb{R}^n), 1 \leq p \leq \infty$
$\mathcal{D}'_{L^p}(\mathbb{R}^n)$	the dual space of $\mathcal{D}_{L^q}(\mathbb{R}^n)$, $1 < p \leq \infty, 1 \leq q < \infty, \frac{1}{p} + \frac{1}{q} = 1$
$\mathcal{O}'_C(\mathbb{R}^n)$	$= \{f \in \mathcal{D}' \; : \; (1 + \|x\|^2)^m f \in \mathcal{D}'_{L^\infty}, \forall m \in \mathbb{N}\}$ (the space of distributions with fast descent)
$f \otimes g$	tensor product of distributions f and g
$E \otimes F$	tensor product of topological vector spaces E and F
$\mathbb{Y}^{n+1}$	$= \{(\mathbf{u}, b, a) \vert \mathbf{u} \in \mathbb{S}^{n-1}, b \in \mathbb{R}, a \in \mathbb{R}^+\}$, where $\mathbb{S}^{n-1}$ is the unit sphere of $\mathbb{R}^n$
$\mathcal{D}(\mathbb{S}^{n-1})$	space of smooth functions on the sphere
$\mathcal{S}(\mathbb{Y}^{n+1})$	$= \mathcal{D}(\mathbb{S}^{n-1})\hat{\otimes}\mathcal{S}(\mathbb{H})$ where $X\hat{\otimes}Y$ is the completion of $X \otimes Y$ in, say, the π-topology or the $\varepsilon-$topology
$\mathcal{S}'(\mathbb{Y}^{n+1})$	the dual of $\mathcal{S}(\mathbb{Y}^{n+1})$
$\mathcal{U}(\mathbb{C}^n)$	space of entire functions which decrease faster than any polynomial in bands
$\mathcal{U}'(\mathbb{C}^n)$	Silva tempered ultradistributions (dual of $\mathcal{U}(\mathbb{C}^n)$)
$\mathcal{K}_1(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}(\mathbb{C}^n)$	space of all smooth functions Φ such that $$\rho_k(\Phi) := \sup_{(x,z) \in \mathbb{R}^n \times \Pi_k, \; \vert\alpha\vert \leq k} e^{k\vert x\vert}(1 + \vert z\vert^2)^{k/2} \left\vert \frac{\partial^\alpha}{\partial x^\alpha}\Phi(x,z)\right\vert < \infty$$
$(\mathcal{K}_1(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}(\mathbb{C}^n))'$	$= \mathcal{K}'_1(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}'(\mathbb{C}^n)$
$L^{p,q}_m(\mathbb{R}^{2n})$	all measurable functions F such that $$\\|F\\|_{L^{p,q}_m} := \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} \vert F(x,\xi)\vert^p m(x,\xi)^p dx\right)^{q/p} d\xi\right)^{1/q} < \infty,$$ where $p, q \in [1, \infty]$
$M^{p,q}_m(\mathbb{R}^n)$	consists of all f such that $V_g f \in L^{p,q}_m(\mathbb{R}^{2n})$ (Modulation space)

$\mathcal{D}_{L^1_\omega}(\mathbb{R}^n)$ $= \{\varphi \in C^\infty(\mathbb{R}^n) : \varphi^{(\alpha)} \in L^1_\omega(\mathbb{R}^n), \forall \alpha \in \mathbb{N}_0^n\}$, provided with the family of norms

$$\|\varphi\|_{1,\omega,k} := \sup_{|\alpha|\le k} \int_{\mathbb{R}^n} |\varphi^{(\alpha)}(t)|\omega(t)dt, \quad k \in \mathbb{N}_0$$

$\mathcal{B}'_\omega(\mathbb{R}^n)$ the space of ω-bounded distributions (the strong dual of $\mathcal{D}_{L^1_\omega}(\mathbb{R}^n)$)

$\sim^q$ (or $\sim$) notation for quasiasymptotics

$\sim^S$ notation for S-asymptotics

$\hat{f}$ Fourier transformation of f

$\mathcal{R}_\psi f$ ridgelet transform of f

Rf Radon transform of f

$V_g f$ Short time Fourier transform of f with respect to a window g

Chapter 1

Preliminaries

Here, we will give the basic notions that will be used in the thesis. We will define the spaces of test functions and distributions that are needed. Our goal is to give a short introduction to the theory of distributions (generalized functions, the other name under which they can be found), i.e. to those aspects who are significant in the theory of their asymptotic analysis. There are several approaches to this theory. We will follow the functional approach known as Schwarz-Sobolev.

1.1 Basic facts and notation

The n-dimensional Cartesian product of natural, integer, positive integer, real numbers, positive real numbers, and complex numbers are denoted by $\mathbb{N}^n$, $\mathbb{Z}^n$, $\mathbb{Z}^n_+$, $\mathbb{R}^n$, $\mathbb{R}^n_+$, and $\mathbb{C}^n$ respectively. Also, $\mathbb{N}^n_0 = \mathbb{N}^n \cup \{0\}$. With $\mathbb{H} = \{(a,b)|a \in \mathbb{R}, b \in \mathbb{R}^+\}$ we denote the upper half plane and we introduce the set $\mathbb{Y}^{n+1} = \mathbb{S}^{n-1} \times \mathbb{H} = \{(\mathbf{u}, b, a)|\mathbf{u} \in \mathbb{S}^{n-1}, b \in \mathbb{R}, a \in \mathbb{R}^+\}$, where $\mathbb{S}^{n-1}$ stands for the unit sphere of $\mathbb{R}^n$. For multi-indexes $\alpha, \beta \in \mathbb{N}^n$, we set

$$|\alpha| = \alpha_1 + \ldots + \alpha_n; \quad \alpha! = \alpha_1! \cdot \ldots \cdot \alpha_n!; \quad \beta \leq \alpha \Leftrightarrow \beta_j \leq \alpha_j, \, \forall j = 1, \ldots, n.$$

For $\beta \leq \alpha$,

$$\binom{\alpha}{\beta} = \prod_{j=1}^{n} \binom{\alpha_j}{\beta_j}.$$

For $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}^n$, we use the symbols

$$x \cdot y = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n; \quad x^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n};$$

$$D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n} \text{ where } D_j^{\alpha_j} = i^{-1} \frac{\partial^{\alpha_j}}{\partial {x_j}^{\alpha_j}}; \quad \partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}.$$

For $x, y \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{N}^n$, the following equalities and inequalities hold

$$(x+y)^\alpha = \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} x^{\alpha-\gamma} y^\gamma.$$

Let $\Omega \subset \mathbb{R}^n$ is open set and let $p \in [1, \infty)$. The vector space $L^p(\Omega)$ consists of all functions f for which $\int_\Omega |f(x)|^p dx < \infty$. With respect to the norm

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p dx \right)^{1/p},$$

$L^p(\Omega)$ is a Banach space. Specially, $L^2(\Omega)$ is a Hilbert space equipped with the inner product $(f, g)_{L^2(\Omega)} = \int_\Omega f(x)\overline{g(x)}dx$. With $L^p_{loc}(\Omega)$ we denote the space of locally integrable functions, that is $f \in L^p_{loc}(\Omega)$ if $f \in L^p(I)$ for every bounded and open interval $I \subset \Omega$. $f \in L^\infty(\Omega)$ if there exists $C > 0$ such that $|f(x)| < C$ for all $x \in \Omega$. With respect to the norm

$$\|f\|_{L^\infty(\Omega)} = ess \sup_{x\in\Omega} |f(x)| := \inf_{C\in\mathbb{R}^+} \{|f(x)| < C\},$$

$L^\infty(\Omega)$ is a Banach space.

For measure and integration theory see, e.g., [80].

We will recall now some concepts and results from functional analysis. For a detailed treatment and proofs see, e.g., [103].

A *topological space* is a nonempty set X in which the collection τ of subsets is defined such that τ contains: X and the empty set $\emptyset$, the intersection of any two elements from τ, and the union of any subcollection from τ. The elements from τ are called *open sets*, and τ is said to define a topology on X (often the pair (X, τ) is called topological space). A set A is called *closed* if its complement is open set.

A set $U \subset X$ is called *neighborhood* of $x \in X$ if and only if there exists open set $O \in \tau$ such that $x \in O \subset U$. Family of sets $\mathcal{B}(x)$ is called *neighborhood basis or local basis* for $x \in X$ if and only if the following conditions hold true:

1. Elements from $\mathcal{B}(x)$ are neighborhoods of $x \in X$;

2. For every neighborhood U of x there exists $B \in \mathcal{B}(x)$ such that $B \subset U$.

The topological space (X, τ) is called *Hausdorf* if distinct points of X have disjoint neighborhoods. A *topological vector space* is a linear space on which a topology is defined in such a way as to preserve the continuity of the operations of the underlying linear space.

A set A is called *absorbing* if for every $x \in X$ there exists $\lambda > 0$ such that $x \in \lambda A$. A set $A \subset X$ is called *balanced* if $\lambda A \subset A$ for every $|\lambda| \leq 1$. A set $A \subset X$ is called *bounded* if for every neighborhood U of zero, there exists $\lambda > 0$ such that $A \subset \lambda U$.

A *locally convex* topological vector space is a topological vector space in which the origin has a local base of convex sets.

Let $\{\| \ \|_j\}_{j\in I}$ be a family of seminorms on X, where I is an arbitrary index set. A locally convex space X is then defined to be a vector space along with a family of seminorms $\{\| \ \|_j\}_{j\in I}$ on X. The space carries a natural topology, the initial topology of the seminorms. In other words, it is the topology for which all mappings $x \to \|x - x_0\|_j, x_0 \in X, j \in I$ are continuous. A base of neighborhoods of x_0 for this topology is obtained in the following way: $U_{B,\varepsilon}(x_0) = \{x \in V : \|x - x_0\|_j < \varepsilon, \quad j \in B\}$, for every finite subset B of I and every $\varepsilon > 0$.

Definition 1.1.1. A subset T of a topological vector space X is called *barrel* if T is absorbing, convex, balanced, and closed. A topological vector space X is said to be *barreled* if every barrel in X is a neighborhood of zero in X.

Definition 1.1.2. Locally convex vector space X is called *bornologycal* space if every balanced convex subset $A \subset X$, who absorbs all bounded sets in X, is a neighborhood of zero.

Definition 1.1.3. *Montel* space is locally convex space who is Hausdorff and barreled, and in which every closed and bounded set is compact.

Definition 1.1.4. A topological vector space is called *Fréchet* space if it is locally convex, metrizable and complete.

Definition 1.1.5. *The support* of a function (distribution) f , denoted by $\operatorname{supp} f$, is defined to be the closure of the set $\{x \in \Omega : f(x) \neq 0\}$.

It is worth mentioning that every Fréchet, Banach and Hilbert spaces are also barreled. The importance of barreled spaces stems mainly from the Banach-Steinhause Theorem.

We denote as X' the dual of X, that is, the space of continuous linear functionals on the topological vector space X. Unless otherwise specified, we shall always provide X' with the strong dual topology [103], that is, the topology of uniform convergence over bounded sets of X.

Theorem 1.1.1 (Banach-Steinhause Theorem)**.** *Let X be barreled topological vector space, and F be locally convex space. The following properties of a subset of X' are equivalent*

- (i) *H is bounded for the topology of pointwise convergence;*
- (ii) *H is bounded for the topology of bounded convergence;*
- (iii) *H is equicontinuous.*

Let X and Y are topological spaces over, and let $f : X \to Y$. f is said to be *isomorphism* from X to Y if f is continuous bijection whose inverse is continuous. The linear map $f : X \to Y$ is said to be *bounded* if $f(A)$ is a bounded subset of Y for every bounded subset A of X. The linear map $f : X \to Y$ is said to be *compact* if there is a neighborhood U of zero in X such that $f(U)$ is relatively compact in Y ($f(U)$ is relatively compact in Y if the closure of $f(U)$ is compact). We shall write $X \hookrightarrow Y$ if $X \subset Y$, the inclusion mapping $X \to Y$ is continuous, and X is dense in Y.

A locally convex space X is called *reflexive* if it coincides with the dual of its dual space, both as linear space and as topological space. It is well known that every Montel space is reflexive.

For the topological vector spaces X, Y and Z, the mapping $f : (x, y) \mapsto f(x, y)$ from $X \times Y$ to Z is called *bilinear* if for every $x_0 \in X$ (resp. $y_0 \in Y$) the mappings $f_{x_0} : y \mapsto f(x_0, y)$ (resp. $f_{y_0} : x \mapsto f(x, y_0)$) from X (resp. Y) into Z are linear. The bilinear map f is said to be *separately continuous* if, for every x_0, y_0, these two linear mappings are continuous.

A projective (injective) sequence of locally convex spaces with continuous linear mappings:

$$X_1 \leftarrow X_2 \leftarrow \ldots \leftarrow X_j \leftarrow \ldots$$

$$(X_1 \rightarrow X_2 \rightarrow \ldots \rightarrow X_j \rightarrow \ldots)$$

is said to be compact if all mappings are compact. The limit $\operatorname{proj}\lim_{j\to\infty} X_j = \cap_j X_j$ ($\operatorname{ind}\lim_{j\to\infty} X_j = \cup_j X_j$) of a compact projective (injective) sequence is said to be Fréchet-Schwartz that is FS space (DFS space). FS and DFS spaces are separable and Montel. Closed subspaces, quotient spaces and projective limits of sequences in FS (DFS) spaces are FS (DFS). The strong dual spaces of FS spaces are DFS spaces, and conversely. More explicitly we have the isomorphism $(\operatorname{proj}\lim_{j\to\infty} X_j)' = \operatorname{ind}\lim_{j\to\infty} X'_j$ and $(\operatorname{ind}\lim_{j\to\infty} X_j)' = \operatorname{proj}\lim_{j\to\infty} X'_j$ when the sequence satisfies certain conditions.

We shall make use of the Landau order symbols.

Definition 1.1.6. Let g and h be two complex-valued functions defined in a pointed neighborhood of x_0. We write

$$g(x) = O(h(x)), x \to x_0,$$

if there exists a positive constant M such that

$$|g(x)| \leq M|h(x)|,$$

for all x sufficiently close to x_0.

Definition 1.1.7. Let g and h be two complex-valued functions defined in a pointed neighborhood of x_0. We write

$$g(x) = o(h(x)), x \to x_0,$$

if for every $\varepsilon > 0$ there exists neighborhood V_{x_0} of x_0 such that

$$|g(x)| \leq \varepsilon|h(x)|, x \in V_{x_0},$$

that is,

$$\lim_{x\to x_0} \frac{g(x)}{h(x)} = 0.$$

Analog definitions holds for "big" O and "small" o when $x \to \infty$. When x_0 is finite, we describe local, and when $x \to \infty$ we describe global behavior of the function g with respect to the function h. Some properties of the "small" o are:

1. If for $g(x) = o(1)$ when $x \to x_0$ ($x \to \infty$), then $g(x) = o(C)$ for arbitrary constant $C \neq 0$ when $x \to x_0$ ($x \to \infty$).

2. If for $g(x) = o(f(x))$ and $f(x) = o(h(x))$ when $x \to x_0$ ($x \to \infty$), then $g(x) = o(h(x))$ when $x \to x_0$ ($x \to \infty$).

We say that g is asymptotic to h as $x \to x_0$ if $g(x) = h(x) + o(h(x))$. In this case, we write

$$g(x) \sim h(x), x \to x_0.$$

When h is a non-zero near x_0, it means that

$$\lim_{x \to x_0} \frac{g(x)}{h(x)} = 1.$$

1.2 Function spaces and distributions

1.2.1 Spaces of distributions $\mathcal{D}'$ and $\mathcal{E}'$

Let Ω be an open subset of $\mathbb{R}^n$ with the usual topology. Then, Ω is subspace of $\mathbb{R}^n$ with topology induced from $\mathbb{R}^n$.
With $C^m(\Omega), m \in \mathbb{N}_0$ or $m = \infty$ we denote the space of complex valued functions defined on Ω with continuous derivatives up to order m. For $m = 0$ we have the space $C^0(\Omega)$ of continuous functions on Ω and for $m = \infty$ we have the space $C^\infty = \cap_{m \geq 0} C^m(\Omega)$ of functions with continuous derivatives of all orders (smooth functions). We have the following inclusions

$$C^\infty(\Omega) \subset ... \subset C^m(\Omega) \subset C^{m-1}(\Omega) \subset ... \subset C^0(\Omega).$$

In $C_0^m(\Omega), m \in \mathbb{N}_0$ or $m = \infty$ belong all those functions from $C^m(\Omega)$ with compact support in Ω. Let K be compact subset of Ω. With $C_0^m(K), m \in \mathbb{N}_0$ or $m = \infty$ we denote the subspace of $C_0^m(\Omega)$ whose elements have support in K. The topology of $C_0^\infty(K)$ is defined by the seminorms

$$p_m(\phi) = \sup_{x \in K}\{|\partial^\alpha \phi(x)| : |\alpha| \leq m\}, m \in \mathbb{N}_0,$$

with the sets $B_m(r) = \{\phi \in C_0^\infty(K) : p_m(\phi) < r\}$ as a local base. But, $C_0^\infty(K)$ is closed subspace of $C_0^\infty(\Omega)$, and we denote the topology of

$$C_0^\infty(\Omega) = \bigcup_{K \subset \Omega} C_0^\infty(K)$$

to be the finest locally convex topology for which the identity map $C_0^\infty(K) \to C_0^\infty(\Omega)$ is continuous for every $K \subset \Omega$. This means that a convex, balanced set $U \subset C_0^\infty(\Omega)$ is a neighborhood of 0 in $C_0^\infty(\Omega)$ if and only if $U \cap C_0^\infty(K)$ is a neighborhood of 0 in $C_0^\infty(K)$ for every $K \subset \Omega$. The collection of all such neighborhoods U constitues a local base for the topology we have defined in $C_0^\infty(\Omega)$, which is known as the *inductive limit* of the topology of $C_0^\infty(K)$. It is clear that if Ω_1 is an open subset of Ω then $C_0^\infty(\Omega_1)$ is a subspace of $C_0^\infty(\Omega)$, because every function in $C_0^\infty(\Omega_1)$ may be extended as a C_0^∞ function into Ω by defining it to be 0 on Ω/Ω_1.

Definition 1.2.1. The locally convex space $C_0^\infty(\Omega)$, endowed with the inductive limit topology, is called *the space of test functions* and is also denoted by $\mathcal{D}(\Omega)$.

For every $\phi \in \mathcal{D}(\Omega)$ we define the norms

$$|\phi|_m = \sup_{x\in\Omega}\{|\partial^\alpha \phi(x)| : |\alpha| \leq m\}, m \in \mathbb{N}_0.$$

We shall use $\mathcal{D}(K)$ to denote the locally convex space $C_0^\infty(K)$ where K is compact subspace of Ω. For every $\phi \in \mathcal{D}(K)$ we define the norms as $p_{K,m}(\phi) = \sum_{|\alpha|\leq m} \sup_{x\in K} |\phi^{(\alpha)}(x)|$. Let us note that $\mathcal{D}(\Omega)$ is a Montel space, and hence it is reflexive.

Theorem 1.2.1. (*i*) *Let E is given locally convex space. The linear mapping $T : \mathcal{D}(\Omega) \to E$ is continuous if and only if the map $T : \mathcal{D}(K) \to E$ is continuous for every compact set $K \subset \Omega$.*

(*ii*) *Set $A \subset \mathcal{D}(\Omega)$ is bounded if and only if there exists compact set $K \subset \Omega$ such that $A \subset \mathcal{D}(K)$ and A is bounded in $\mathcal{D}(K)$.*

(*iii*) *The sequence $\{\psi_k\}_{k\in\mathbb{N}}$ from $\mathcal{D}(\Omega)$ converge in $\mathcal{D}(\Omega)$ if and only if there exists compact set $K \subset \Omega$ such that the sequence $\{\psi_k\}_{k\in\mathbb{N}}$ converges in $\mathcal{D}(K)$.*

Definition 1.2.2. A distribution on Ω is a continuous linear functional on $\mathcal{D}(\Omega)$.

The linear space of all distributions on Ω is denoted by $\mathcal{D}'(\Omega)$, the topological dual of $\mathcal{D}(\Omega)$, and its element are denoted as $f, g, \ldots$. Every distribution f maps $\mathcal{D}(\Omega)$ to the field of complex numbers $\mathbb{C}^n$. Symbolically, this is written as:

$$f : \varphi \to \langle f, \varphi\rangle, \quad \varphi \in \mathcal{D}(\Omega).$$

Example 1.2.1. The Delta distribution $\delta(x - x_0) \in \mathcal{D}'(\Omega),\ x_0 \in \Omega$ is defined in the following manner:

$$\langle \delta(x - x_0), \varphi(x)\rangle = \varphi(x_0),\ \varphi \in \mathcal{D}(\Omega).$$

Example 1.2.2. The Heaviside distribution H:

$$\langle H(x), \varphi(x)\rangle = \int_0^\infty \varphi(x)dx.$$

The following theorem is often taken as a definition for distributions.

Theorem 1.2.2. *A linear functional f on $\mathcal{D}(\Omega)$ is a distribution if and only if, for every compact set $K \subset \Omega$, there exists a nonnegative integer $m \in \mathbb{N}_0$ and a constant $C > 0$ such that*

$$|\langle f, \varphi\rangle| \leq C p_{K,m}(\varphi). \tag{1.1}$$

Definition 1.2.3. Let $f \in \mathcal{D}'(\Omega)$. Suppose that there is $m \in \mathbb{N}_0$ for which (1.1) holds for every compact set $K \subset \Omega$. Then the smallest number m with this property is called *the order* of the distribution f and then the distribution f is said to be of *finite order*. If no finite number m satisfies the inequality (1.1) for all K, then f is said to be of *infinite order*.

In the space $\mathcal{D}'(\Omega)$ we will introduce the weak topology τ_s, the topology of compact convergence τ_k and the strong topology τ_b. The weak topology is defined with the following family of seminorms

$$\|f\|_\varphi = |\langle f, \varphi\rangle|, \quad \varphi \in \mathcal{D}(\Omega);$$

the topology of compact convergence is defined with the family of seminorms:

$$\|f\|_K = \sup\{|\langle f, \varphi\rangle| : \varphi \in K\};$$

where K is compact subspace of $\mathcal{D}(\Omega)$, and the strong topology is defined with the family of seminorms:

$$\|f\|_B = \sup\{|\langle f, \varphi\rangle| : \varphi \in B\};$$

where B is bounded subspace of $\mathcal{D}(\Omega)$.

Observe that the space $\mathcal{D}(\Omega)$ is barreled. The Banach-Steinhaus theorem then leads to the following simply descriptions of convergent sequences and bounded sets in $\mathcal{D}'(\Omega)$.

Theorem 1.2.3. *The convergence of sequences in $\mathcal{D}'(\Omega)$ with respect to the weak and strong topology coincide.*

From the last theorem it follows that is easier to investigate convergence in the weak topology.

Theorem 1.2.4. *Let $B' \subset \mathcal{D}'(\Omega)$. The following are equivalent:*

(*i*) *B' is bounded set with respect to the weak topology;*

(*ii*) *B' is bounded set with respect to the strong topology;*

(*iii*) *B' is uniformly continuous subset from $\mathcal{D}'(\Omega)$.*

Let $f \in L^1_{loc}(\Omega)$ that is $\int_K |f(x)|dx < \infty$ for evert compact set $K \subset \Omega$. It is easy to show that with

$$\varphi \to \langle f, \varphi\rangle = \int_{\mathbb{R}^n} f(x)\varphi(x)dx, \quad \varphi \in \mathcal{D}(\Omega)$$

is defined a distribution from $\mathcal{D}'(\Omega)$.

Definition 1.2.4. Let $f \in L^1_{loc}(\Omega)$. We shall identify f with a distribution, denoted also as f, in the following way:

$$\langle f, \varphi\rangle := \int_{\mathbb{R}^n} f(x)\varphi(x)dx = \int_K f(x)\varphi(x)dx,$$

where K is the support of $\varphi \in \mathcal{D}(\Omega)$. Distributions arising in this way are called *regular distribution.*

Definition 1.2.5. Let μ be a regular Borel measure on Ω. We identify μ with a distribution, denoted also as μ, as follows:

$$\langle \mu, \varphi \rangle = \int_\Omega \varphi(x) d\mu(x), \quad \varphi \in \mathcal{D}(\Omega).$$

Distributions arising in this way are called *Radon measures.*

There exist distributions that are not regular and they are called *singular* distributions. An example of singular distribution is δ.

For every $f \in \mathcal{D}'(\Omega), \varphi \in \mathcal{D}(\Omega)$ it holds:

(i) $\langle f(ax+b), \varphi(x) \rangle := \left\langle f(x), \frac{1}{|a|}\varphi\left(\frac{x-b}{a}\right) \right\rangle, a, b \in \mathbb{R}, a \neq 0,$

(ii) $\langle g(x)f(x), \varphi(x) \rangle := \langle f(x), g(x)\varphi(x) \rangle, g \in C^\infty(\Omega)$

(iii) $\langle f^{(\alpha)}(x), \varphi(x) \rangle := (-1)^{|\alpha|} \langle f(x), \varphi^{(\alpha)}(x) \rangle, \alpha \in \mathbb{N}_0^n.$

We will now give brief introduction to homogenous distributions.

Definition 1.2.6. A distributions $f \in \mathcal{D}'(\mathbb{R}^n)$ is called *homogeneous of degree* α if

$$f(\lambda x) = \lambda^\alpha f(x), \lambda > 0.$$

Example 1.2.3. Dirac delta distribution $\delta(x)$ is homogeneous of degree $-n$, i.e. $\delta(\lambda x) = \lambda^{-n}\delta(x), \lambda > 0.$

Example 1.2.4. The function

$$x_+^\alpha = \begin{cases} x^\alpha, & x > 0 \\ 0, & x \leq 0 \end{cases} = H(x)x^\alpha, \alpha \in \mathbb{C}.$$

is locally integrable over $\mathbb{R}$ if $Re\alpha > -1$, and in this case, it defines the regular distribution

$$\langle x_+^\alpha, \varphi(x) \rangle = \int_0^\infty x^\alpha \varphi(x) dx,$$

where $\varphi \in \mathcal{D}(\mathbb{R})$. ($H$ is the Heaviside function that it $H(x) = 1$ if $x > 0$ and 0 if $x \leq 0$.) Using regularization procedures for analytic continuation (see [17, Chapter 2.4.]), for every $\alpha \in \mathbb{C}, \alpha \neq -1, -2, \ldots$, we can define a regular distribution

$$\langle x_+^\alpha, \varphi(x) \rangle = \int_0^1 x^\alpha \left(\varphi(x) - \varphi(0) - \ldots - \frac{x^{n-1}}{(n-1)!}\varphi(0) \right) dx$$
$$+ \int_1^\infty x^\alpha \varphi(x) dx + \sum_{k=1}^n \frac{\varphi^{(k-1)}(0)}{(k-1)!(\alpha+k)}.$$

It is easy to show that $x_+^\alpha, \alpha \neq -1, -2, \ldots$ is homogeneous with degree α. Analogous results hold for the distribution $x_-^\alpha, \alpha \neq -1, -2, \ldots$ defined as

$$x_-^\alpha = \begin{cases} (-x)^\alpha, & x < 0 \\ 0, & x \geq 0 \end{cases}.$$

We remark that all homogeneous distributions on the real line are explicitly known [17, Thrm. 2.6.1]; indeed, they are of the form

$$g(x) = C_- x_-^\alpha + C_+ x_+^\alpha \quad \text{if} \quad \alpha \notin \mathbb{N},$$

$$g(x) = \gamma \delta^{(k-1)}(x) + \beta x^{-k} \quad \text{if} \quad \alpha = -k \in \mathbb{N}$$

where $C_-, C_+, \gamma, \beta \in \mathbb{C}$ and the distribution x^{-k} stands for the standard regularization of the corresponding function i.e. $x^{-1} = (\log|x|)'$, $-kx^{-k-1} = (x^{-k})'$

In dimension n, every homogeneous distributions g of degree $\alpha \neq -n, -n-1, -n-2, \dots$ has the form

$$\langle g(x), \varphi(x) \rangle = \text{F. p.} \int_0^\infty r^{\alpha+n-1} \langle G(\omega), \varphi(r\omega) \rangle dr,$$

where G is a distribution on the unit sphere of $\mathbb{R}^n$ and F.p. stands for the Hadamard finite part of the integral. See the book [17] for the definition of the finite part and more over homogeneous distributions in several variables. We refer to [17, Sect. 2.6] for properties of homogeneous distributions. Observe also that when f is a positive measure, then $\alpha \geq -n$ and $g = \upsilon$, where υ is also a positive Radon measure that must necessarily satisfy $\upsilon(aB) = a^{\alpha+n}\upsilon(B)$ for all Borel set B.

If there exists a constant C such that for every $\varphi \in \mathcal{D}(\Omega)$

$$\lim_{\varepsilon \to 0} \langle f(x_0 + \varepsilon x), \varphi(x) \rangle = \langle C, \varphi \rangle$$

then we say that f has *value C at the point $x = x_0$*, and we denote $f(x)|_{x=x_0} = C$; that is f has *value at the point $x = x_0$* if and only if

$$\lim_{\varepsilon \to 0} \left\langle f(x), \varphi\left(\frac{x - x_0}{\varepsilon}\right) \right\rangle = C \int_{\mathbb{R}^n} \varphi(x) dx.$$

In general case, distributions does not have value at a point. For example, $\delta(x)$ does not have value at $x_0 = 0$ which follows from

$$\frac{1}{\varepsilon} \left\langle \delta(x), \varphi\left(\frac{x}{\varepsilon}\right) \right\rangle = \frac{1}{\varepsilon} \varphi(0).$$

Moreover, the regular distribution defined with the Heaviside function does not have value at $x_0 = 0$.

But, if $\widetilde{f}(x)$ is a regular distribution defined with the continuous function $f(x)$ for which $f(x_0) = C$, then, from Lebesque Dominated convergence theorem we have $\widetilde{f}(x)|_{x=x_0} = C$.

Important class of distributions is the space $\mathcal{E}'(\Omega)$ formed by those distributions of $\mathcal{D}'(\Omega)$ whose support is compact. The notion $\mathcal{E}'(\Omega)$ suggests that this space is the dual of certain space $\mathcal{E}(\Omega)$. Indeed, let $\mathcal{E}(\Omega)$ be the space of all smooth functions defined in Ω with the topology generated by the family of seminorms

$$\|\phi\|_{K,\alpha} = \sup_{x \in K} |\phi^{(\alpha)}(x)|,$$

for $\alpha \in \mathbb{N}^n$ and K a compact subset of Ω. Then, the space $\mathcal{D}(\Omega)$ is dense subspace of $\mathcal{E}(\Omega)$ and the inclusion is continuous. It follows that the dual space $\mathcal{E}'(\Omega)$ can be identified with a subspace of $\mathcal{D}'(\Omega)$, and this subspace is precisely the space of distributions with compact support.

1.2.2 Tempered distributions

Definition 1.2.7. A function $\varphi \in C^\infty(\Omega)$ is said to be *rapidly decreasing* if

$$\sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta \varphi(x)| < \infty,$$

for all pairs of multi-indices α and β.

We shall use $\mathcal{S}$ to denote the set of all rapidly decreasing function, which is clearly a linear space under the usual operations of addition and multiplications by scalars. A function in $\mathcal{S}(\mathbb{R}^n)$ approaches to 0 as $|x| \to \infty$ faster than any power of $1/|x|$. An example of such a function is $e^{-|x|^2}$.

For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we define the seminorms

$$\|\varphi\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta \varphi(x)| \tag{1.2}$$

with $\alpha, \beta \in \mathbb{N}_0^n$. The countable family $\{\|\varphi\|_{\alpha,\beta}\}$ defines a Hausdorff, locally convex topology on $\mathcal{S}(\mathbb{R}^n)$ which is metrizable and complete. With this topology, $\mathcal{S}(\mathbb{R}^n)$ is a Fréchet space, and a sequence $\{\varphi_k\}$ converges to 0 in $\mathcal{S}(\mathbb{R}^n)$ if and only if $x^\alpha\partial^\beta\varphi_k(x) \to 0$ uniformly on $\mathbb{R}^n$ as $k \to \infty$.

Theorem 1.2.5. *Let $\varphi \in C^\infty(\mathbb{R}^n)$. The following are equivalent:*

(*i*) $\varphi \in \mathcal{S}(\mathbb{R}^n)$;

(*ii*) *for every $k \in \mathbb{N}_0, \beta \in \mathbb{N}_0^n$ it holds*

$$\|\varphi\|_{k,\beta} = \sup_{x\in\mathbb{R}^n} \{(1+|x|^2)^{k/2}|\partial^\beta\varphi(x)|\} < \infty; \tag{1.3}$$

(*iii*) $P(x)(Q(\partial)\varphi(x)) \in \mathcal{S}(\mathbb{R}^n)$;

(*iv*) $Q(\partial)(P(x)\varphi(x)) \in \mathcal{S}(\mathbb{R}^n)$,

In this theorem $P(x)$ and $Q(x), x \in \mathbb{R}^n$ are polynomials with constant coefficients and $Q(\partial)$ is differential operator obtained when in $Q(x)$ one replace x with $\frac{d}{dx}$.

It is easy to show that the topology of $\mathcal{S}(\mathbb{R}^n)$ obtained with the family of seminorms (1.2) is equivalent with the one defined with (1.3), and that both this family of seminorms are equivalent with the family

$$\|\varphi\|_N = \sup_{x\in\mathbb{R}^n, |\alpha|\le N} \{(1+|x|^2)^{N/2}|\partial^\alpha\varphi(x)|\},\ N \in \mathbb{N}_0.$$

Definition 1.2.8. The space of all continuous and linear functionals over $\mathcal{S}(\mathbb{R}^n)$ is denoted as $\mathcal{S}'(\mathbb{R}^n)$. The elements of $\mathcal{S}'(\mathbb{R}^n)$ are called *tempered distributions* or *distributions of slow growth.*

The following inclusions hold true

$$\mathcal{D}(\mathbb{R}^n) \hookrightarrow \mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{E}(\mathbb{R}^n) \hookrightarrow \mathcal{E}'(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n) \hookrightarrow \mathcal{D}'(\mathbb{R}^n).$$

Moreover,

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n), 1 \le p < \infty.$$

Remark 1.2.1. Any locally integrable function f such that $|x|^{-m}|f(x)|$ is bounded as $|x| \to \infty$, for some positive integer m, defines a distribution in $\mathcal{S}'(\mathbb{R}^n)$.

Theorem 1.2.6. *A distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ belongs to $f \in \mathcal{S}'(\mathbb{R}^n)$ if and only if if and only if there exists constants $C > 0$ and $p \in \mathbb{N}_0$ such that*

$$|\langle f, \varphi \rangle| \leq C\|\varphi\|_N, \quad \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{1.4}$$

Theorem 1.2.7. *A distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ belongs to $f \in \mathcal{S}'(\mathbb{R}^n)$ if and only if if and only if there exists constants $\beta \in \mathbb{N}_0^n, m \in \mathbb{N}_0$ and continuous and bounded function F on $\mathbb{R}^n$ such that*

$$f(x) = [(1 + |x|^2)^{m/2} F(x)]^{(\beta)}.$$

We shall also employ two important subspaces of $\mathcal{S}'(\mathbb{R}^n)$. They are the Schwartz spaces of integrable distributions $\mathcal{D}'_{L^1}(\mathbb{R}^n)$ and the spaces of convolutors $\mathcal{O}'_C(\mathbb{R}^n)$.

The space of integrable distributions $\mathcal{D}'_{L^1}(\mathbb{R}^n)$ is, by definition, the strong dual of the space $\dot{B}$ of smooth functions $\varphi : \mathbb{R}^n \to \mathbb{C}$ for which $\partial^\alpha \varphi \to 0$ as $|x| \to \infty$, for each multi index α. The space $\dot{B}$ is a closed subspace of the space B consisting of those smooth functions φ with the property that $\partial^\alpha \varphi$ is bounded for every multi-index α, endowed with the topology of the uniform convergence on $\mathbb{R}^n$ of each derivative. The space C_0^∞ is dense in $\dot{B}$ but not in B. Thus, the space $\mathcal{D}'_{L^1}(\mathbb{R}^n)$ is a subspace of $\mathcal{D}'(\mathbb{R}^n)$. Moreover, according to [89, p.201], the elements of $\mathcal{D}'_{L^1}(\mathbb{R}^n)$ are precisely those distributions that can be written as finite sums of derivatives of L^1-functions. Also, the space $\mathcal{D}'_{L^1}(\mathbb{R}^n)$ is closed under multiplication by functions in B.

It is possible to consider $\mathcal{D}'_{L^1}(\mathbb{R}^n)$ as the strong dual of the space B, provided that we endow B with a topology that gives rise to the following notion of sequence convergence: A sequence $\{\varphi_j\}$ converges to φ if, for every multi index α, $\sup_j \|\partial^\alpha \varphi_j\|_\infty < \infty$ and the sequence $\{\partial^\alpha \varphi_j\}$ converges to $\partial^\alpha \varphi$ uniformly on compact sets. If we denote as B_c the resulting topological space, it can be proved that C_0^∞, and so $\dot{B}$, is dense in B_c. Since every distribution $f \in \mathcal{D}'_{L^1}(\mathbb{R}^n)$ is well defined on C_0^∞ and it is continuous on C_0^∞ with respect to the topology of B_c, it turns out that $\mathcal{D}'_{L^1}(\mathbb{R}^n)$ is also the dual of B_c.

They satisfy the dense and continuous inclusions:

$$\mathcal{E}'(\mathbb{R}^n) \hookrightarrow \mathcal{O}'_C(\mathbb{R}^n) \hookrightarrow \mathcal{D}'_{L^1}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

We refer to Schwartz' book [89, p.200 and p.244] for the precise definition of these distribution spaces.

1.2.3 The Fourier transform of distributions

Now, we will study the Fourier transform. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$. Then, its *Fourier transform* is given by

$$\mathcal{F}(\varphi(x))(\omega) = \hat{\varphi}(x) = \int_{\mathbb{R}^n} \varphi(x) e^{2\pi i x \cdot \omega} dx. \tag{1.5}$$

It is easy to see that $\hat{\varphi}$ belongs to $\mathcal{S}(\mathbb{R}^n)$ and that the Fourier transform is an isomorphism of $\mathcal{S}(\mathbb{R}^n)$ to itself, [112].

Theorem 1.2.8. *Let $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$. Then*

(*i*) $\hat{\hat{\varphi}}(x) = \varphi(-x), x \in \mathbb{R}^n$;

(*ii*) *Parseval relation*

$$\int_{\mathbb{R}^n} \hat{\varphi}(x)\psi(x)dx = \int_{\mathbb{R}^n} \varphi(x)\hat{\psi}(x)dx;$$

(*iii*) $\int_{\mathbb{R}^n} |\varphi(x)|^2 dx = \int_{\mathbb{R}^n} |\hat{\varphi}(x)|^2 dx$.

Definition 1.2.9. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Then, the Fourier transform of a tempered distribution f is given by

$$\langle \hat{f}(\omega), \varphi(\omega)\rangle := \langle f(x), \hat{\varphi}(x)\rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{1.6}$$

In a similar way we can define the inverse Fourier transform in $\mathcal{S}'(\mathbb{R}^n)$

Theorem 1.2.9. *$\mathcal{F}$ and $\mathcal{F}^{-1}$ are isomorphism of $\mathcal{S}'(\mathbb{R}^n)$ into itself.*

Theorem 1.2.10. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Then*

(*i*) $\mathcal{F}(f^{(\alpha)}(x))(\omega) = (2\pi i\omega)^\alpha \hat{f}(\omega), \alpha \in \mathbb{N}^n$;

(*ii*) $\mathcal{F}((-2\pi i x)^\alpha f(x))(\omega) = \hat{f}^{(\alpha)}(\omega), \alpha \in \mathbb{N}^n$;

(*iii*) $\mathcal{F}(f(x-k))(\omega) = e^{-2\pi i\omega\cdot k}\hat{f}(\omega), k \in \mathbb{R}^n$;

(*iv*) $\mathcal{F}(f(ax))(\omega) = \frac{1}{|a|^n}\hat{f}\left(\frac{\omega}{a}\right), a \in \mathbb{R}\setminus\{0\}$.

Example 1.2.5. (*i*) $\hat{\delta}(\omega) = 1$ because

$$\langle \hat{\delta}(\omega), \varphi(\omega)\rangle = \langle \delta(x), \hat{\varphi}(x)\rangle = \hat{\varphi}(0) = \int_{\mathbb{R}^n} \varphi(\omega)d\omega = \langle 1, \varphi(\omega)\rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

(*ii*) From Theorem (1.2.10) it follows that $\hat{1}(\omega) = \delta(\omega)$.

(*ii*) From Theorem (1.2.10) it follows that $\mathcal{F}(\delta^{(\alpha)}(x))(\omega) = (2\pi i\omega)^\alpha, \quad \alpha \in \mathbb{N}^n$.

Example 1.2.6. If f is a homogeneous distribution of order λ, then $\hat{f}$ is a homogeneous distribution of order $-\lambda - 1$. Indeed,

$$a^\lambda \mathcal{F}(f(x))(\omega) = \mathcal{F}(a^\lambda f(x))(\omega) = \mathcal{F}(f(ax))(\omega) = \frac{1}{a}\mathcal{F}(f(x))\left(\frac{\omega}{a}\right), \quad a > 0.$$

i.e. $a^{-\lambda-1}\hat{f}\left(\frac{\omega}{a}\right) = \hat{f}(\omega)$.

1.2.4 Tensor products of distributions

Most of this subsection is taken from [103]. Let Y be a open subset of $\mathbb{R}^n$ and let E be some topological vector space (or a space of distributions). We shall denote by $C^k(Y,E)$ the vector space of C^k mappings of Y into E $(0 \leq k \leq \infty)$, and with $C^k_c(Y,E)$ the subspace of $C^k(Y,E)$ consisting of functions with compact support.

Definition 1.2.10. The C^k topology on $C^k(Y,E)$ is the topology of uniform convergence of the functions together with the derivatives of order $< k+1$ on every compact subsets of Y.

Consider the sequence $\Omega_1 \subset \Omega_2 \subset \ldots \subset \Omega_j \subset \ldots$ of relatively compact open subsets of Y such that $\bigcup_j \Omega_j = Y$; consider an arbitrary integer $l < k+1$, and a basis of neighborhoods of zero in E, $\{U_\alpha\}$. As j, l and α vary in all possible ways, the subsets of $C^k(Y,E)$,

$$U_{j,l,\alpha} = \{f \ : \ \frac{\partial^q}{\partial y^q} f(y) \in U_\alpha \ \text{ for all } \ y \in \Omega_j \ \text{ and all } \ q \in \mathbb{N}^n, |q| \leq l\},$$

form a basis of neighborhoods of zero for the C^k topology. Noting that $U_{j,l,\alpha}$ is convex whenever U is a convex set, we see also that $C^k(Y,E)$ is a locally convex space whenever this is true for E. When E is locally convex, it is easy to obtain a basis of continuous seminorms on $C^k(Y,E)$. It suffices to select a basis of continuous seminorms on E, $\{p_k\}$ and to form the seminorms

$$P_{j,l,k}(f) = \sup_{y \in \Omega_j} \left(\sum_{|q| \leq l} p_k(\frac{\partial^q}{\partial y^q} f(y)) \right). \tag{1.7}$$

For all j, l and k,, the $P_{j,l,k}$ form a basis of continuous seminorms for the topology C^k.

Given an arbitrary compact subset K of Y, we denote by $C^k_c(K,E)$ the subspace of $C^k(Y,E)$ consisting of those functions with support contained in K. We provide $C^k_c(K,E)$ with the topology induced by $C^k(Y,E)$.

Proposition 1.2.1. *[103, Prop. 40.1] Let Y be a open subset of $\mathbb{R}^n$ and let E be locally Hausdorff topological vector space, and k an integer, possibly infinite. A linear map g of $C^k_c(Y,E)$ into a locally convex topological vector space F is continuous if and only if the restriction of f to every subspace $C^k_c(K,E)$ is continuous.*

Corollary 1.2.1. *A linear functional on $C^k_c(Y,E)$ is continuous if and only if its restriction to every subspace $C^k_c(K,E)$ is continuous.*

Example 1.2.7 (Tensor product of functions)**.** Let X, Y be two sets, and f and g a complex-valued functions defined in X and Y, respectively. We shall denote by $f \otimes g$ the function defined in $X \times Y$

$$(x,y) \mapsto f(x)g(y).$$

Now, let E (resp. F) be an arbitrary linear space of complex-valued functions defined in X (resp. Y). We shall denote by $E \otimes F$ the linear subspace of the space of all complex-valued functions defined in $X \times Y$ spanned by the elements of the form $f \otimes g$ where f varies over E and g over F. ($E \otimes F$ is a tensor product of E and F.)

Theorem 1.2.11. *[103, Thrm. 39.2.] Let X (resp Y) be an open subset of $\mathbb{R}^m$ (resp. $\mathbb{R}^n$). Then $C_c^\infty(X) \otimes C_c^\infty(Y)$ is sequentially dense in $C_c^\infty(X \times Y)$.*

Example 1.2.8 (Projective tensor product)**.** Let E and F be two Hausdorff locally convex topological vector spaces and let $\mathcal{P}$ and $\mathcal{Q}$ be the filtering systems of seminorms defining the topology of E and F respectively. The general elements χ in $E \otimes F$ are of the form

$$\chi = \sum_{i=1}^{m} e_i \otimes f_i \text{ with } e_i \in E \text{ and } f_i \in F, i = 1, \ldots, m, m \in \mathbb{N}.$$

Note that this representation is not unique. Now, given two seminorms $p \in \mathcal{P}$ and $q \in \mathcal{Q}$ the projective limit tensor product $p \otimes_\pi q$ of p and q is defined by

$$p \otimes_\pi q(\chi) = \inf \left\{ \sum_{i=1}^{m} p(e_i)q(f_i) : \chi = \sum_{i=1}^{m} e_i \otimes f_i \right\}.$$

The system $\mathcal{P} \otimes_\pi \mathcal{Q} = \{p \otimes_\pi q : p \in \mathcal{P}, q \in \mathcal{Q}\}$ of seminorms of $E \otimes F$ is filtering and thus defines a locally convex topology on $E \otimes F$, called the projective tensor product topology. The vector space $E \otimes F$ equipped with this topology is denoted by $E \otimes_\pi F$, and is called *the projective tensor product of the spaces E and F.*

Similarly, we define the injective tensor product of p and q

$$p \otimes_\varepsilon q(\xi) = \sup \Big\{ \sum_{i=1}^{m} \langle e_i, u \rangle \langle e_i, u \rangle : \xi = \sum_{i=1}^{m} e_i \otimes f_i$$
$$u \in E', v \in F', \sup_{p(x) \leq 1} |\langle u, x \rangle| \leq 1, \sup_{q(y) \leq 1} |\langle v, y \rangle| \leq 1 \Big\}.$$

Then, $E \otimes F$ provided with this topology is denoted as $E \otimes_\varepsilon F$, and it is called *the injective tensor product of the spaces E and F.*

The completions of the algebraic tensor product in these two norms are called the projective and injective tensor products, and are denoted by $E \hat{\otimes}_\pi F$ and $E \hat{\otimes}_\varepsilon F$. It is easy to see that $p \otimes_\pi q(\chi) \geq p \otimes_\varepsilon q(\chi)$ for all χ in $E \otimes F$.

Theorem 1.2.12. *Assume that $X \subseteq \mathbb{R}^m, Y \subseteq \mathbb{R}^n$ are nonempty open sets. The completion of the projective tensor product $\mathcal{D}(X) \otimes_\pi \mathcal{D}(Y)$ of the test function spaces is equal to the test function space $\mathcal{D}(X \times Y)$ over the product $X \times Y$ of the sets X and Y:*

$$\mathcal{D}(X) \hat{\otimes}_\pi \mathcal{D}(Y) = \mathcal{D}(X) \hat{\otimes}_\varepsilon \mathcal{D}(Y) = \mathcal{D}(X \times Y)$$

A nuclear space E is a locally convex Hausdorff space in which $E\otimes_\pi F = E\otimes_\pi F$ for any arbitrary locally convex space F (see [103] for equivalent definitions of nuclear spaces). For nuclear spaces we shall use the common notation

$$E\hat{\otimes}F = E\hat{\otimes}_\pi F = E\hat{\otimes}_\pi F$$

for the topological tensor product.

The inductive limit of a sequence of nuclear spaces, the strong dual of a nuclear Fréchet space and the product of a family of nuclear spaces is also a nuclear space. Let us note that there are no infinite-dimensional Banach spaces that are nuclear. On the other hand the spaces $\mathcal{S}$,$\mathcal{S}'$, $\mathcal{D}'$, $\mathcal{E}'$ are examples of nuclear spaces.

Example 1.2.9. Let $d = dimE$ be finite, $\mathbf{e}_1, \dots, \mathbf{e}_d$ a basis of E, and $\mathbf{e}'_1, \dots, \mathbf{e}'_d$ the dual basis in E' (this means that the functions $\langle \mathbf{e}'_i, \mathbf{e}_j\rangle = 0$ if $i \neq j$ and $\langle \mathbf{e}'_i, \mathbf{e}_j\rangle = 1$ if $i = j$). Consider a function $f \in C^k(Y, E)$. For each $y \in Y$, we may write

$$f(y) = \sum_{j=1}^{n} f_j(y)\mathbf{e}_j;$$

we have

$$f_j(y) = \langle \mathbf{e}'_j, f(y)\rangle \in C^k(Y).$$

Conversely, let g be such a function and $\mathbf{e}$ a vector in E. Let us denote by $g \otimes \mathbf{e}$ the function, valued in E ($y \mapsto g(y)\mathbf{e}$). This means that the function of the form $g \otimes \mathbf{e}$ span $C^k(Y, E)$ when g varies over $C^k(Y)$ and $\mathbf{e}$ over E; thus it is true if and only if E is finite dimensional. Then, the bilinear map $(g, \mathbf{e}) \mapsto g\otimes\mathbf{e}$ of $C^k(Y)\times E$ into $C^k(Y, E)$ turns the latter into a tensor product of $C^k(Y)$ and E.

Now, let E be a vector space over the field of complex numbers, f a complex-valued function defined in $Y \subset \mathbb{R}^n$, and $\mathbf{e}$ a vector belonging to E. We denote by $f \otimes \mathbf{e}$ the function defined in Y and valued in E.

Proposition 1.2.2. *[103, Prop. 40.2.] Let E be a Hausdorff topological vector space. The bilinear mapping $(f, \mathbf{e}) \mapsto f \otimes \mathbf{e}$ of $C^k(Y) \times E$ into the subspace of $C^k(Y, E)$, consisting of the functions whose image is contained in a finite dimensional subspace of E, turns this subspace into a tensor product of $C^k(Y)$ and E (denoted by $C^k(Y) \otimes E$.)*

We shall use the notation $C^k_c(Y) \otimes E$ to denote the subspace of $C^k(Y) \otimes E$ consisting of the functions with compact support.

Theorem 1.2.13. *[103, Thrm. 40.1.] Let X and Y be open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. The mapping*

$$\phi \mapsto (y \mapsto (x \mapsto \phi(x, y))) \tag{1.8}$$

is an isomorphism from $C^\infty(X \times Y)$ onto $C^\infty(Y, C^\infty(X))$.

Corollary 1.2.2. *The restriction of* (1.8) *to $C^\infty_c(X \times Y)$ is an isomorphism of this space into $C^\infty_c(Y, C^\infty_c(X))$.*

Theorem 1.2.14. *[103, Thrm. 40.3.] Let* f *be a distribution in* X*. Then,* $\phi \mapsto (y \mapsto \langle f_x, \phi(x,y)\rangle)$ *is a continuous linear map of* $C_c^\infty(X \times Y)$ *into* $C_c^\infty(Y)$*. If the support of* f *is compact, then it is a continuous linear map of* $C^\infty(X \times Y)$ *into* $C^\infty(Y)$*.*

Here, $\langle f_x, \phi(x,y)\rangle$ means that f is acting on the test function $x \mapsto \phi(x,y)$, with y playing the role of parameter.

Now, let g be a function in Y. By the previous theorem,

$$C_c^\infty(X \times Y) \ni \phi \mapsto \langle g_y, \langle f_x, \phi(x,y)\rangle\rangle$$

defines a distribution in $X \times Y$. Similarly,

$$C_c^\infty(X \times Y) \ni \phi \mapsto \langle f_x, \langle g_y, \phi(x,y)\rangle\rangle$$

is a distribution in $X \times Y$. The next result states that these two distributions are equal. It can be viewed as a kind of rule of interchanging integration with respect to x and y. In analogy with the integration theory, it is often referred to as *Fubini's theorem for distributions.*

Theorem 1.2.15. *[103, Thrm. 40.4.] Let* f *be a distribution in* X *and* g *in* Y*. For every* $\phi \in C_c^\infty(X \times Y)$*, we have*

$$\langle f_x, \langle g_y, \phi(x,y)\rangle\rangle = \langle g_y, \langle f_x, \phi(x,y)\rangle\rangle.$$

Definition 1.2.11. Let f be a distribution in X and g a distribution in Y. The distribution in $X \times Y$

$$C_c^\infty(X \times Y) \ni \phi \mapsto \langle g_y, \langle f_x, \phi(x,y)\rangle\rangle = \langle f_x, \langle g_y, \phi(x,y)\rangle\rangle$$

is called *the tensor product of* f *and* g (or *of* g *and* f) and denoted by

$$f \otimes g \quad \text{or} \quad g \otimes f.$$

Some basic properties of the tensor product are stated with the following Proposition.

Proposition 1.2.3. *(i)* $(f,g) \mapsto f \otimes g$ *is a bilinear map of* $\mathcal{D}'(X) \times \mathcal{D}'(Y)$ *into* $\mathcal{D}'(X \times Y)$*;*

(ii) $\operatorname{supp}(f \otimes g) = (\operatorname{supp} f) \times (\operatorname{supp} g)$;

(iii) $(f,g) \mapsto f \otimes g$ *is a bilinear map of* $\mathcal{E}'(X) \times \mathcal{E}'(Y)$ *into* $\mathcal{E}'(X \times Y)$*;*

(iv) $D_x^\alpha(f_x \otimes g_y) = (D_x^\alpha f_x) \otimes g_y$*;*

Definition 1.2.12. We shall denote by $\mathcal{D}'(X) \otimes \mathcal{D}'(Y)$ the linear subspace of $\mathcal{D}'(X \times Y)$ spanned by the distributions of the form $f \otimes g, f \in \mathcal{D}'(X), g \in \mathcal{D}'(Y)$.

$\mathcal{D}'(X) \otimes \mathcal{D}'(Y)$ is obviously a tensor product of $\mathcal{D}'(X)$ and $\mathcal{D}'(Y)$.

Proposition 1.2.4. *[103, Prop. 40.4.]* $\mathcal{D}'(X) \otimes \mathcal{D}'(Y)$ *(resp.* $\mathcal{E}'(X) \otimes \mathcal{E}'(Y)$*) is a dense subspace of* $\mathcal{D}'(X \times Y)$ *(resp.* $\mathcal{E}'(X \times Y)$*).*

The tensor product of distributions $(f, g) \mapsto f \otimes g$ is a continuous bilinear mapping of $\mathcal{S}'(\mathbb{R}^m) \times \mathcal{S}'(\mathbb{R}^n)$ into $\mathcal{S}'(\mathbb{R}^{m+n})$. (The same holds if we replace $\mathcal{S}'$ with $\mathcal{E}'$).

Proposition 1.2.5. *[103, Prop. 50.7] Let E and F be two Fréchet spaces. If F is nuclear, then*

$$E' \hat{\otimes} F' \cong (E \hat{\otimes} E)'.$$

By [103, Thrm. 51.6] we have the following isomorphisms

(*i*) $\mathcal{S}(\mathbb{R}^m)\hat{\otimes}\mathcal{S}(\mathbb{R}^n) \cong \mathcal{S}(\mathbb{R}^{m+n})$;

(*ii*) $\mathcal{S}'(\mathbb{R}^m)\hat{\otimes}\mathcal{S}'(\mathbb{R}^n) \cong L(\mathcal{S}(\mathbb{R}^m); \mathcal{S}'(\mathbb{R}^n)) \cong \mathcal{S}'(\mathbb{R}^{m+n})$;

(*iii*) $C^\infty(X)\hat{\otimes}\mathcal{D}'(Y) \cong C^\infty(X; \mathcal{D}'(Y)) \quad (\mathcal{D}'(X)\hat{\otimes}C^\infty(Y) \cong C^\infty(Y; \mathcal{D}'(X)))$.

Moreover, if E is locally convex, Hausdorff and complete, then

(*i*) $C^\infty(X; E) \cong C^\infty(X)\hat{\otimes}E$;

(*ii*) $\mathcal{S}(\mathbb{R}^n; E) \cong \mathcal{S}(\mathbb{R}^n)\hat{\otimes}E$;

(*iii*) $\mathcal{S}'(\mathbb{R}^n; E) \cong \mathcal{S}'(\mathbb{R}^n)\hat{\otimes}E \cong L(\mathcal{S}(\mathbb{R}^n); E)$;

(*iv*) (The space of E-valued distributions in X) $\mathcal{D}'(X; E) = L(C_c^\infty; E) \cong \mathcal{D}'(X)\hat{\otimes}E$.

where $L(F, E)$ is the space of all continuous linear mapping from F to E provided with the strong topology [103].

1.2.5 Lizorkin distributions

In Chapter 4, of crucial importance will be the space of Lizorkin test functions $\mathcal{S}_0(\mathbb{R}^n)$ of highly time-frequency localized functions over $\mathbb{R}^n$ [37].

The space $\mathcal{S}_0(\mathbb{R}^n)$ consists all those elements of $\mathcal{S}(\mathbb{R}^n)$ having all moments equal to 0, namely, $\phi \in \mathcal{S}_0(\mathbb{R}^n)$ if

$$\int_{\mathbb{R}^n} x^m \phi(x) dx = 0,$$

for all $m \in \mathbb{N}_0^n$. It is a closed subspace of $\mathcal{S}(\mathbb{R}^n)$. Let us point out that other authors use a different notion for this space. For instance, Helgason [33] denotes $\mathcal{S}_0(\mathbb{R}^n)$ by $\mathcal{S}^*(\mathbb{R}^n)$. Its dual space $\mathcal{S}_0'(\mathbb{R}^n)$, known as the space of Lizorkin distributions, is canonically isomorphic to the quotient of $\mathcal{S}'(\mathbb{R}^n)$ by the space of polynomials; the quotient projection $\mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}_0'(\mathbb{R}^n)$ is explicitly given by the restriction of tempered distributions to $\mathcal{S}_0(\mathbb{R}^n)$. This quotient projection is injective on $\mathcal{D}'_{L^1}(\mathbb{R}^n)$; therefore, we can regard $\mathcal{D}'_{L^1}(\mathbb{R}^n)$, $\mathcal{E}'(\mathbb{R}^n)$, and $\mathcal{O}'_C(\mathbb{R}^n)$ as (dense) subspaces of $\mathcal{S}_0'(\mathbb{R}^n)$.

We denote by $\mathcal{D}(\mathbb{S}^{n-1})$ the space of smooth functions on the sphere. Given a locally convex space $\mathcal{A}$ of smooth test functions on $\mathbb{R}$, we write $\mathcal{A}(\mathbb{S}^{n-1}\times\mathbb{R})$ for the space of functions $\varrho(\mathbf{u},p)$ having the properties of $\mathcal{A}$ in the variable $p\in\mathbb{R}$ and being smooth in $\mathbf{u}\in\mathbb{S}^{n-1}$.

We introduce $\mathcal{S}(\mathbb{Y}^{n+1})$ as the space of functions $\Phi\in C^{\infty}(\mathbb{Y}^{n+1})$ satisfying the decay conditions

$$\rho_{s,r}^{l,m,k}(\Phi)=\sup_{(\mathbf{u},b,a)\in\mathbb{Y}^{n+1}}\left(a^{s}+\frac{1}{a^{s}}\right)(1+b^{2})^{r/2}\left|\frac{\partial^{l}}{\partial a^{l}}\frac{\partial^{m}}{\partial b^{m}}\triangle_{\mathbf{u}}^{k}\Phi\left(\mathbf{u},b,a\right)\right|<\infty \qquad (1.9)$$

for all $l,m,k,s,r\in\mathbb{N}_0$, where $\triangle_{\mathbf{u}}$ is the Laplace-Beltrami operator on the unit sphere $\mathbb{S}^{n-1}$. The topology of this space is defined by means of the seminorms (1.9). Its dual $\mathcal{S}'(\mathbb{Y}^{n+1})$ will be fundamental in our definition of the ridgelet transform of Lizorkin distributions, as it contains the range of this transform (cf. Section 4.4). We follow the ensuing convention. We fix $a^{-n}d\mathbf{u}dbda$ as the *standard measure* on $\mathbb{Y}^{n+1}$. Here $d\mathbf{u}$ stands for the surface measure on the sphere $\mathbb{S}^{n-1}$. Accordingly, our convention for identifying a locally integrable function F on $\mathbb{Y}^{n+1}$ with a distribution on $\mathbb{Y}^{n+1}$ is as follows. If it is of slow growth on $\mathbb{Y}^{n+1}$, namely, it satisfies the bound

$$|F\left(\mathbf{u},b,a\right)|\leq C(1+|b|)^{s}\left(a^{s}+\frac{1}{a^{s}}\right),\quad (\mathbf{u},b,a)\in\mathbb{Y}^{n+1},$$

for some $s,C>0$, we shall always identify F with an element of $\mathcal{S}'(\mathbb{Y}^{n+1})$ via

$$\langle F,\Phi\rangle:=\int_{0}^{\infty}\int_{-\infty}^{\infty}\int_{\mathbb{S}^{n-1}}F\left(\mathbf{u},b,a\right)\Phi\left(\mathbf{u},b,a\right)\frac{d\mathbf{u}dbda}{a^{n}},\quad \Phi\in\ \mathcal{S}\left(\mathbb{Y}^{n+1}\right). \qquad (1.10)$$

A related space is $\mathcal{S}(\mathbb{H})$, the space of highly localized test functions on the upper half-plane [37]. Its elements are smooth functions Ψ on $\mathbb{H}$ that satisfy

$$\sup_{(b,a)\in\mathbb{H}}\left(a^{s}+\frac{1}{a^{s}}\right)(1+b^{2})^{r/2}\left|\frac{\partial^{m}}{\partial b^{m}}\frac{\partial^{l}}{\partial a^{l}}\Psi\left(b,a\right)\right|<\infty,$$

for all $l,m,s,r\in\mathbb{N}_0$; its topology being defined in the canonical way [37]. The dual space is denotes with $\mathcal{S}'(\mathbb{H})$ and any locally integrable function F of slow growth on $\mathbb{H}$, that is

$$|F(b,a)|\leq C\left(a+\frac{1}{a}\right)^{m}(1+b^{2})^{\frac{l}{2}},\quad (b,a)\in\mathbb{H},$$

for some $C>0$ and integers $m,l\in\mathbb{N}$ can be identified with an element of $\mathcal{S}'(\mathbb{H})$

Observe that the nuclearity of the Schwartz spaces [103] immediately yields the equalities $\mathcal{S}(\mathbb{Y}^{n+1})=\mathcal{D}(\mathbb{S}^{n-1})\hat{\otimes}\mathcal{S}(\mathbb{H})$, $\mathcal{S}(\mathbb{S}^{n-1}\times\mathbb{R})=\mathcal{D}(\mathbb{S}^{n-1})\hat{\otimes}\mathcal{S}(\mathbb{R})$, and $\mathcal{S}_0(\mathbb{S}^{n-1}\times\mathbb{R})=\mathcal{D}(\mathbb{S}^{n-1})\hat{\otimes}\mathcal{S}_0(\mathbb{R})$, where $X\hat{\otimes}Y$ is the topological tensor product space obtained as the completion of $X\otimes Y$ in, say, the π-topology or the $\varepsilon-$topology [103].

1.2.6 Distributions of exponential type

The Hasumi-Silva [90, 31] test function space $\mathcal{K}_1(\mathbb{R}^n)$ consists of those $\varphi \in C^\infty(\mathbb{R}^n)$ for which all norms

$$\nu_k(\varphi) := \sup_{x\in\mathbb{R}^n,\ |\alpha|\leq k} e^{k|x|}|\varphi^{(\alpha)}(x)|, \quad k \in \mathbb{N}_0,$$

are finite. The elements of $\mathcal{K}_1(\mathbb{R}^n)$ are called exponentially rapidly decreasing smooth functions. It is easy to see that $\mathcal{K}_1(\mathbb{R}^n)$ is an FS-space and therefore Montel and reflexive. The space $\mathcal{K}_1(\mathbb{R}^n)$ is also nuclear [31]. The following topological inclusions are clear

$$\mathcal{D}(\mathbb{R}^n) \hookrightarrow \mathcal{K}_1(\mathbb{R}^n) \hookrightarrow \mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n) \hookrightarrow \mathcal{K}_1'(\mathbb{R}^n) \hookrightarrow \mathcal{D}'(\mathbb{R}^n).$$

Note that if $\varphi \in \mathcal{K}_1(\mathbb{R}^n)$, then the Fourier transform (1.5) extends to an entire function. In fact, the Fourier transform is a topological isomorphism from $\mathcal{K}_1(\mathbb{R}^n)$ onto $\mathcal{U}(\mathbb{C}^n)$, the space of entire functions which decrease faster than any polynomial in bands. More precisely, a entire function $\phi \in \mathcal{U}(\mathbb{C}^n)$ if and only if

$$\dot{\nu}_k(\phi) := \sup_{x\in\Pi_k}(1+|x|^2)^{k/2}|\phi(x)| < \infty, \quad \forall k \in \mathbb{N}_0,$$

where Π_k is the tube $\Pi_k = \mathbb{R}^n + i[-k,k]^n$.

The dual space $\mathcal{K}_1'(\mathbb{R}^n)$ consists of all distributions f of exponential type, i.e., those of the form $f = \sum_{|\alpha|\leq l}(e^{s|\,\cdot\,|}f_\alpha)^{(\alpha)}$, where $f_\alpha \in L^\infty(\mathbb{R}^n)$ [31]. The Fourier transform extends to a topological isomorphism $\mathcal{F} : \mathcal{K}_1'(\mathbb{R}^n) \to \mathcal{U}'(\mathbb{C}^n)$, the latter space is known as the space of Silva tempered ultradistributions [31] (also called the space of tempered ultra-hyperfunctions [58]). The space $\mathcal{U}'(\mathbb{C}^n)$ contains the space of analytic functionals. See also the textbook [38] for more information about these spaces.

In Chapter 3, we will study the short-time Fourier transform in the context of the space $\mathcal{K}_1'(\mathbb{R}^n)$, and we will obtain various characterizations of $\mathcal{K}_1'(\mathbb{R}^n)$ and related spaces via the short-time Fourier transform. The space $\mathcal{K}_1'(\mathbb{R}^n)$ was introduced by Silva [90] and Hasumi [31] in connection with the so-called space of Silva tempered ultradistributions $\mathcal{U}'(\mathbb{C}^n)$. Let us mention that $\mathcal{K}_1'(\mathbb{R}^n)$ and $\mathcal{U}'(\mathbb{R}^n)$ were also studied by Morimoto through the theory of ultra-hyperfunctions [58] (cf. [60]). We refer to [19, 38, 91, 121] for some applications of the Silva spaces.

We introduce a generalization of the Schwartz space of bounded distributions $\mathcal{B}'(\mathbb{R}^n)$ [89, p. 200]. Let $\omega : \mathbb{R}^n \to (0,\infty)$ be an exponentially moderate weight, namely, ω is measurable and satisfies the estimate

$$\omega(x+y) \leq A\omega(y)e^{a|x|}, \quad x,y \in \mathbb{R}^n, \tag{1.11}$$

for some constants $A > 0$ and $a \geq 0$. For instance, any positive measurable function ω which is submultiplicative, i.e., $\omega(x+y) \leq \omega(x)\omega(y)$, and integrable near the origin must necessarily satisfy (1.11), as follows from the standard results about subadditive functions [1, 36]. Extending the Schwartz space $\mathcal{D}_{L^1}(\mathbb{R}^n)$, we define the Fréchet space $\mathcal{D}_{L^1_\omega}(\mathbb{R}^n) = \{\varphi \in C^\infty(\mathbb{R}^n) : \varphi^{(\alpha)} \in L^1_\omega(\mathbb{R}^n), \forall \alpha \in \mathbb{N}_0^n\}$, provided with the family of norms

$$\|\varphi\|_{1,\omega,k} := \sup_{|\alpha|\le k} \int_{\mathbb{R}^n} |\varphi^{(\alpha)}(x)|\omega(x)dx, \quad k \in \mathbb{N}_0.$$

Then, $\mathcal{B}'_\omega(\mathbb{R}^n)$ stands for the strong dual of $\mathcal{D}_{L^1_\omega}(\mathbb{R}^n)$, i.e., $\mathcal{B}'_\omega(\mathbb{R}^n) = (\mathcal{D}_{L^1_\omega}(\mathbb{R}^n))'$. Since we have the dense embedding $\mathcal{K}_1(\mathbb{R}^n) \hookrightarrow \mathcal{D}_{L^1_\omega}(\mathbb{R}^n)$, we have $\mathcal{B}'_\omega(\mathbb{R}^n) \subset \mathcal{K}'_1(\mathbb{R}^n)$. We call $\mathcal{B}'_\omega(\mathbb{R}^n)$ the space of ω-bounded distributions. We also define $\dot{\mathcal{B}}'_\omega(\mathbb{R}^n)$ as the closure of $\mathcal{D}(\mathbb{R}^n)$ in $\mathcal{B}'_\omega(\mathbb{R}^n)$.

Next, we shall consider $\mathcal{K}_1(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}(\mathbb{C}^n)$, the topological tensor product space obtained as the completion of $\mathcal{K}_1(\mathbb{R}^n) \otimes \mathcal{U}(\mathbb{C}^n)$ in, say, the π- or the ε- topology [103]. Explicitly, the nuclearity of $\mathcal{K}_1(\mathbb{R}^n)$ implies that

$$\mathcal{K}_1(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}(\mathbb{C}^n) = \mathcal{K}_1(\mathbb{R}^n)\widehat{\otimes}_\pi\mathcal{U}(\mathbb{C}^n) = \mathcal{K}_1(\mathbb{R}^n)\widehat{\otimes}_\varepsilon\mathcal{U}(\mathbb{C}^n).$$

Thus, the topology of $\mathcal{K}_1(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}(\mathbb{C}^n)$ is given by the family of the norms

$$\rho_k(\Phi) := \sup_{(x,z)\in\mathbb{R}^n\times\Pi_k,\ |\alpha|\le k} e^{k|x|}(1+|z|^2)^{k/2}\left|\frac{\partial^\alpha}{\partial x^\alpha}\Phi(x,z)\right|, \quad k \in \mathbb{N}_0,$$

and we also obtain $(\mathcal{K}_1(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}(\mathbb{C}^n))' = \mathcal{K}'_1(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}'(\mathbb{C}^n)$.

1.2.7 Distributions of M-exponential type

Let us introduce the distribution space $\mathcal{K}'_M(\mathbb{R}^n)$. We begin with the test function space $\mathcal{K}_M(\mathbb{R}^n)$. We shall assume that $M : [0,\infty) \to [0,\infty)$ is a continuous increasing function satisfying the following two conditions:

1. $M(t) + M(s) \le M(t+s)$,
2. $M(t+s) \le M(2t) + M(2s)$.

We extend M to $\mathbb{R}^n$ and for simplicity we write $M(x) := M(|x|)$, $x \in \mathbb{R}^n$. It follows from (1) that $M(0) = 0$ and the existence of $A > 0$ such that $M(x) \ge A|x|$. Examples of M are $M(x) = |x|^p$ with any $p \ge 1$. More generally, any function of the form

$$M(t) = \int_0^t \eta(s)ds, \quad t \ge 0,$$

satisfies (1) and (2), provided that η is a continuous non-decreasing function with $\eta(0) = 0$ and $\lim_{t\to\infty}\eta(t) = \infty$.

Using the function M, we define the following family of norms:

$$\nu_{r,l}(\varphi) := \sup_{|\alpha|\le r,\, x\in\mathbb{R}^n} e^{M(lx)}|\varphi^{(\alpha)}(x)|, \quad r, l \in \mathbb{N}. \tag{1.12}$$

The test function space $\mathcal{K}_M(\mathbb{R}^n)$ consists of all those smooth functions $\varphi \in C^\infty(\mathbb{R}^n)$ for which all the norms (1.12) are finite. We call its strong dual $\mathcal{K}'_M(\mathbb{R}^n)$, the space of distributions of "M-exponential" growth at infinity. A standard argument shows that a distributions $f \in \mathcal{D}'(\mathbb{R}^n)$ belongs to $\mathcal{K}'_M(\mathbb{R}^n)$ if and only if it has the form $f = \partial_x^\alpha \left(e^{M(kx)}F(x)\right)$, where $k \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$, and $F \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$.

Denote as $\mathcal{K}_{M,r,l}(\mathbb{R}^n)$ the Banach space obtained as the completion of $\mathcal{D}(\mathbb{R}^n)$ in the norm (1.12). It is clear that

$$\mathcal{K}_{M,r,l}(\mathbb{R}^n) = \{\varphi \in C^r(\mathbb{R}^n) : \lim_{|x|\to\infty} e^{M(lx)}\varphi^{(\alpha)}(x) = 0,\ |\alpha| \leq r\}.$$

Set $\mathcal{K}_{M,r}(\mathbb{R}^n) = \operatorname{proj}\lim_{l\to\infty} \mathcal{K}_{M,r,l}(\mathbb{R}^n)$. Note that

$$\mathcal{K}_M(\mathbb{R}^n) \hookrightarrow \cdots \hookrightarrow \mathcal{K}_{M,r+1}(\mathbb{R}^n) \hookrightarrow \mathcal{K}_{M,r}(\mathbb{R}^n) \hookrightarrow \cdots \hookrightarrow \mathcal{K}_{M,0}(\mathbb{R}^n),$$

where each embedding in this projective sequence is compact, due to the Arzelà-Ascoli theorem and the property (1). Consequently, the embeddings $\mathcal{K}'_{M,r}(\mathbb{R}^n) \to \mathcal{K}'_{M,r+1}(\mathbb{R}^n)$ are also compact,

$$\mathcal{K}_M(\mathbb{R}^n) = \operatorname{proj}\lim_{r\to\infty} \mathcal{K}_{M,r}(\mathbb{R}^n),$$

and

$$\mathcal{K}'_M(\mathbb{R}^n) = \bigcup_{r\in\mathbb{N}} \mathcal{K}'_{M,r}(\mathbb{R}^n) = \operatorname{ind}\lim_{r\to\infty} \mathcal{K}'_{M,r}(\mathbb{R}^n). \tag{1.13}$$

Therefore $\mathcal{K}_M(\mathbb{R}^n)$ is an FS-space and $\mathcal{K}'_M(\mathbb{R}^n)$ a DFS-space. In particular, they are Montel and hence reflexive.

For the Schwartz spaces $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$, we can also employ useful projective and inductive presentations with similar compact inclusion relations. Define $\mathcal{S}_{r,l}(\mathbb{R}^n)$ as the completion of $\mathcal{D}(\mathbb{R}^n)$ with the norm

$$\rho_{r,\,l}(\varphi) := \sup_{|\alpha|\leq r,\, x\in\mathbb{R}^n} (1+|x|)^l |\varphi^{(\alpha)}(x)|, \quad r, l \in \mathbb{N},$$

and set $\mathcal{S}_r(\mathbb{R}^n) = \operatorname{proj}\lim_{l\to\infty} \mathcal{S}_{r,l}(\mathbb{R}^n)$. Thus, $\mathcal{S}(\mathbb{R}^n) = \operatorname{proj}\lim_{r\to\infty} \mathcal{S}_r(\mathbb{R}^n)$ and

$$\mathcal{S}'(\mathbb{R}^n) = \bigcup_{r\in\mathbb{N}} \mathcal{S}'_r(\mathbb{R}^n) = \operatorname{ind}\lim_{r\to\infty} \mathcal{S}'_r(\mathbb{R}^n). \tag{1.14}$$

The following simple but useful lemma describes convergence of filters with bounded bases in the Fréchet space $\mathcal{K}_{M,r}(\mathbb{R}^n)$ (resp. $\mathcal{S}_r(\mathbb{R}^n)$), we leave the proof to the reader. Recall that the canonical topology on $C^r(\mathbb{R}^n)$ is that of uniform convergence of functions and all their derivatives up to order r on compact subsets.

Lemma 1.2.1. *Let $\mathcal{F}$ be a filter with bounded basis over $\mathcal{K}_{M,r}(\mathbb{R}^n)$ (resp. $\mathcal{S}_r(\mathbb{R}^n)$). Then $\mathcal{F} \to \varphi$ in $\mathcal{K}_{M,r}(\mathbb{R}^n)$ (resp. in $\mathcal{S}_r(\mathbb{R}^n)$) if and only if $\mathcal{F} \to \varphi$ in $C^r(\mathbb{R}^n)$.*

Chapter 2

Quasiasymptotics and S-asymptotics

In the last five decades many definition of the asymptotic behavior of distributions have been given by different authors. The main goal of this chapter if to give a short survey to those results that are of most relevance for their application in this dissertation, mainly to Abelian and Tauberian type theorems. The analysis of this notion is given using regularly varying functions [93].
The quasiasymptotic behavior of distributions was introduced by Zavialov as a result of his investigations in Quantum Field Theory, and further developed by him, Vladimirov and Drozhzhinov, [113]. Great contribution to this theory is made by Pilipović and his coworkers, [69, 65, 67, 62, 105, 106, 110, 86, 87, 88].
In the last few decades a theory of S-asymptotics (shift-asymptotics) has been presented and developed. It has an origin in the book of L. Schwartz [89], and more about the good properties of this notion can be found in [65, 73].

2.1 Slowly varying functions

We will measure the behavior of a distribution by comparison with Karamata regularly varying functions [93], defined in the early thirties as a natural generalization of power functions. We start with regularly varying functions at infinity.

Definition 2.1.1. A function $\rho : (A, \infty) \to \mathbb{R}, A \in \mathbb{R}$ is called *regularly varying at infinity* if it is a positive, measurable and there exists a real number α such that for every $x > 0$

$$\lim_{k\to\infty} \frac{\rho(kx)}{\rho(k)} = x^{\alpha}. \tag{2.1}$$

The number α is called index of ρ. Specially, if $\alpha = 0$, then ρ is called slowly varying function and for such a function it will be used the letter "L". In fact, the following assertion is obvious.

Proposition 2.1.1. *A positive and measurable function $\rho : (A, \infty) \to \mathbb{R}$ is regularly varying at infinity if and only if it can be written as*

$$\rho(x) = x^{\alpha} L(x), x > A, \tag{2.2}$$

for some real number α and some slowly varying function L at infinity.

Hence, it is enough to explore the properties of slowly varying functions in order to study those of regularly varying functions.

Example 2.1.1. The functions

$$x^\alpha, x^\alpha \ln x, x^\alpha \ln\ln x, x^\alpha \left(1 + \frac{1}{2}\sin\ln\ln x\right), x^\alpha \left(2 + \sin\sqrt{\ln x}\right), \alpha \in \mathbb{R},$$

are slowly varying with index α.

One can prove that the convergence in (2.1) is uniform on every fixed interval $[a', b'], A < a' < b' < \infty$, and that ρ is necessarily bounded (hence integrable) on it. Some properties of slowly varying functions are stated with the next proposition.

Let L be slowly varying function at infinity. Then, for each $\varepsilon > 0$

(i) there exist constants $C_1, C_2 > 0$ and $X > A$ such that

$$C_1 x^{-\varepsilon} \leq L(x) \leq C_2 x^\varepsilon, \quad \text{for} \quad x \geq X; \tag{2.3}$$

(ii) we have $\lim_{x\to\infty} x^\varepsilon L(x) = +\infty,\ \lim_{x\to\infty} x^{-\varepsilon} L(x) = 0$;

The first two statements together with relation (2.2) explain the relation of the regularly varying functions to power functions, while the third shows that such functions with positive index are asymptotically equal at infinity to monotone ones. It is also known that if $L_2(x) \to \infty, x \to \infty$, and L_1, L_2 are slowly varying, then their composition $L_1(L_2)$ is slowly varying, as well. Hence, for $x > -\infty$

$$\lim_{h\to\infty} \frac{L(x+h)}{L(h)} = \lim_{u\to\infty} \frac{L(\log ut)}{L(\log u)} = 1.$$

Definition 2.1.2. A function $\rho : (0, A) \to \mathbb{R}, A > 0$ is regularly varying at zero (from the right) if the function $\rho_1(x) = \rho(1/x)$ is regularly varying at infinity.

The following representation holds: L is slowly varying function at zero on the interval $(0, A], A > 0$ if and only if there exists measurable functions u and ω defined on the interval $(0, B], B \leq A$, such that u is bounded and $\lim_{x\to 0} u(x) = M < \infty$, and ω is continuous on $[0, B]$ and $\lim_{x\to 0} \omega(x) = 0$, and for which

$$L(x) = \exp\left(u(x) + \int_x^B \frac{\omega(t)}{t} dt\right),\ x \in (0, B].$$

A similar representation formula holds for slowly varying functions at infinity.

Some useful properties of a slowly varying function L are:

(i) For every $\alpha > 0$

$$L(\varepsilon) = o\left(\frac{1}{\varepsilon^\alpha}\right) \quad \text{when} \quad \varepsilon \to 0^+, \tag{2.4}$$

$$L(\lambda) = o\left(\lambda^\alpha\right) \quad \text{when} \quad \lambda \to \infty, \tag{2.5}$$

(ii) For every $\alpha > 0$

$$\frac{1}{C}\min\{x^{-\alpha}, x^\alpha\} < \frac{L(\varepsilon x)}{L(\varepsilon)} < C\max\{x^{-\alpha}, x^\alpha\}, \quad x, \varepsilon \in (0, \infty),$$

for some $C > 0$.

2.2 Quasiasymptotic behavior of distributions

2.2.1 Quasiasymptotic behavior at zero

In general, we cannot talk about pointwise behavior of distributions, therefore, if we want to study asymptotic properties of distributions, we should usually introduce new parameters in order to give sense to asymptotic relations.

The quasiasymptotic behavior will be a fundamental concept in the subsequent chapters in this thesis. It is a very convenient notion to describe the local behavior of a distribution around a point, or its asymptotic behavior at infinity. One gains generality by considering quasiasymptotics rather than ordinary asymptotics of functions because they are directly applicable to the nature of a distribution; moreover, one might say that every distribution shows, in one way or another, quasiasymptotic properties. Despite its generality, the concept is extremely useful in practice; in fact, it has an evident advantage over the asymptotics of ordinary function: its flexibility under analytical manipulations such as differentiation or integral transforms. We now define the concept of quasiasymptotic behavior.

Definition 2.2.1. Let L be a slowly varying function at the origin and let $x_0 \in \mathbb{R}^n$. We say that the distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ has *quasiasymptotic behavior* (*quasi-asymptotics*) of degree $\alpha \in \mathbb{R}$ at the point $x_0 \in \mathbb{R}^n$ with respect to L if the following limit

$$\lim_{\varepsilon \to 0^+} \left\langle \frac{f(x_0 + \varepsilon x)}{\varepsilon^\alpha L(\varepsilon)},\ \varphi(x) \right\rangle . \tag{2.6}$$

exists and is finite for every $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

From the Banach-Steinhaus theorem follows that if $f \in \mathcal{D}'(\mathbb{R}^n)$ has quasi-asymptotic behavior in sense of Definition 2.2.1, then there exists distribution $g \in \mathcal{D}'(\mathbb{R}^n)$ such that

$$\lim_{\varepsilon \to 0^+} \left\langle \frac{f(x_0 + \varepsilon x)}{\varepsilon^\alpha L(\varepsilon)},\ \varphi(x) \right\rangle = \langle g(x), \varphi(x) \rangle. \tag{2.7}$$

for every $\varphi \in \mathcal{D}(\mathbb{R}^n)$. We will use the following convenient notation for the quasiasymptotic behavior,

$$f(x_0 + \varepsilon x) \sim^q \varepsilon^\alpha L(\varepsilon) g(x) \quad \text{as} \quad \varepsilon \to 0^+ \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n)\ ,$$

or,

$$f(x_0 + \varepsilon x) = \varepsilon^\alpha L(\varepsilon) g(x) + o(\varepsilon^\alpha L(\varepsilon)) \quad \text{as} \quad \varepsilon \to 0^+ \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n)\ ,$$

which should always be interpreted in the weak topology of $\mathcal{D}'(\mathbb{R}^n)$, i.e., in the sense of (2.7). A trivial observation is that, by shifting to x_0, in most cases it is enough to consider $x_0 = 0$.

One can prove that g cannot have an arbitrary form; indeed, it must be homogeneous with degree of homogeneity α, i.e., $g(ax) = a^\alpha g(x)$, for all $a \in \mathbb{R}_+$ [65, 113].

Proposition 2.2.1. *([73, Prop. 2.8]) Let* $f \in \mathcal{D}'(\mathbb{R})$ *and let* $f \sim^q g$ *at* 0 *related to* $\varepsilon^v L(\varepsilon)$ *in* $\mathcal{D}'(\mathbb{R})$*. Then:*

(*i*) $f^{(m)} \sim^q g^{(m)}$ *at* 0 *related to* $\varepsilon^{v-m}L(\varepsilon)$ *in* $\mathcal{D}'(\mathbb{R}), m \in \mathbb{N}$;

(*ii*) $x^m f(x) \sim^q x^m g(x)$ *at* 0 *related to* $\varepsilon^{v+m}L(\varepsilon)$ *in* $\mathcal{D}'(\mathbb{R}), m \in \mathbb{N}$.

The next proposition asserts that the quasi-asymptotics at 0 is a local property.

Proposition 2.2.2. *([73, Prop. 2.9]) Let* $f \in \mathcal{D}'(\mathbb{R})$ *and* $f \sim^q g$ *at* 0 *related to* $\varepsilon^v L(\varepsilon)$ *in* $\mathcal{D}'(\mathbb{R})$, *and let* $f_1 \in \mathcal{D}'(\mathbb{R})$ *be such that* $f = f_1$ *in some neighborhood of zero. Then* $f_1 \sim^q g$ *at* 0 *related to* $\varepsilon^v L(\varepsilon)$, *as well.*

The same assertions from Propositions (2.2.1) and (2.2.2) hold for quasi-asymptotics at 0 in $\mathcal{S}'(\mathbb{R})$.

The following proposition gives the relation between the asymptotic behavior at zero of a locally integrable function and the quasi-asymptotics at zero of the distribution defined by it.

Proposition 2.2.3. *([73, Prop. 2.10]) Let* $f \in \mathcal{S}'(\mathbb{R})$ *be a locally integrable function in* $(-a, a), a > 0$, *and let* $c(\varepsilon) = \varepsilon^\alpha L(\varepsilon), \alpha > -1$ *where* L *be slowly varying function at* 0^+. *If*

$$\lim_{x \to 0^\pm} f(x)/c(|x|) = C_\pm,$$

then f *has the quasi-asymptotics at zero in* $\mathcal{S}'(\mathbb{R})$ *related to* c *and*

$$\lim_{\varepsilon \to 0^+} f(\varepsilon x)/c(\varepsilon) = C_+ x_+^\alpha + C_- x_-^\alpha \quad \text{in} \quad \mathcal{S}'(\mathbb{R}).$$

Proposition 2.2.4. *([73, Prop. 2.12]) Let* $f \in \mathcal{S}'(\mathbb{R})$ *and* $f = F^{(m)}$ *in some neighborhood of* 0, *where* $m \in \mathbb{N}_0$ *and* F *is a locally integrable function such that for some* $v > -1$ *and some slowly varying function* L,

$$\lim_{x \to 0^\pm} \frac{F(x)}{|x|^v L(|x|)} = C_\pm.$$

Then $f \sim^q g$ *at zero related to* $\varepsilon^v L(\varepsilon)$ *in* $\mathcal{S}'(\mathbb{R})$ *where* $g = (C_+ x_+^v + C_- x_-^v)^{(m)}$.

A deeper result of Pilipović and Vindas asserts that the converse to Proposition 2.2.4 also holds true, see [73, Sect. 2.10] or [106].

It is obvious that if a tempered distribution has quasiasymptotics at 0 in $\mathcal{S}'$, then it will have it in $\mathcal{D}'$. It was shown in [64] that if L is bounded near the origin and $\alpha < 0, \alpha \notin \{-1, -2, -3, \ldots\}$, then the converse is true. The following theorem shows that the converse is true without any restrictions on L and α.

Theorem 2.2.1. *([73, Thrm. 2.35])Let* $f \in \mathcal{S}'(\mathbb{R}^n)$. *If* f *has quasi-asymptotic behavior at* 0 *in* $\mathcal{D}'(\mathbb{R}^n)$, *then* f *has the same quasi-asymptotic behavior at* 0 *in the space* $\mathcal{S}'(\mathbb{R}^n)$.

We will considerably extend Theorem 2.2.1 in Section 5.4.

Let us note that in [105, 106, 111, 73] authors give complete structural theorems for quasiasymptotics at the origin and at infinity. Their results are based on the concept of asymptotically and associate asymptotically homogeneous functions. This concept is used for obtaining easier proofs for various structural theorems when the degree of the quasiasymptotics in not negative, but also when the degree of the quasiasymptotic is a negative integer.

If $\alpha = 0$ and $L \equiv 1$, the definition of quasiasymptotics at x_0 in $\mathcal{D}'(\mathbb{R}^n)$ is a slight generalization of the Łojasiewicz definition of the distributional "value at x_0". Actually, as explained in Section 1.1, f has the distributional point value γ in the sense of Łojasiewicz, [49] and we write

$$f(x_0) = \gamma$$

if and only if

$$\lim_{\varepsilon\to 0} f(x_0 + \varepsilon x) = \gamma, \tag{2.8}$$

distributionaly, that is, if and only if

$$\lim_{\varepsilon\to 0}\langle f(x_0+\varepsilon x), \phi(x)\rangle = \lim_{\varepsilon\to 0}\left\langle f(x), \frac{1}{\varepsilon}\phi\left(\frac{x-x_0}{\varepsilon}\right)\right\rangle = \gamma\int_{-\infty}^{\infty}\phi(x)dx, \qquad \phi\in\mathcal{D}(\mathbb{R}^n).$$

In particular, this is a local concept. So, in the notation of quasiasymptotics, the limit (2.8) may be written as

$$f(x_0 + \varepsilon x) = \gamma + o(1) \quad \text{as} \quad \varepsilon \to 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n).$$

Łojasiewicz gave himself a structural characterization of distributional point values. It was shown by him [50] that the existence of the point value $f(x_0) = \gamma$, distributionaly, is equivalent to the existence of $m \in \mathbb{N}$, and a primitive of order m of f, that is $F^{(m)} = f$, which is continuous in a neighborhood of x_0 and satisfies

$$\lim_{x\to x_0}\frac{m!F(x)}{(x-x_0)^m} = \gamma.$$

2.2.2 Quasiasymptotic behavior at infinity

The quasiasymptotics of distributions at infinity with respect to a slowly varying function L at infinity is defined in a similar manner,

Definition 2.2.2. Let L be a slowly varying function at infinity. We say that the distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ has *quasiasymptotic behavior* (*quasiasymptotics*) of degree $\alpha \in \mathbb{R}$ at infinity with respect to L if there exists a distribution $g \in \mathcal{D}'(\mathbb{R}^n)$ such that

$$\lim_{\varepsilon\to\infty}\left\langle \frac{f(\lambda x)}{\lambda^\alpha L(\lambda)},\ \varphi(x)\right\rangle = \langle g(x), \varphi(x)\rangle, \tag{2.9}$$

for every $\varphi \in \mathcal{D}(\mathbb{R}^n)$, and the notation

$$f(\lambda x) \sim^q \lambda^\alpha L(\lambda) g(x) \quad \text{as} \quad \lambda\to\infty \quad \text{in} \quad \mathcal{D}'(\mathrm{R}^\mathrm{n})$$

or,

$$f(\lambda x) = \lambda^\alpha L(\lambda) g(x) + o(\lambda^\alpha L(\lambda)) \quad \text{as} \quad \lambda\to\infty \quad \text{in} \quad \mathcal{D}'(\mathrm{R}^\mathrm{n})$$

will be used in this case. Let us note that the quasiasymptotic behavior at infinity is a global property. Several properties of the quasi-asymptotics at $\pm\infty$ are given with the following theorems.

Theorem 2.2.2. *([73, Thrm. 2.11]) Let $f \in \mathcal{D}'(\mathbb{R})$ and let $f \sim^q g$ at $\pm\infty$ related to $\lambda^v L(\lambda)$ in $\mathcal{D}'(\mathbb{R})$. Then:*

(i) $f^{(m)} \sim^q g^{(m)}$ *at* $\pm\infty$ *related to* $\lambda^{v-m}L(\lambda),\ \ \lambda > A, m \in \mathbb{N}$;

(ii) $x^m f(x) \sim^q x^m g(x)$ *at* $\pm\infty$ *related to* $\lambda^{v+m}L(\lambda),\ \ m \in \mathbb{N}$;

Theorem 2.2.3. *([73, Thrm. 2.12]) Let $f \in \mathcal{E}(\mathbb{R})$ and $f \sim^q g$ at $\pm\infty$ related to $\lambda^v L(\lambda), g \neq 0$. Then $L(\lambda) = 1, \lambda > A, v \in -\mathbb{N}$, and $g(x) = C\delta^{(-v-1)}(x)$, for some constant C.*

Theorem 2.2.4. *([73, Th. 2.13]) Let F be a locally integrable function on $\mathbb{R}$ such that for some $v \in \mathbb{R}, v > -1$,*

$$\lim_{x\to\pm\infty} \frac{F(x)}{|x|^v L(|x|)} = C_\pm,$$

where L is some slowly varying function at ∞. Then $F \sim^q g$ at $\pm\infty$ related to $\lambda^v L(\lambda)$ where $g = (C_+ x_+^v + C_- x_-^v)^{(m)}$.

Theorem 2.2.5. *([73, Thrm. 2.14]) Let $f \in \mathcal{D}'(\mathbb{R})$ and $f \sim^q g$ at $\pm\infty$ related to $\lambda^v L(\lambda)$, where $g \neq 0$ and $v \in \mathbb{R}\backslash(-\mathbb{N})$. There are $m \in \mathbb{N}_0$ and a locally integrable function F such that*

$$f = F^{(m)} \quad \textit{and} \quad \lim_{x\to\pm\infty} \frac{F(x)}{|x|^{v+m} L(|x|)} = C_\pm,$$

where $C_+, C_- \neq (0,0)$.

In the following theorem, we compare the quasiasymptotics at zero and infinity via the Fourier transform.

Theorem 2.2.6. *([73, Thrm. 2.16]) Let $f \in \mathcal{D}'(\mathbb{R})$ and $v \in \mathbb{R}\backslash(-\mathbb{N})$. If*

$$f \sim^q g \quad \textit{at} \quad \pm\infty \quad \textit{related} \quad \textit{to} \quad \lambda^v L(\lambda), \tag{2.10}$$

with $g \neq 0$, then

$$\lim_{\lambda\to\infty} \frac{\hat{f}(x/\lambda)}{(1/\lambda)^{-v-1} L_1(1/\lambda)} = \hat{g}(x) \quad \textit{in} \quad \mathcal{S}'(\mathbb{R}) \tag{2.11}$$

where $L_1(\,\cdot\,) = L(1/\cdot)$ is slowly varying at the origin.

Conversely, if $f \in \mathcal{S}'(\mathbb{R})$ and (2.11) holds with $v \in \mathbb{R}$, then (2.10) holds, as well.

Remark 2.2.1. We may also consider quasiasymptotics in other distribution spaces. The relation $f(x_0 + \varepsilon x) \sim \varepsilon^\alpha L(\varepsilon) u(x)$ as $\varepsilon \to 0^+$ in $\mathcal{A}'(\mathbb{R})$ means that (2.7) is satisfied just for each $\varphi \in \mathcal{A}(\mathbb{R})$; and analogously for quasiasymptotics at infinity in $\mathcal{A}'(\mathbb{R})$.

Example 2.2.1. Let $f(x) = H(x-a)F(x), x \in \mathbb{R}$ for $a > 0$, where F i a locally integrable function such that $F(x) \sim x^\alpha L(x)$ as $x \to \infty$ for $\alpha > -1$. Then f has the quasiasymptotics related to $\lambda^\alpha L(\lambda),\ \lambda \to \infty$.

2.3 S-asymptotic behavior of distributions

2.3.1 Definition of S-asymptotics and basic properties

Here we will discussed the so-called S-asymptotic behavior of distributions (also known as shift-asymptotics or Schwartz-asymptotics). We briefly explain this notion; we refer to [73] for a complete treatment of the subject. The idea of the S-asymptotics is to study the asymptotic properties of the translates $T_{-h}f$ with respect to a locally bounded and measurable comparison function $c : \mathbb{R} \to (0, \infty)$.

Definition 2.3.1. Let $f \in \mathcal{D}'(\mathbb{R}^n)$. It is said that f has S-asymptotic behavior with respect to c if there is $g \in \mathcal{D}'(\mathbb{R}^n)$ such that

$$\lim_{|h|\to\infty} \left\langle \frac{f(x+h)}{c(h)}, \varphi(x) \right\rangle = \langle g(x), \varphi(x) \rangle \quad \varphi \in \mathcal{D}(\mathbb{R}^n). \tag{2.12}$$

We will use the more suggestive notation

$$f(t+h) \sim^S c(h)g(t) \quad \text{ in } \mathcal{D}'(\mathbb{R}^n) \text{ as } |h| \to \infty \tag{2.13}$$

for denoting (2.12), which of course means that $(f * \check{\varphi})(h) \sim c(h) \int_{\mathbb{R}^{\ltimes}} \varphi(t)g(t)dt$ as $|h| \to \infty$, for each $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

This definition is valid also for some subspaces of $\mathcal{D}'(\mathbb{R}^n)$. One only have to suppose that in relation (2.12) φ belongs to the corresponding test function space. The following proposition gives the characteristic properties of the S- asymptotics.

Proposition 2.3.1. *Let $f \in \mathcal{D}'(\mathbb{R})$. If for every $r > 0$ exists h_r such that the sets $\{x \in \mathbb{R} | x \in \operatorname{supp} f \cap (h-r, h+r)\}, |h| \geq h_r$ are empty, then for every $c(h)$:*

$$\lim_{|h|\to\infty} \left\langle \frac{f(x+h)}{c(h)}, \varphi(x) \right\rangle = 0, \quad \varphi \in \mathcal{D}(\mathbb{R}). \tag{2.14}$$

Proposition 2.3.1 shows that the S-asymptotics preserved the natural property of the asymptotics of numerical functions.

Example 2.3.1. From Proposition (2.3.1) it follows that δ which is distribution with support $\{0\}$ has S-asymptotics with limit 0, with respect to every $c(h)$, i.e.

$$\delta(x+h) \sim^S c(h) \cdot 0, \ |h| \to \infty.$$

The quasiasymptotics at infinity does not have the same property. For example, δ has quasiasymptotics of degree -1, i.e.

$$\delta(\lambda x) \sim^q \lambda^{-1} \delta(x), \lambda \to \infty.$$

Theorem 2.3.1. *Let f_1 and f_2 be two distributions equal on a open set $\Omega \in \mathbb{R}$, where Ω has the following property: for every $r > 0$, there exists h_r such that $(-r, r) \subset \{\Omega - h, |h| \geq h_r\}$. If $f_1(x+h) \sim^S c(h)g(x), |h| \to \infty$, then $f_2(x+h) \sim^S c(h)g(x), |h| \to \infty$.*

This last theorem shows that S-asymptotics is a local property, which is not true for quasiasymptotics at infinity. For example, the supports of δ and δ' are the same and equal to $\{0\}$, but δ has quasiasimptotics of degree -1 and δ' of degree -2.

The distribution g is not arbitrary; in fact, one can show [73] that the relation (2.12) forces it to have the form $g(t) = Ce^{\beta\cdot t}$, for some $C \in \mathbb{R}$ and $\beta \in \mathbb{R}$. If $C \neq 0$, one can also prove [73] that c must satisfy the asymptotic relation

$$\lim_{|h|\to\infty} \frac{c(t+h)}{c(h)} = e^{\beta\cdot t}, \quad \text{uniformly for } t \text{ in compact subsets of } \mathbb{R}. \tag{2.15}$$

From now on, we shall always assume that c satisfies (2.15). A typical example of such a c is any function of the form $c(t) = e^{\beta\cdot t}L(e^{|t|})$, where L is a Karamata slowly varying function [2]. The assumption (2.15) implies [73] that (2.12) actually holds in the space $\mathcal{D}'(\mathbb{R})$.

Let us note that an explicit form of the function c is not known in n-dimensional case, $n \geq 2$. This problem is related to the extension of the definition of a regularly varying function to the multi-dimensional case, [65].

Example 2.3.2. [65, 73]

1. $e^{a(x+h)} \sim^s e^{ah}e^{ax}, h \in \mathbb{R}^n$.

2. For a slowly varying function $L(t), t \geq \alpha > 0$, we have

$$L(t+h) \sim^s L(h)\cdot 1, h \in \mathbb{R}_+.$$

3. Let $f \in L^1(\mathbb{R})$. Then, the distribution defined by f has S-asymptotic behavior related to $c = 1$ and limit $g = 0$.

4. If $f \in \mathcal{S}'$, then there exists a real number k such that f has S-asymptotic behavior related to $c(h)\|h\|^k$, where $c(h)$ tends to infinity as $\|h\| \to \infty, h \in \mathbb{R}^n$ and with limit $g = 0$.

5. If $f \in \mathcal{K}_1'$, then there exists $k \in \mathbb{N}_0$ such that f has S-asymptotic behavior related to $c(h)e^{k\|h\|}$, where $c(h)$ tends to infinity as $\|h\| \to \infty, h \in \mathbb{R}^n$ and with limit $g = 0$.

2.3.2 S-asymptotics and asymptotics of a function

In this section we will compare the asymptotic behavior of a locally integrable function f and the S-asymptotic behavior of the generalized function generated by it. Let us note that a function f has *asymptotics* at infinity if there exists a positive function c such that

$$\lim_{x\to\infty} \frac{f(x)}{c(x)} = A \neq 0, \quad (\text{in} \quad \text{short} \quad f(x) \sim Ac(x), x \to \infty).$$

The following example point out that a continuous and L^1-integrable function can have S-asymptotics as a distribution without having an ordinary asymptotics.

Example 2.3.3. Let $G \in L^1(\mathbb{R}) \cap C(\mathbb{R})$ have the property $G(n) = n, n \in \mathbb{N}$ and it is equal to zero outside suitable small intervals $I_n \ni n, n \in \mathbb{N}$. Denote by $f(t) = e^t \int_0^t G(x)dx, t \in \mathbb{R}$. It is easy to see that

$$f(t+h) \sim^S e^h \cdot e^t \int_0^\infty g(x)dx, \ h \in \mathbb{R}_+.$$

Then, $f'(t)$ has S-asymptotics related to e^h and with the same limit. But, in view of the properties of G, $f'(t) + e^t g(t)$ has not the same asymptotics. Moreover, G can be chosen so that f' has no asymptotics at all.

The following example shows that a function f can have asymptotic behavior without having S-asymptotics with limit g different from zero.

Example 2.3.4. Let $f(x) = e^{x^2}, x \in \mathbb{R}$ and let us assume that f has S-asymptotics related to a $c(h) > 0, h \in \mathbb{R}_+$ with limit g different from zero. As mentioned previously, g has the form $g(x) = Ce^{ax}, C > 0$. Then, for every $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\varphi > 0$ we have

$$\lim_{h\to\infty} \frac{1}{c(h)} \int \exp\left[(x+h+h_0)^2\right]\varphi(x)dx = e^{ah_0}\langle Ce^{ax}, \varphi(x)\rangle.$$

Therefore,

$$\begin{aligned} e^{ah_0}\langle g, \varphi\rangle &= e^{h_0^2} \lim_{h\to\infty} \frac{1}{c(h)} \int e^{(x+h)^2} e^{2h_0(x+h)}\varphi(x)dx \\ &\geq e^{h_0^2}\langle g, \varphi\rangle, \quad \text{for every} \quad h_0 > 0. \end{aligned}$$

But this inequality is absurd. Consequently, e^{x^2} cannot have such an S-asymptotic behavior.

One can prove a more general assertion.

Proposition 2.3.2. *([73, Prop.1.3]) Let $f \in L^1_{loc}(\mathbb{R}) \subset \mathcal{D}'(\mathbb{R})$ have one of the four properties for $\alpha > 0, \beta > 0, x \geq x_0, h > 0, M > 0$ and $N > 0$:*

(*i*) $f(x+h) \geq Me^{(\beta h^\alpha)} f(x) \geq 0,$

(*i'*) $-f(x+h) \geq -Me^{(\beta h^\alpha)} f(x) \geq 0,$

(*ii*) $0 \leq f(x+h) \leq Ne^{(-\beta h^\alpha)} f(x) \geq 0,$

(*iii*) $0 \leq -f(x+h) \leq -Ne^{(-\beta h^\alpha)} f(x) \geq 0,$

Then, f cannot have S-asymptotics with limit $g \neq 0$, but the function f can have asymptotics.

It is easy to show that for some classes of real functions f on $\mathbb{R}$ the asymptotic behavior at infinity implies the S-asymptotics.

Proposition 2.3.3. *([73, Prop.1.4]) a) Let c be a positive function and let* $T \in L^1_{loc}(\mathbb{R})$. *Suppose that there exist locally integrable functions* $u(x)$ *and* $v(x), x \in \mathbb{R}$, *such that for every compact set* $K \subset \mathbb{R}^n$

$$|T(x+h)/c(h)| \leq v(x), x \in K, |h| > r_K,$$

$$\lim_{|h|\to\infty} T(x+h)/c(h) = u(x), x \in K.$$

Then, $T(x+h) \sim^S c(h)u(x), |h| \to \infty$ *in* $\mathcal{D}'(\mathbb{R})$.
b) Let $T \in L^1_{loc}(\mathbb{R})$ *have the ordinary asymptotic behavior*

$$T(x) \sim e^{\alpha x} L(e^x), x \to \infty, \alpha \in \mathbb{R},$$

where L is slowly varying function. Then

$$T(x+h) \sim^S e^{\alpha h} L(e^h) e^{\alpha x}, h \in \mathbb{R}_+, \quad \text{in} \quad \mathcal{D}'(\mathbb{R}).$$

The following proposition gives a sufficient condition under which the S-asymptotics of $f \in L^1_{loc}(\mathbb{R})$, in $\mathcal{D}'(\mathbb{R})$, implies the ordinary asymptotic behavior of f.

Proposition 2.3.4. *([73, Prop.1.6]) Let* $f \in L^1_{loc}(\mathbb{R}), c(h) = h^\beta L(h)$, *where* $\beta > -1$ *and L be a slowly varying function. If for some* $m \in \mathbb{N}, x^m f(x), x > 0$, *is monotonous and* $f(x+h) \sim^S c(h)\cdot 1, h \in \mathbb{R}_+$ *in* $\mathcal{D}'(\mathbb{R})$, *then* $\lim_{h\to\infty} f(h)/c(h) = 1$. *If we suppose that L is monotonous, then we can omit the hypothesis* $\beta > -1$.

The space of tempered distributions is a natural one for the quasiasymptotics while for the S-asymptotics the space $\mathcal{K}'_1$ has this role. The following example illustrate the problem of comparison of these two types of asymptotic behavior in $\mathcal{S}(\mathbb{R})$.

Example 2.3.5. The regular distribution $f(x) = H(x)e^{iax}, x \in \mathbb{R}, a \neq 0$ has quasiasymptotics $\frac{i}{a}\delta$ in $\mathcal{S}'(\mathbb{R})$ related to $c(k) = k^{-1}$:

$$\begin{aligned} k\langle H(kx)e^{ikax}, \varphi(x)\rangle &= k\int_0^\infty e^{ikax}\varphi(x)dx = \frac{1}{ia}\int_0^\infty \varphi\left(\frac{x}{k}\right) d(e^{iax}) \\ &= \frac{-1}{ia}\varphi(0) - \frac{1}{k}\int_0^\infty \varphi'\left(\frac{x}{k}\right) e^{iax}dx \to \frac{i}{a}\varphi(0), \quad k \to \infty. \end{aligned}$$

But the distribution f has no S-asymptotics related to h^α with $g \neq 0$ for any $\alpha \in \mathbb{R}$. We start with

$$\begin{aligned} \langle H(t+h)e^{ia(t+h)}, \varphi(x)\rangle &= e^{iah}\int_0^\infty e^{iax}\varphi(x)dx \\ &\sim e^{iah}\int_0^\infty e^{iax}\varphi(x)dx, \quad n \in \mathbb{R}_+. \end{aligned}$$

This distribution has the S-asymptotic but related to the oscillatory function $c(h) = e^{iah}$.

More on this relations between the S-asymptotics and quasiasymptotics of distribution can be found in [73, 63, 66, 68].

2.3.3 Characterization of some generalized function spaces

Theorem 2.3.2. *([73, Thrm.1.7]) A necessary and sufficient condition for a distribution f to belong to $\mathcal{E}'$ is that $f(x+h) \sim^S c(h) \cdot 0, h \in \mathbb{R}^n$ for every positive function c.*

Proposition 2.3.5. *([73, Prop.1.7])*

a) If for every rapidly decreasing function c, f has the S-asymptotic behavior related to $c^{-1}(h)$ and with limit u_c ($u_c = 0$ included), then $f \in \mathcal{S}'(\mathbb{R})$.

b) If for every rapidly exponentially decreasing function c (for every $k > 0$, $c(h)e^{k\|h\|} \to 0, \|h\| \to \infty$) a distribution f has the S-asymptotic behavior related to c^{-1} with limit u_c ($u_c = 0$ included), then $f \in \mathcal{K}_1'(\mathbb{R})$.

Proposition 2.3.6. *Let $f \in \mathcal{K}_1'(\mathbb{R})$. If for every rapidly exponentially decreasing function r on $\mathbb{R}^n$ the set $\{r(h)f(x+h)|h \in \mathbb{R}^n\}$ is bounded in $\mathcal{D}'$ then, $f \in \mathcal{K}_1'(\mathbb{R})$.*

2.4 Quasi-asymptotic boundedness

Definition 2.4.1. Let L be a slowly varying function at infinity (respectively at the origin). We say that $f \in \mathcal{D}'(\mathbb{R}^n)$ is *quasi-asymptotically bounded at infinity (at the origin)* in $\mathcal{D}'(\mathbb{R}^n)$ with respect to $\lambda^\alpha L(\lambda), \alpha \in \mathbb{R}$ if,

$$\langle f(\lambda x), \phi(x)\rangle = O(\lambda^\alpha L(\lambda)), \quad \text{as} \quad \lambda \to \infty, \quad \forall \phi \in \mathcal{D}(\mathbb{R}^n), \tag{2.16}$$

(resp. $\lambda \to 0^+$). If (2.16) holds, it is also said that f is *quasi-asymptotically bounded of degree α at infinity (at the origin)* with respect to the slowly varying function L. We express (2.16) by

$$f(\lambda x) = O(\lambda^\alpha L(\lambda)), \quad \text{as} \quad \lambda \to \infty \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n), \tag{2.17}$$

(resp. $\lambda \to 0^+$).

Note that in analogy to the quasi-asymptotic behavior of distributions we may talk about (2.17) in other spaces of distributions.

Theorem 2.4.1. *[73, Thrm. 2.44] Let $f \in \mathcal{S}'(\mathbb{R})$. If f is quasi-asymptotically bounded at 0, with respect to a slowly varying function L, in $\mathcal{D}'(\mathbb{R})$, then f is quasi-asymptotically bounded at 0 of the same degree with respect to L in the space $\mathcal{S}'(\mathbb{R})$.*

Chapter 3

The short time Fourier transform of distribution spaces

The short-time Fourier transform (STFT) is a very effective device in the study of function spaces. The investigation of major test function spaces and their duals through time-frequency representations has attracted much attention. For example, the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ and the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ were studied in [29] (cf. [26]). Characterizations of Gelfand-Shilov spaces and ultradistribution spaces by means of the short-time Fourier transform and modulation spaces are also known [30, 59, 102].

The purpose of this Section is three folded. On the one hand we analyze the quasiasymptotic behavior of tempered distributions via the STFT. Next, we study the short-time Fourier transform in the context of the space $\mathcal{K}_1'(\mathbb{R}^n)$ of distributions of exponential type, the dual of the space of exponentially rapidly decreasing smooth functions $\mathcal{K}_1(\mathbb{R}^n)$. The third aim is to present a new kind of Tauberian theorems. In such theorems the exponential asymptotics of functions and distributions can be obtained from those of the short-time Fourier transform.

3.1 The short time Fourier transform

Gabor (1946) adapted the Fourier transform to analyze only a small section of the signal at a time - a technique called windowing the signal. Gabor's adaptation, called the short-time Fourier transform (STFT), maps a signal into a two-dimensional function of time and frequency. The STFT represents a sort of compromise between the time- and frequency-based views of a signal. It provides some information about both when and at what frequencies a signal event occurs. However, you can only obtain this information with limited precision, and that precision is determined by the size of the window.

The translation and modulation operators are defined by $T_x f(\,\cdot\,) = f(\,\cdot\, - x)$ and $M_\xi f(\,\cdot\,) = e^{2\pi i \xi \,\cdot}\, f(\,\cdot\,)$, $x, \xi \in \mathbb{R}^n$. The operators $M_\xi T_x$ and $T_x M_\xi$ are called time-frequency shifts and for $x, \xi \in \mathbb{R}^n$, $f \in L^2(\mathbb{R}^n)$ we have

$$M_\xi T_x f = e^{2\pi i x\cdot\xi} T_x M_\xi f \quad \text{and} \quad (M_\xi T_x f)\hat{} = T_\xi M_{-x} \hat{f}. \tag{3.1}$$

Definition 3.1.1. The *short-time Fourier transform* (STFT) of a function $f \in L^2(\mathbb{R}^n)$ with respect to a window function $g \in L^2(\mathbb{R}^n)$ is defined as

$$V_g f(x,\xi) := \langle f, M_\xi T_x g\rangle = \int_{\mathbb{R}^n} f(t)\overline{g(t-x)}e^{-2\pi i\xi\cdot t}\,dt,\ \ x,\xi \in \mathbb{R}^n \tag{3.2}$$

Remark 3.1.1. If g is compactly supported with its support centered at the origin, then $V_g f(x,\cdot)$ is the Fourier transform of a segment of f centered in a neighborhood of x. As x varies, the window slides along the x-axis to different positions. For this reason, the STFT if often called the "sliding window Fourier transform".

Remark 3.1.2. In signal analysis, at least in dimension $n = 1$, $\mathbb{R}^2$ is called *the time-frequnecy plane* and in physics *the phase space.*

The following lemma lists several useful equivalent forms of the STFT.

Lemma 3.1.1. *[26, Lemma 3.1.1] If $f, g \in L^2(\mathbb{R}^n)$, then $V_g f$ is uniformly continuous on $\mathbb{R}^n$ and*

$$\begin{aligned} & V_g f(x,\xi) = (f\cdot T_x\overline{g})\hat{}(\xi) \\ = \ & \langle f, M_\xi T_x g\rangle \\ = \ & \langle \hat{f}, T_\xi M_{-x}\hat{g}\rangle \\ = \ & e^{-2\pi i x\cdot\xi}V_{\hat{g}}\hat{f}(\xi,x). \end{aligned}$$

The STFT may be considered as the sesquilinear form $(f,g) \mapsto V_g f$. Let $f\otimes g$ be the tensor product $f\otimes g(x,t) = f(x)g(t)$, and let $\mathcal{T}_a$ be the asymmetric coordinate transform $\mathcal{T}_a F(t_1,t_2) = F(t_2, t_2 - t_1)$, and let $\mathcal{F}_2$ be the partial Fourier transform $\mathcal{F}_2 F(t_1,t_2) = \int_{\mathbb{R}^n} F(t_1,t_2)e^{-2\pi i t_2\cdot\xi}dt_2$ of a function F on $\mathbb{R}^{2n}$. Then

Lemma 3.1.2. *[26, Lemma 3.1.2] If $f \in L^2(\mathbb{R}^n)$, then*

$$V_g f = \mathcal{F}_2\mathcal{T}_a(f\otimes\overline{g}).$$

Theorem 3.1.1. *[26, Th. 3.2.1] (Orthogonality relation for STFT) Let $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^n)$, then $V_{g_j} f_j \in L^2(\mathbb{R}^{2n})$ for $j = 1, 2$, and*

$$(V_{g_1}f_1, V_{g_2}f_2) = (f_1,f_2)\overline{(g_1,g_2)}.$$

Corollary 3.1.1. *[26, Co. 3.2.2] If $f, g \in L^2(\mathbb{R}^n)$, then*

$$\|V_g f\|_2 = \|f\|_2\|g\|_2,$$

In particular, if $\|g\|_2 = 1$ then

$$\|f\|_2 = \|V_g f\|_2 \tag{3.3}$$

for all $f \in L^2(\mathbb{R}^n)$. Thus, in this case the STFT is an isometry from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^{2n})$.

It follows from (3.3) that f can be completely determined by $V_g f$. Furthermore, the implication $(f, M_\xi T_x g) = 0, \forall x, \xi \in \mathbb{R}^n \Rightarrow f = 0$ is equivalent to saying that for each fixed $g \in L^2(\mathbb{R}^n)$ the set $\{M_\xi T_x g : x, \xi \in \mathbb{R}^n\}$ spans a dense subspace of $L^2(\mathbb{R}^n)$.

The adjoint of V_ψ is given by the mapping

$$V_\psi^* F(t) = \iint_{\mathbb{R}^{2n}} F(x,\xi)\psi(t-x)e^{2\pi i\xi\cdot t}dxd\xi,$$

interpreted as an $L^2(\mathbb{R}^n)$-valued weak integral. If $\psi \neq 0$ and $\gamma \in L^2(\mathbb{R}^n)$ is a synthesis window for ψ, namely, $(\gamma,\psi)_{L^2} \neq 0$, then for any $f \in L^2(\mathbb{R}^n)$, the following inversion formula holds

$$f = \frac{1}{(\gamma,\psi)_{L^2}} \iint_{\mathbb{R}^{2n}} V_\psi f(x,\xi) M_\xi T_x \gamma d\xi dx. \tag{3.4}$$

Whenever the dual pairing in (3.3) is well-defined, the definition of $V_\psi f$ can be generalized for f in larger classes than $L^2(\mathbb{R}^n)$, for instance: $f \in \mathcal{D}'(\mathbb{R}^n)$ and $\psi \in \mathcal{D}(\mathbb{R}^n)$. In fact, it is enough to have $\psi \in \mathcal{A}(\mathbb{R}^n)$ and $f \in \mathcal{A}'(\mathbb{R}^n)$, where $\mathcal{A}(\mathbb{R}^n)$ is a time-frequency shift invariant topological vector space. Note also that the inversion formula (3.4) holds pointwisely when f is sufficiently regular, for instance, for function in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$. It is obvious that for $g \in \mathcal{S}(\mathbb{R}^n)$ the set

$$\{M_\xi T_x g \ : \ x, \xi \in K\} \tag{3.5}$$

is compact in $\mathcal{S}(\mathbb{R}^n)$, where K is a compact subset of $\mathbb{R}^n$.

Lemma 3.1.3. *[26, Lemma. 11.3.3] Let $g_0, g, \gamma \in \mathcal{S}(\mathbb{R}^n)$ such that $\langle \gamma, g\rangle \neq 0$ and let $f \in \mathcal{S}'(\mathbb{R}^n)$. Then*

$$|V_{g_0} f(x,\omega)| \leq \frac{1}{|\langle \gamma, g\rangle|} \left(|V_g f| * |V_{g_0}\gamma|\right)(x,\omega),$$

for all $(x,\omega) \in \mathbb{R}^{2n}$.

Note that [26, Thrm. 11.2.3] for each used window $g \in \mathcal{S}(\mathbb{R}^n)\backslash\{0\}$ and $f \in \mathcal{S}'(\mathbb{R}^n)$ there exist constants $C > 0$ and $N \geq 0$ such that

$$|V_g f(x,\xi)| \leq C(1+|x|+|\xi|)^N \ \text{ for all } \ x, \xi \in \mathbb{R}^n. \tag{3.6}$$

It is also known that [26, Thrm. 11.2.5] if $f, g \in \mathcal{S}(\mathbb{R}^n)$ then for all $N \geq 0$, there exists constant $C_N > 0$ such that

$$|V_g f(x,\xi)| \leq C_N(1+|x|+|\xi|)^{-N} \ \text{ for all } \ x, \xi \in \mathbb{R}^n. \tag{3.7}$$

In the proof of our results we use the relations (3.8) and (3.9) regarding the use of an adapted STFT window. In particular, we apply dilation to adapt the window (or any function) and we use the notation

$$f_\varepsilon(x) = f(\varepsilon x), \quad \varepsilon > 0.$$

It turns out that dilating the window is equivalent to the inverse dilation of the function of interest or,

$$V_g f_\varepsilon(x,\xi) = \frac{1}{\varepsilon} V_{g_{1/\varepsilon}} f(\varepsilon x, \xi/\varepsilon). \tag{3.8}$$

Indeed, using the substitution $t = y/\varepsilon$ we have

$$\begin{aligned} V_g f_\varepsilon(x,\xi) = \langle f_\varepsilon, M_\xi T_x g\rangle = \int_{\mathbb{R}^n} f_\varepsilon(t)\overline{g(t-x)}e^{-2\pi i\xi\cdot t}dt \\ = \frac{1}{\varepsilon}\int_{\mathbb{R}^n} f(y)\overline{g\left(\frac{y-\varepsilon x}{\varepsilon}\right)}e^{-2\pi i\frac{\xi}{\varepsilon}\cdot y}dy \\ = \frac{1}{\varepsilon}\langle f, M_{\xi/\varepsilon}T_{\varepsilon x}g_{1/\varepsilon}\rangle = \frac{1}{\varepsilon}V_{g_{1/\varepsilon}}f(\varepsilon x,\xi/\varepsilon). \end{aligned}$$

We will also prove the following relation

$$\varepsilon V_g f_\varepsilon\left(\frac{x_0}{\varepsilon}+x,\varepsilon^2\xi\right) = V_{g_{1/\varepsilon}}f(x_0+\varepsilon x,\varepsilon\xi),\ x_0\in\mathbb{R}^n. \tag{3.9}$$

Indeed, using the substitution $y = t/\varepsilon$ we obtain

$$\begin{aligned} V_{g_{1/\varepsilon}}f(x_0+\varepsilon x,\varepsilon\xi) = \langle f, M_{\varepsilon\xi}T_{x_0+\varepsilon x}g_{1/\varepsilon}\rangle \\ = \int_{\mathbb{R}^n} f(t)\overline{g\left(\frac{t-x_0-\varepsilon x}{\varepsilon}\right)}e^{-2\pi i\xi\cdot\varepsilon t}dt \\ = \varepsilon\int_{\mathbb{R}^n} f(\varepsilon y)\overline{g\left(y-\frac{x_0}{\varepsilon}-x\right)}e^{-2\pi i\xi\cdot\varepsilon^2 y}dy \\ = \varepsilon\langle f_\varepsilon, M_{\varepsilon^2\xi}T_{\frac{x_0}{\varepsilon}+x}g\rangle = \varepsilon V_g f_\varepsilon\left(\frac{x_0}{\varepsilon}+x,\varepsilon^2\xi\right). \end{aligned}$$

Following [26], we now give a brief introduction to modulation space. In general a weight function is a simply a non-negative, locally integrable function on $\mathbb{R}^{2n}$ and they are used for describing the decay or growth of functions.

Definition 3.1.2. A weight function v on $\mathbb{R}^{2n}$ is called *submultiplicative* if

$$v(z_1+z_2) \le v(z_1)v(z_2) \quad \text{for all} \quad z_1,z_2\in\mathbb{R}^{2n}.$$

A weight function m on $\mathbb{R}^{2n}$ is *v-moderate* if

$$m(z_1+z_2) \le Cv(z_1)m(z_2) \quad \text{for all} \quad z_1,z_2\in\mathbb{R}^{2n}.$$

Two weights m_1 and m_2 are equivalent if

$$C_{-1}m_1(z) \le m_2(z) \le Cm_1(z) \quad \text{for all} \quad z\in\mathbb{R}^{2n}.$$

For simplicity we will assume without loss of generality that v is continuous and symmetric in each coordinate, formally that

$$v(x,\omega) = v(-x,\omega) = v(x,-\omega) = v(-x,-\omega) \quad \text{for all} \quad x,\omega\in\mathbb{R}^n.$$

Example 3.1.1. The standard class of weights on $\mathbb{R}^{2n}$ are weights of polynomial type

$$v_s(z) = (1+|z|)^s = (1+(x^2+\omega^2)^{1/2})^s,$$

where $z = (x, \omega) \in \mathbb{R}^{2n}$ and $s \geq 0$. $v_s(z)$ is equivalent to the weights

$$(1+|x|+|\omega|)^s \quad \text{and} \quad (1+|z|^2)^{s/2}.$$

Let m be a weight on $\mathbb{R}^{2n}$, that is, $m : \mathbb{R}^{2n} \to (0,\infty)$ is measurable and locally bounded. Then, if $p, q \in [1,\infty]$, the weighted Banach space $L^{p,q}_m(\mathbb{R}^{2n})$ consists of all measurable functions F such that

$$\|F\|_{L^{p,q}_m} := \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|F(x,\xi)|^p m(x,\xi)^p dx\right)^{q/p} d\xi\right)^{1/q} < \infty.$$

If $p = \infty$ or $q = \infty$, then the corresponding p-norm is replaced by the essential supremum. Thus

$$\|F\|_{L^{\infty,q}_m} := \left(\int_{\mathbb{R}^n} (\operatorname{esssup}_{x\in\mathbb{R}^n}|F(x,\xi)|m(x,\xi)dx)^q \, d\xi\right)^{1/q} < \infty,$$

and

$$\|F\|_{L^{p,\infty}_m} := \left(\operatorname{esssup}_{x\in\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|F(x,\xi)|^p m(x,\xi)^p dx\right)^p d\xi\right)^{1/p} < \infty.$$

$L^{p,q}_m$ arises by taking a weighted L^p norm with respect to x and an L^q norm with respect to ω. Since $\omega \mapsto F(\,\cdot\,,\omega)m(\,\cdot\,,\omega)$ takes values in L^p, the mixed norm space $L^{p,q}_m$ may be viewed as a vector-valued L^q space. If $p = q$, then $L^{p,q}_m = L^p_m$ is the usual weighted L^p space.

Definition 3.1.3. Fix a non-zero window $g \in \mathcal{S}(\mathbb{R}^n)$, a polynomial moderate weight function m on $\mathbb{R}^{2n}$, and $1 \leq p, q \leq \infty$. Then, the modulation space $M^{p,q}_m(\mathbb{R}^n)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $V_g f \in L^{p,q}_m(\mathbb{R}^{2n})$.

The norm on $M^{p,q}_m$ is $\|f\|_{M^{p,q}_m} = \|V_g f\|_{L^{p,q}_m}$. If $p = q$, we write M^p_m and if $m(z) \equiv 1$ on $\mathbb{R}^{2n}$, then we write $M^{p,q}$.

Proposition 3.1.1. *[26, Pr.11.3.2] The definition of $M^{p,q}_m$ is independent of the window $g \in \mathcal{S}(\mathbb{R}^n)$. Different windows yield equivalent norms.*

Proposition 3.1.2. *[26, Pr.11.3.1] Writing $v_s(z) = (1+|z|)^s, z \in \mathbb{R}^{2n}$, we have*

$$\mathcal{S}(\mathbb{R}^n) = \bigcap_{s\geq 0} M^\infty_{v_s} \quad \text{and} \quad \mathcal{S}'(\mathbb{R}^n) = \bigcup_{s\geq 0} M^\infty_{1/v_s}.$$

Proposition 3.1.3. *[26, Pr.11.3.4] If $|m(z)| \leq C(1+|z|)^N$ and $1 \leq p, q \leq \infty$, then $\mathcal{S}(\mathbb{R}^n)$ is a dense subspace of $M^{p,q}_m$.*

3.2 Abelian and Tauberian results on spaces of tempered distributions

Our main goal in this paper is to provide Abelian and Tauberian type results relating asymptotics of STFT and the quasiasymptotic behavior of tempered distributions.

Theorem 3.2.1. *Let L be slowly varying function at the origin, $\alpha \in \mathbb{R}$ and $f \in \mathcal{S}'(\mathbb{R})$. Suppose that*

$$f(\varepsilon x) \sim \varepsilon^{\alpha} L(\varepsilon) u(x) \quad \text{as} \quad \varepsilon \to 0^+ \quad \text{in} \quad \mathcal{S}'(\mathbb{R}).$$

Then for its STFT with respect to window $g \in \mathcal{S}(\mathbb{R})\backslash\{0\}$ we have

$$V_{g_{1/\varepsilon}} f(\varepsilon x, \xi/\varepsilon) \sim \varepsilon^{\alpha+1} L(\varepsilon) V_g u(x,\xi) \quad \text{as} \quad \varepsilon \to 0^+.$$

uniformly for x, ξ in compact subsets of $\mathbb{R}$.

Proof. By relation (3.8) we have

$$\begin{aligned} \frac{V_{g_{1/\varepsilon}} f(\varepsilon x, \xi/\varepsilon)}{\varepsilon^{\alpha+1} L(\varepsilon)} &= \frac{\varepsilon V_g f_\varepsilon(x,\xi)}{\varepsilon^{\alpha+1} L(\varepsilon)} = \Big\langle \frac{f_\varepsilon(t)}{\varepsilon^{\alpha} L(\varepsilon)}, M_\xi T_x g(t) \Big\rangle \\ &= \Big\langle \frac{f(\varepsilon t)}{\varepsilon^{\alpha} L(\varepsilon)},\ M_\xi T_x g(t) \Big\rangle. \end{aligned}$$

Using the compactness of the set given by (3.5) and the Banach-Steinhaus theorem we obtain

$$\begin{aligned} \lim_{\varepsilon \to 0^+} \frac{V_{g_{1/\varepsilon}} f(\varepsilon x, \xi/\varepsilon)}{\varepsilon^{\alpha+1} L(\varepsilon)} &= \lim_{\varepsilon \to 0^+} \Big\langle \frac{f(\varepsilon t)}{\varepsilon^{\alpha} L(\varepsilon)}, M_\xi T_x g(t) \Big\rangle \\ &= \langle u(t), M_\xi T_x g(t) \rangle = V_g u(x,\xi). \end{aligned}$$

□

Remark 3.2.1. Let $f, g_1, g_2 \in \mathcal{S}(\mathbb{R})\backslash\{0\}$ and

$$g_1(\varepsilon x) \sim \varepsilon^{\alpha} L(\varepsilon) g_2(x) \quad \text{as} \quad \varepsilon \to 0^+ \quad \text{in} \quad \mathcal{S}'(\mathbb{R}). \tag{3.10}$$

According to Theorem 3.1 it follows

$$V_{f_{1/\varepsilon}} g_1(\varepsilon x, \xi/\varepsilon) \sim \varepsilon^{\alpha+1} L(\varepsilon) V_f g_2(x,\xi) \quad \text{as} \quad \varepsilon \to 0^+.$$

By relation $V_g f(x,\xi) = e^{-2\pi i x\xi} \overline{V_f g(x,\xi)}$, $x, \xi \in \mathbb{R}$ we obtain

$$e^{-2\pi i x\xi} \overline{V_{g_1} f_{1/\varepsilon}\left(\varepsilon x, \frac{\xi}{\varepsilon}\right)} \sim \varepsilon^{\alpha+1} L(\varepsilon) e^{-2\pi i x\xi} \overline{V_{g_2} f(x,\xi)} \quad \text{as} \quad \varepsilon \to 0^+,$$

i.e.

$$V_{g_1} f_{1/\varepsilon}(\varepsilon x, \xi/\varepsilon) \sim \varepsilon^{\alpha+1} L(\varepsilon) V_{g_2} f(x,\xi) \quad \text{as} \quad \varepsilon \to 0^+.$$

This is an expected result, given that the choice of STFT window is causing no significant change in the quality of the STFT; that is, two windows with the same quasiasymptotic property result with STFTs with related quasiasymptotics.

Theorem 3.2.2. *Let L be a slowly varying function at the origin, $\alpha \in \mathbb{R}$ and $f \in \mathcal{S}'(\mathbb{R})$, $g \in \mathcal{S}(\mathbb{R})\backslash\{0\}$. The following two conditions:*

(i) the limits

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{\alpha+1}L(\varepsilon)} V_{g_{1/\varepsilon}} f(\varepsilon x, \xi/\varepsilon) = M_{x,\xi} < \infty, \tag{3.11}$$

uniformly for x, ξ in compact subsets of $\mathbb{R}$.

(ii) there exist $C > 0$ and $N \geq 0$ such that

$$\frac{|V_{g_{1/\varepsilon}} f(\varepsilon x, \xi/\varepsilon)|}{\varepsilon^{\alpha+1}L(\varepsilon)} < C(1 + |x| + |\xi|)^N, \tag{3.12}$$

for all $x, \xi \in \mathbb{R}$ and $0 < \varepsilon \leq 1$, are necessary and sufficient conditions for existence of a homogeneous distribution u such that

$$f(\varepsilon x) \sim \varepsilon^\alpha L(\varepsilon) u(x) \quad as \quad \varepsilon \to 0^+ \quad in \quad \mathcal{S}'(\mathbb{R}). \tag{3.13}$$

Proof. (3.11) and (3.12) imply that the function given by $J(x, \xi) = M_{x,\xi}$, $x, \xi \in \mathbb{R}$ is measurable and satisfies the estimate

$$|J(x, \xi)| = |M_{x,\xi}| \leq C(1 + |x| + |\xi|)^N,$$

for all $x, \xi \in \mathbb{R}$ and some constant $C > 0$. Moreover, by relation (3.8) and the inversion formula we obtain

$$\lim_{\varepsilon\to 0^+} \Big\langle \frac{f(\varepsilon t)}{\varepsilon^\alpha L(\varepsilon)}, \varphi(t) \Big\rangle = \frac{1}{\langle \gamma, g\rangle} \lim_{\varepsilon\to 0^+} \int\int_{\mathbb{R}^2} \frac{V_{g_{1/\varepsilon}} f(\varepsilon x, \xi/\varepsilon)}{\varepsilon^{\alpha+1}L(\varepsilon)} \overline{V_\gamma \varphi(x, \xi)} d\xi dx,$$

where γ is synthesis window for g such that $\langle g, \gamma\rangle \neq 0$. Because of (3.11) and (3.12) we can use Lebesque dominated convergence theorem

$$\lim_{\varepsilon\to 0^+} \Big\langle \frac{f(\varepsilon t)}{\varepsilon^\alpha L(\varepsilon)}, \varphi(t) \Big\rangle = \frac{1}{\langle \gamma, g\rangle} \int\int_{\mathbb{R}^2} J(x, \xi) \overline{V_\gamma \varphi(x, \xi)} d\xi dx.$$

Observe that the last integral converges absolutely because $|J(x, \xi)| = O((1+|x|+|\xi|)^N)$ for some $N > 0$ and $|V_\gamma \varphi(x, \xi)| = O((1+|x|+|\xi|)^{-n})$ for all $n \geq 0$, whenever $\varphi, \gamma \in \mathcal{S}(\mathbb{R})$ [[26], Theorem 11.2.5]. It follows that the limit $\lim_{\varepsilon\to 0^+} \langle \frac{f(\varepsilon t)}{\varepsilon^\alpha L(\varepsilon)}, \varphi(t)\rangle$ exists for each $\varphi \in \mathcal{S}(\mathbb{R})$. So, we conclude that f has quasiasymptotic behavior at the origin in $\mathcal{S}'(\mathbb{R})$.

We now prove the converse. If (3.13) holds then (3.11) follows from the Abelian type result given in Theorem 3.1. Also, from (3.8), (3.13) and (3.6) it follows that there exist constants C_1, $C_2 > 0$ and $N \geq 0$ such that

$$\frac{|V_{g_{1/\varepsilon}} f(\varepsilon x, \xi/\varepsilon)|}{\varepsilon^{\alpha+1}L(\varepsilon)} = \frac{|V_g f_\varepsilon(x, \xi)|}{\varepsilon^\alpha L(\varepsilon)} = \frac{|\langle f(\varepsilon t), M_\xi T_x g(t)\rangle|}{\varepsilon^\alpha L(\varepsilon)}$$

$$
\begin{aligned}
&< C_1|\langle u, M_\xi T_x g\rangle| = C_1|V_g u(x,\xi)| \\
&\leq C_2(1+|x|+|\xi|)^N.
\end{aligned}
$$

□

Remark 3.2.2. Clearly, the STFT $V_g u(x,\xi)$ in Theorem 3.6.1 is given by the limits (3.11).

A similar assertion as previous theorem holds for quasiasymptotics at infinity.

Theorem 3.2.3. *Let L be a slowly varying function at infinity, $\alpha \in \mathbb{R}$ and $f \in \mathcal{S}'(\mathbb{R})$, $g \in \mathcal{S}(\mathbb{R})\backslash\{0\}$ The following two conditions:*

(i) the limits

$$\lim_{\lambda\to\infty} \frac{1}{\lambda^{\alpha+1}L(\lambda)} V_{g_{1/\lambda}} f(\lambda x, \xi/\lambda) < \infty,$$

uniformly for x,ξ in compact subsets of $\mathbb{R}$.

(ii) there exist $C>0$ and $N\geq 0$ such that

$$\frac{|V_{g_{1/\lambda}} f(\lambda x, \xi/\lambda)|}{\lambda^{\alpha+1}L(\lambda)} < C(1+|x|+|\xi|)^N,$$

for all $x,\xi\in\mathbb{R}$ and $\lambda\geq 1$, are necessary and sufficient conditions for existence of a homogeneous distribution u such that

$$f(\lambda x) \sim \lambda^\alpha L(\lambda) u(x) \quad \text{as} \quad \lambda\to\infty \quad \text{in} \quad \mathcal{S}'(\mathbb{R}).$$

Remark 3.2.3. The same consideration of Remark 3.2.2 apply to the case of infinity by analogy.

Theorem 3.2.4. *Let L be a slowly varying function at the origin, $\alpha\in\mathbb{R}$, $x_0\in\mathbb{R}$ and $f\in\mathcal{S}'(\mathbb{R})$. Suppose that*

$$f(x_0+\varepsilon x) \sim \varepsilon^\alpha L(\varepsilon) u(x) \quad \text{as} \quad \varepsilon\to 0^+ \quad \text{in} \quad \mathcal{S}'(\mathbb{R}).$$

Then for its STFT with respect to window $g\in\mathcal{S}(\mathbb{R})\backslash\{0\}$ we have

$$V_{g_{1/\varepsilon}} f(x_0+\varepsilon x, \varepsilon\xi) \sim \varepsilon^{\alpha+1} L(\varepsilon) V_g u(x,0) \quad \text{as} \quad \varepsilon\to 0^+.$$

uniformly for x,ξ in compact subsets of $\mathbb{R}$.

Proof. Using the substitution $t-x_0=\varepsilon y$ we obtain

$$
\begin{aligned}
&\lim_{\varepsilon\to 0^+} \frac{V_{g_{1/\varepsilon}} f(x_0+\varepsilon x, \varepsilon\xi)}{\varepsilon^{\alpha+1}L(\varepsilon)} = \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{\alpha+1}L(\varepsilon)} \langle f, M_{\varepsilon\xi} T_{x_0+\varepsilon x} g_{1/\varepsilon}\rangle \\
&= \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{\alpha+1}L(\varepsilon)} \left\langle f(t), g\left(\frac{t-x_0-\varepsilon x}{\varepsilon}\right) e^{2\pi i\varepsilon\xi t}\right\rangle \\
&= \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{\alpha}L(\varepsilon)} \left\langle f(x_0+\varepsilon y), g(y-x) e^{2\pi i\varepsilon\xi(x_0+\varepsilon y)}\right\rangle
\end{aligned}
$$

$$= \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^\alpha L(\varepsilon)} \Big\langle f(x_0+\varepsilon y), M_0 T_x g(y) e^{2\pi i\varepsilon\xi(x_0+\varepsilon y)} \Big\rangle.$$

In view of (3.2.4), the Banach-Steinhaus theorem and the compactness of the set given by (3.5) we have

$$\lim_{\varepsilon\to 0^+} \frac{V_{g_{1/\varepsilon}} f(x_0+\varepsilon x, \varepsilon\xi)}{\varepsilon^{\alpha+1} L(\varepsilon)} = \langle u(y), M_0 T_x g(y)\rangle = V_g u(x,0).$$

□

We now investigate the inverse (Tauberian) theorem related to Theorem 3.2.4.

Theorem 3.2.5. *Let L be a slowly varying function at the origin, $\alpha \in \mathbb{R}$, $x_0 \in \mathbb{R}$, and $f \in \mathcal{S}'(\mathbb{R})$, $g \in \mathcal{S}(\mathbb{R})\backslash\{0\}$. Suppose that the limits*

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{\alpha-1} L(\varepsilon)} V_{g_{1/\varepsilon}} f(x_0+\varepsilon x, \varepsilon\xi) = M_{x,\xi} < \infty, \tag{3.14}$$

uniformly for x,ξ in compact subsets of $\mathbb{R}$, and that there exist $C>0$ and $N \geq 0$ such that

$$\frac{|V_{g_{1/\varepsilon}} f(x_0+\varepsilon x, \varepsilon\xi)|}{\varepsilon^{\alpha-1} L(\varepsilon)} < C(1+|x|)^N, \tag{3.15}$$

for all $x,\xi \in \mathbb{R}$ and $0<\varepsilon\leq 1$. Then, there exists a homogeneous distribution u such that

$$f(x_0+\varepsilon x) \sim \varepsilon^\alpha L(\varepsilon) u(x) \quad as \quad \varepsilon \to 0^+ \quad in \quad \mathcal{S}'(\mathbb{R}). \tag{3.16}$$

Proof. (3.14) and (3.15) imply that the function $M_{x,\xi} = J(x,\xi)$ satisfies the estimate

$$|J(x,\xi)| = |M_{x,\xi}| \leq C(1+|x|)^N,$$

for every $x,\xi \in \mathbb{R}$ and for some constants $C>0$ and $N\geq 0$. Let $\varphi \in \mathcal{S}(\mathbb{R})$ and $\gamma \in \mathcal{S}(\mathbb{R})\backslash\{0\}$ be a synthesis window for g such that $\langle g,\gamma\rangle \neq 0$. By inversion formula (3.4) and the substitution $\xi = \varepsilon^2\xi_1$, $t = t_1 - \dfrac{x_0}{\varepsilon}$ we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+} \langle \frac{f(x_0+\varepsilon t)}{\varepsilon^\alpha L(\varepsilon)}, \varphi(t)\rangle \\
= \ & \frac{1}{\langle\gamma,g\rangle} \lim_{\varepsilon\to 0^+} \int\int_{\mathbb{R}^2} \frac{\langle f(x_0+\varepsilon t), M_\xi T_x g(t)\rangle}{\varepsilon^\alpha L(\varepsilon)} \langle M_\xi T_x\gamma, \varphi\rangle d\xi dx \\
= \ & \frac{1}{\langle\gamma,g\rangle} \lim_{\varepsilon\to 0^+} \int\int_{\mathbb{R}^2} \frac{\langle f(\varepsilon t_1), M_{\varepsilon^2\xi_1} T_x g(t_1 - \frac{x_0}{\varepsilon})\rangle}{\varepsilon^{\alpha-2} L(\varepsilon)} \langle M_{\varepsilon^2\xi_1} T_x\gamma, \varphi\rangle d\xi_1 dx \\
= \ & \frac{1}{\langle\gamma,g\rangle} \lim_{\varepsilon\to 0^+} \int\int_{\mathbb{R}^2} \frac{\langle f(\varepsilon t_1), M_{\varepsilon^2\xi_1} T_{x+\frac{x_0}{\varepsilon}} g(t_1)\rangle}{\varepsilon^{\alpha-2} L(\varepsilon)} \langle M_{\varepsilon^2\xi_1} T_x\gamma, \varphi\rangle d\xi_1 dx \\
= \ & \frac{1}{\langle\gamma,g\rangle} \lim_{\varepsilon\to 0^+} \int\int_{\mathbb{R}^2} \frac{V_g f_\varepsilon(x+\frac{x_0}{\varepsilon}, \varepsilon^2\xi_1)}{\varepsilon^{\alpha-2} L(\varepsilon)} \overline{V_\gamma\varphi(x,\varepsilon^2\xi_1)} d\xi_1 dx.
\end{aligned}$$

By relation (3.9) we have

$$\lim_{\varepsilon\to 0^+} \langle \frac{f(x_0+\varepsilon t)}{\varepsilon^\alpha L(\varepsilon)}, \varphi(t)\rangle$$

$$= \frac{1}{\langle\gamma, g\rangle}\lim_{\varepsilon\to 0^+}\int\int_{\mathbb{R}^2}\frac{V_{g_{1/\varepsilon}}f(x_0+\varepsilon x,\varepsilon\xi_1)}{\varepsilon^{\alpha-1}L(\varepsilon)}\overline{V_\gamma\varphi(x,\varepsilon^2\xi_1)}d\xi_1 dx.$$

Because of (3.14), (3.15) and (3.6) we can use Lebesque dominated convergence theorem

$$\lim_{\varepsilon\to 0^+}\langle\frac{f(x_0+\varepsilon t)}{\varepsilon^\alpha L(\varepsilon)},\varphi(t)\rangle = \frac{1}{\langle\gamma, g\rangle}\int\int_{\mathbb{R}^2}J(x,\xi)\overline{V_\gamma\varphi(x,0)}d\xi dx.$$

Observe that the last integral converges absolutely because $|J(x,\xi)| = O((1+|x|)^N)$ for some $N>0$ and $|V_\gamma\varphi(x,0)| = O((1+|x|)^{-n})$ for all $n\geq 0$, whenever $\varphi\in\mathcal{S}(\mathbb{R})$. It follows that the limit $\lim_{\varepsilon\to 0^+}\left\langle\frac{f(x_0+\varepsilon t)}{\varepsilon^{\alpha+1}L(\varepsilon)},\varphi(t)\right\rangle$ exists for each $\varphi\in\mathcal{S}(\mathbb{R})$. So, we conclude that f has quasiasymptotic behavior in $\mathcal{S}'(\mathbb{R})$. □

3.3 Short-time Fourier transform of distributions of exponential type

In this section we study the mapping properties of the STFT on the space of distributions of exponential type. Note that the STFT extends to the sesquilinear mapping $(f,\psi)\mapsto V_\psi f$ and its adjoint induces the bilinear mapping $(F,\psi)\mapsto V_\psi^* F$.

We start with the test function space $\mathcal{K}_1(\mathbb{R}^n)$. If $f,\psi\in\mathcal{K}_1(\mathbb{R}^n)$, then we immediately get that (3.3) extends to a holomorphic function in the second variable, namely, $V_\psi f(x,z)$ is entire in $z\in\mathbb{C}^n$. We write in the sequel $z=\xi+i\eta$ with $\xi,\eta\in\mathbb{R}^n$. Observe also that an application of the Cauchy theorem shows that if $\Phi\in\mathcal{K}_1(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}(\mathbb{C}^n)$ and $\psi\in\mathcal{K}_1(\mathbb{R}^n)$, then for arbitrary $\eta\in\mathbb{R}^n$ we may write $V_\psi^*\Phi$ as

$$V_\psi^*\Phi(t) = \iint_{\mathbb{R}^{2n}}\Phi(x,\xi+i\eta)\psi(t-x)e^{2\pi i(\xi+i\eta)\cdot t}dx d\xi. \tag{3.17}$$

Our first proposition deals with the range and continuity properties of V and V^* on test function spaces.

Proposition 3.3.1. *The following mappings are continuous:*

(i) $V:\mathcal{K}_1(\mathbb{R}^n)\times\mathcal{K}_1(\mathbb{R}^n)\to\mathcal{K}_1(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}(\mathbb{C}^n)$.

(ii) $V^*:(\mathcal{K}_1(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}(\mathbb{C}^n))\times\mathcal{K}_1(\mathbb{R}^n)\to\mathcal{K}_1(\mathbb{R}^n)$.

Proof. For part (i), let $\varphi,\psi\in\mathcal{K}_1(\mathbb{R}^n)$. Let k be an even integer. If $(x,z)\in\mathbb{R}^n\times\Pi_k$ and $|\alpha|\leq k$, then

$$e^{k|x|}(1+|z|^2)^{k/2}\left|\frac{\partial^\alpha}{\partial x^\alpha}V_\psi\varphi(x,z)\right|$$
$$= e^{k|x|}(1+|z|^2)^{k/2}\left|\int_{\mathbb{R}^n}(-1)^\alpha\varphi(t)\overline{\psi^{(\alpha)}(t-x)}e^{-2\pi iz\cdot t}dt\right|$$

$$\leq (1+nk^2)^{k/2} e^{k|x|} \left| \int_{\mathbb{R}^n} (1-\Delta_t)^{k/2} (\varphi(t) \overline{\psi^{(\alpha)}(t-x)} e^{2\pi\eta\cdot t}) dt \right|$$
$$\leq \tilde{C}_k \sum_{|\beta_1|+|\beta_2|\leq k} e^{k|x|} \int_{\mathbb{R}^n} \left| \varphi^{(\beta_1)}(t) \overline{\psi^{(\alpha+\beta_2)}(t-x)} \right| e^{2\pi k|t|} dt,$$

which shows that $\rho_k(V_\psi \varphi) \leq C_k \nu_{8k}(\varphi)\nu_k(\psi)$. For (ii), if $\Phi \in \mathcal{K}_1(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}(\mathbb{C}^n)$, $\psi \in \mathcal{K}_1(\mathbb{R}^n)$, and $|\alpha| \leq k$, from the Leibniz formula we obtain

$$e^{k|t|} \left| \frac{\partial^\alpha}{\partial t^\alpha} V_\psi^* \Phi(t) \right| = e^{k|t|} \left| \frac{\partial^\alpha}{\partial t^\alpha} \iint_{\mathbb{R}^{2n}} \Phi(x,z)\psi(t-x) e^{-2\pi i \xi\cdot t} \right|$$
$$\leq (2\pi)^{|\alpha|} \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} e^{k|t|} \iint_{\mathbb{R}^{2n}} |\xi|^k |\Phi(x,\xi)||\psi^{(\beta)}(t-x)| dx d\xi$$
$$\leq (4\pi)^{|\alpha|} \nu_k(\psi) \iint_{\mathbb{R}^{2n}} |\xi|^k e^{k|x|} |\Phi(x,\xi)| dx d\xi$$
$$\leq A_{k,n} \nu_k(\psi) \rho_{k+n+1}(\Phi);$$

hence $\rho_k(V_\psi^* \Phi) \leq A_{k,n}\nu_k(\psi)\rho_{k+n+1}(\Phi)$.

□

Observe that if the window $\psi \in \mathcal{K}_1(\mathbb{R}^n) \setminus \{0\}$ and $\gamma \in \mathcal{K}_1(\mathbb{R}^n)$ is a synthesis window, the reconstruction formula (3.4) reads as:

$$\frac{1}{(\gamma,\psi)_{L^2}} V_\gamma^* V_\psi = \mathrm{id}_{\mathcal{K}_1(\mathbb{R}^n)}. \tag{3.18}$$

We now study the STFT on $\mathcal{K}_1'(\mathbb{R}^n)$. Notice that the modulation operators M_z operate continuously on $\mathcal{K}_1(\mathbb{R}^n)$ even when $z \in \mathbb{C}^n$. Thus, if $f \in \mathcal{K}_1'(\mathbb{R}^n)$ and $\psi \in \mathcal{K}_1(\mathbb{R}^n)$ then $V_\psi f$, defined by the dual pairing in (3.3), also extends in the second variable as an entire function $V_\psi f(x,z)$ in $z \in \mathbb{C}^n$. Furthermore, it is clear that $V_\psi f(x,z)$ is C^∞ in $x \in \mathbb{R}^n$. We begin with a lemma.

Lemma 3.3.1. *Let $\psi \in \mathcal{K}_1(\mathbb{R}^n)$.*

(a) *Let $B' \subset \mathcal{K}_1'(\mathbb{R}^n)$ be a bounded set. There is $k = k_{B'} \in \mathbb{N}_0$ such that*

$$\sup_{f\in B',\,(x,z)\in\mathbb{R}^n\times\Pi_\lambda} e^{-k|x|-2\pi x\cdot\Im m\, z|}(1+|z|)^{-k}|V_\psi f(x,z)| < \infty, \quad \forall \lambda \geq 0. \tag{3.19}$$

(b) *For every $f \in \mathcal{K}_1'(\mathbb{R}^n)$ and $\Phi \in \mathcal{K}_1(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}(\mathbb{C}^n)$,*

$$\langle V_\psi f, \Phi\rangle = \left\langle f, \overline{V_\psi^* \overline{\Phi}} \right\rangle. \tag{3.20}$$

Proof. Part (a). By the Banach-Steinhaus theorem, B' is equicontinuous, so that there are $C > 0$ and $k \in \mathbb{N}_0$ such that $|\langle f,\varphi\rangle| \leq C\nu_k(\varphi)$, $\forall f \in B', \forall \varphi \in \mathcal{K}_1(\mathbb{R}^n)$. Hence, for all $f \in B'$ $(z = \xi + i\eta)$,

$$|V_\psi f(x,z)| \leq C \sup_{t\in\mathbb{R}^n, |\alpha|\leq k} e^{k|t|} \left| \frac{\partial^\alpha}{\partial t^\alpha} \left(e^{-2\pi i z\cdot t} \overline{\psi(t-x)} \right) \right|$$

$$\le (2\pi)^k C(1+|z|^2)^{k/2} \sup_{t\in\mathbb{R}^n,|\alpha|\le k} e^{k|t|+2\pi\eta\cdot t}\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\left|\psi^{(\beta)}(t-x)\right|$$
$$\le (4\pi)^k C(1+|z|^2)^{k/2} e^{k|x|+2\pi\eta\cdot x}\nu_{k+1+\lfloor 2\pi|\eta|\rfloor}(\psi),$$

where $\lfloor 2\pi|\eta|\rfloor$ stands for the integral part of $2\pi|\eta|$.

Part (b). We first remark that the left hand side of (4.21) is well defined because of part (a). To show (4.21), notice that the integral in (3.17), with $\eta = 0$, can be approximated by a sequence of convergent Riemann sums in the topology of $\mathcal{K}_1(\mathbb{R}^n)$; this justifies the exchange of integral and dual pairing in

$$\left\langle f(t), \iint_{\mathbb{R}^{2n}} \Phi(x,\xi)e^{-2\pi i\xi\cdot t}\overline{\psi(t-x)}dxd\xi\right\rangle_t = \iint_{\mathbb{R}^{2n}} \Phi(x,\xi)\langle f, \overline{M_\xi T_x\psi}\rangle dxd\xi,$$

which is the same as (4.21). □

In particular, if B' is a singleton, part (a) of Lemma 3.3.1 gives the growth order of the function $V_\psi f$ on every set $\mathbb{R}^n\times\Pi_\lambda$.

Let us define the adjoint STFT on $\in \mathcal{K}_1'(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}'(\mathbb{C}^n)$.

Definition 3.3.1. Let $\psi\in\mathcal{K}_1(\mathbb{R}^n)$. The adjoint STFT V_ψ^* of $F\in\mathcal{K}_1'(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}'(\mathbb{C}^n)$ is the distribution $V_\psi^* F\in\mathcal{K}_1'(\mathbb{R}^n)$ whose action on test functions is given by

$$\langle V_\psi^* F,\varphi\rangle := \left\langle F, \overline{V_\psi\overline{\varphi}}\right\rangle, \quad \varphi\in\mathcal{K}_1(\mathbb{R}^n). \tag{3.21}$$

The next theorem summarizes our results.

Theorem 3.3.1. *The two STFT mappings*

(i) $V:\mathcal{K}_1'(\mathbb{R}^n)\times\mathcal{K}_1(\mathbb{R}^n)\to\mathcal{K}_1'(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}'(\mathbb{C}^n)$

(ii) $V^*:(\mathcal{K}_1'(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}'(\mathbb{C}^n))\times\mathcal{K}_1(\mathbb{R}^n)\to\mathcal{K}_1'(\mathbb{R}^n)$

are hypocontinuous. Let $\psi\in\mathcal{K}_1(\mathbb{R}^n)\setminus\{0\}$ *and let* $\gamma\in\mathcal{K}_1(\mathbb{R}^n)$ *be a synthesis window for it. The following inversion and desingularization formulas hold:*

$$\frac{1}{(\gamma,\psi)_{L^2}}V_\gamma^* V_\psi = \mathrm{id}_{\mathcal{K}_1'(\mathbb{R}^n)}, \tag{3.22}$$

and, for all $f\in\mathcal{K}_1'(\mathbb{R}^n)$, $\varphi\in\mathcal{K}_1(\mathbb{R}^n)$, *and* $\eta\in\mathbb{R}^n$,

$$\langle f,\varphi\rangle = \frac{1}{(\gamma,\psi)_{L^2}}\iint_{\mathbb{R}^{2n}} V_\psi f(x,\xi+i\eta)V_{\overline{\gamma}}\varphi(x,-\xi-i\eta)dxd\xi. \tag{3.23}$$

Proof. That V and V^* are hypocontinuous on these spaces follows from Proposition 3.3.1 and the formula (4.21) from Lemma 3.3.1; we leave the details to the reader. By the Cauchy theorem, it is enough to show (3.23) for $\eta = 0$. Using (4.4.1), (4.21), and (3.18), we have $\langle V_\gamma^* V_\psi f,\varphi\rangle = \langle V_\psi f,\overline{V_\gamma\overline{\varphi}}\rangle = \langle f,\overline{V_\psi^* V_\gamma\overline{\varphi}}\rangle = (\gamma,\psi)_{L^2}\langle f,\varphi\rangle$, namely, (3.22) and (3.23). □

The next corollary gives the converse to part (a) of Lemma 3.3.1 under a weaker inequality than (3.19), namely, a characterization of bounded sets in $\mathcal{K}_1'(\mathbb{R}^n)$ in terms of the STFT.

Corollary 3.3.1. *Let $B' \subset \mathcal{K}'_1(\mathbb{R}^n)$ and $\psi \in \mathcal{K}_1(\mathbb{R}^n) \setminus \{0\}$. If there are $\eta \in \mathbb{R}^n$ and $k \in \mathbb{N}_0$ such that*

$$\sup_{f\in B',(x,\xi)\in\mathbb{R}^{2n}} e^{-k|x|}(1+|\xi|)^{-k}|V_\psi f(x,\xi+i\eta)| < \infty, \tag{3.24}$$

then the set B' is bounded in $\mathcal{K}'_1(\mathbb{R}^n)$. Conversely, if B' is bounded in $\mathcal{K}'_1(\mathbb{R}^n)$ there is $k \in \mathbb{N}_0$ such that (3.19) holds.

Proof. In view of the Banach-Steinhaus theorem, we only need to show that B' is weakly bounded. Let γ be a synthesis window for ψ and let $\varphi \in \mathcal{K}_1(\mathbb{R}^n)$. Then, by the desingularization formula (3.23), we have

$$\sup_{f\in B'} |\langle f,\varphi\rangle| \le \frac{C_\eta}{(\gamma,\psi)_{L^2}} \iint_{\mathbb{R}^{2n}} e^{k|x|}(1+|\xi|)^k \, |V_{\overline{\gamma}}\varphi(x,-\xi-i\eta)| \, dxd\xi < \infty,$$

because $V_\gamma\varphi \in \mathcal{K}_1(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}(\mathbb{C}^n)$. The converse was already shown in Lemma 3.3.1. □

3.4 Characterizations of $\mathcal{B}'_\omega(\mathbb{R}^n)$ and $\dot{\mathcal{B}}'_\omega(\mathbb{R}^n)$

We now turn our attention to the characterization of the space of ω-bounded distributions $\mathcal{B}'_\omega(\mathbb{R}^n)$ and its subspace $\dot{\mathcal{B}}'_\omega(\mathbb{R}^n)$. Recall that ω stands for an exponentially moderate weight, i.e., a positive and measurable function satisfying (1.11).

Theorem 3.4.1. *Let $f \in \mathcal{K}'_1(\mathbb{R}^n)$ and $\psi \in \mathcal{K}_1(\mathbb{R}^n) \setminus \{0\}$.*

(i) *The following statements are equivalent:*

- (a) $f \in \mathcal{B}'_\omega(\mathbb{R}^n)$.
- (b) *The set $\{T_{-h}f/\omega(h) : h \in \mathbb{R}^n\}$ is bounded in $\mathcal{K}'_1(\mathbb{R}^n)$.*
- (c) *There is $s \in \mathbb{R}$ such that*

$$\sup_{(x,\xi)\in\mathbb{R}^{2n}} (1+|\xi|)^{-s}\frac{|V_\psi f(x,\xi)|}{\omega(x)} < \infty. \tag{3.25}$$

(ii) *The next three conditions are equivalent:*

- $(a)'$ $f \in \dot{\mathcal{B}}'_\omega(\mathbb{R}^n)$.
- $(b)'$ $\lim_{|h|\to\infty} T_{-h}f/\omega(h) = 0$ *in $\mathcal{K}'_1(\mathbb{R}^n)$.*
- $(c)'$ *There is $s' \in \mathbb{R}$ such that*

$$\lim_{|(x,\xi)|\to\infty} (1+|\xi|)^{-s'}\frac{|V_\psi f(x,\xi)|}{\omega(x)} = 0. \tag{3.26}$$

Remark 3.4.1. Theorem 3.4.1 remains valid if we replace $\mathcal{K}_1'(\mathbb{R}^n)$ and $\mathcal{K}_1(\mathbb{R}^n)$ by $\mathcal{D}'(\mathbb{R}^n)$ and $\mathcal{D}(\mathbb{R}^n)$ everywhere in the statement. Schwartz has shown in [89, p. 204] the equivalence between (a) and (b) for $\omega = 1$ by using a much more complicated method involving a parametrix technique.

Proof. Part (i). $(a) \Rightarrow (b)$. Let $f \in \mathcal{B}'_\omega(\mathbb{R}^n)$, since $\mathcal{D}_{L^1_\omega}(\mathbb{R}^n)$ is barreled, we only need to show that the $f * \varphi$ is bounded by ω for fixed $\varphi \in \mathcal{K}_1(\mathbb{R}^n)$. Let $B := \{\phi \in \mathcal{D}(\mathbb{R}^n) : \int_{\mathbb{R}^n} |\phi(x)|\omega(x)dx \leq 1\}$. By the assumption (1.11),

$$\|\check{\varphi} * \phi\|_{1,\omega,k} \leq A \max_{|\alpha|\leq k} \int_{\mathbb{R}^n} |\varphi^{(\alpha)}(x)| e^{a|x|} dx, \quad \forall k \in \mathbb{N}_0,\ \forall \phi \in B,$$

namely, the set $\check{\varphi} * B$ is bounded in $\mathcal{K}_1(\mathbb{R}^n)$. Consequently, $\sup_{\phi\in B} |\langle f * \varphi, \phi\rangle| = \sup_{\phi\in B} |\langle f, \check{\varphi} * \phi\rangle| < \infty$. Since $\mathcal{D}(\mathbb{R}^n)$ is dense in $L^1_\omega(\mathbb{R}^n)$, this implies that $f * \varphi \in (L^1_\omega(\mathbb{R}^n))'$, i.e., $\sup_{h\in\mathbb{R}^n} |(f * \varphi)(h)|/\omega(h) < \infty$, as claimed.

$(b) \Rightarrow (c)$. Notice that $(V_\psi T_{-h} f)(x, z) = e^{2\pi i z\cdot h} V_\psi f(x + h, z)$. Fix $\lambda \geq 0$. By Corollary 3.3.1 (cf. (3.19)), there are $k \in \mathbb{N}_0$ and $C_\lambda > 0$ such that, for all $x, h \in \mathbb{R}^n$ and $z \in \Pi_\lambda$,

$$|e^{2\pi i z\cdot h} V_\psi f(x + h, z)| \leq C_\lambda \omega(h)(1 + |z|)^k e^{(k+2\pi\lambda)|x|}.$$

Taking $x = 0$ and $\Im m\, z = 0$, one gets (3.25).

$(c) \Rightarrow (a)$. Fix a synthesis window $\gamma \in \mathcal{K}_1(\mathbb{R}^n)$. In view of (1.11), one has that if j is any non-negative even integer and $\lambda \geq 0$, then, for all $\varphi \in \mathcal{D}_{L^1_\omega}(\mathbb{R}^n)$,

$$\begin{aligned}
&\sup_{z\in\Pi_\lambda} (1 + |z|^2)^{j/2} \int_{\mathbb{R}^n} e^{-2\pi x\cdot \Im m\, z} \omega(x) |V_{\overline{\gamma}}\varphi(x, z)| dx \\
&\leq \tilde{C}_j \sum_{|\beta_1|+|\beta_2|\leq j} \iint_{\mathbb{R}^{2n}} \omega(x) |\varphi^{(\beta_1)}(t)\gamma^{(\beta_2)}(t - x)| e^{2\pi\lambda|t-x|} dt dx \\
&\leq A\tilde{C}_j \|\varphi\|_{1,\omega,j} \max_{|\beta|\leq j} \int_{\mathbb{R}^n} |\gamma^{(\beta)}(x)| e^{(2\pi\lambda+a)|x|} dx \leq C_{j,\lambda} \|\varphi\|_{1,\omega,j}.
\end{aligned}$$

We may assume that s is an even integer. By (3.25) and the previous estimate, we obtain, for every $\varphi \in \mathcal{K}_1(\mathbb{R}^n)$,

$$\begin{aligned}
|\langle f, \varphi\rangle| &\leq \frac{C}{(\psi, \gamma)_{L^2}} \iint_{\mathbb{R}^{2n}} (1 + |\xi|)^s \omega(x) \left|V_{\overline{\gamma}}\varphi(x, -\xi)\right| dx d\xi \\
&\leq C_s \|\varphi\|_{1,\omega,s+n+1},
\end{aligned}$$

which yields $f \in \mathcal{B}'_\omega(\mathbb{R}^n)$.

Part (ii). Any of the conditions implies that $f \in \mathcal{B}'(\mathbb{R}^n)$. $(a)' \Rightarrow (b)'$. Fix $\varphi \in \mathcal{K}_1(\mathbb{R}^n)$. Given fixed $\varepsilon > 0$, we must show that $\limsup_{|h|\to\infty} |\langle T_{-h} f, \varphi\rangle|/\omega(h) \leq \varepsilon$. Notice that $\{T_h\varphi/\omega(h) : h \in \mathbb{R}^n\}$ is a bounded set in $\mathcal{D}_{L^1_\omega}(\mathbb{R}^n)$. Since f is in the closure of $\mathcal{D}(\mathbb{R}^n)$ in $\mathcal{B}'_\omega(\mathbb{R}^n)$, there is $\phi \in \mathcal{D}(\mathbb{R}^n)$ such that $|\langle T_{-h}(f - \phi), \varphi| \leq \varepsilon\omega(h)$ for every $h \in \mathbb{R}^n$. Consequently,

$$\limsup_{|h|\to\infty} \frac{|\langle T_{-h} f, \varphi\rangle|}{\omega(h)} \leq \varepsilon + \lim_{|h|\to\infty} \frac{1}{\omega(h)} \left| \int_{\mathbb{R}^n} \varphi(t - h)\phi(t) dt \right| \leq \varepsilon.$$

$(b)' \Rightarrow (c)'$. If ξ remains on a compact of $K \subset \mathbb{R}^n$, then $\{\overline{M_\xi \psi} : \xi \in K\}$ is compact in $\mathcal{K}_1(\mathbb{R}^n)$, thus, by the Banach-Steinhaus theorem,

$$0 = \lim_{|x|\to\infty} \frac{|\langle T_{-x} f, \overline{M_\xi \psi}\rangle|}{\omega(x)} = \lim_{|x|\to\infty} \frac{|V_\psi f(x,\xi)|}{\omega(x)}, \text{ uniformly in } \xi \in K.$$

There is s such that (3.25) holds. Taking into account that the above limit holds for arbitrary K, we obtain that (3.26) is satisfied for any $s' > s$.

$(c)' \Rightarrow (a)'$. We may assume that s' is a non-negative even integer. Consider the weight $\omega_{s'}(x,\xi) = \omega(x)(1+|\xi|)^{s'}$. The limit relation (3.26) implies that $V_\psi f$ is in the closure of $\mathcal{K}_1(\mathbb{R}^n) \otimes \mathcal{S}(\mathbb{R}^n)$ with respect to the norm $\| \, \|_{L^{\infty,\infty}_{1/\omega_{s'}}}$. Since we have the dense embedding $\mathcal{U}(\mathbb{C}^n) \hookrightarrow \mathcal{S}(\mathbb{R}^n)$, there is a sequence $\{\Phi_j\}_{j=1}^\infty \subset \mathcal{K}_1(\mathbb{R}^n) \widehat{\otimes} \mathcal{U}(\mathbb{C}^n)$ such that $\lim_{j\to\infty} \Phi_j = V_\psi f$ in $L^{\infty,\infty}_{1/\omega_{s'}}(\mathbb{R}^{2n})$. Let $\gamma \in \mathcal{K}_1(\mathbb{R}^n)$ be a synthesis window and set $\phi_j = V_\gamma^* \Phi_j \in \mathcal{K}_1(\mathbb{R}^n)$ (cf. Proposition 3.3.1). By the relations (3.23) and (4.4.1), we have for any $\varphi \in \mathcal{K}_1(\mathbb{R}^n)$,

$$|\langle f - \phi_j, \varphi\rangle| \leq \frac{C\|\varphi\|_{1,\omega,s+n+1}}{(\gamma,\psi)_{L^2}} \|V_\psi f - \Phi_j\|_{L^{\infty,\infty}_{1/\omega_{s'}}},$$

where C does not depend on j. Thus, $\phi_j \to f$ in $\mathcal{B}'_\omega(\mathbb{R}^n)$, which in turn implies that $f \in \dot{\mathcal{B}}'_\omega(\mathbb{R}^n)$ because $\mathcal{D}(\mathbb{R}^n) \hookrightarrow \mathcal{K}_1(\mathbb{R}^n)$.

□

We immediately get the ensuing result, a corollary of Theorem 3.4.1.

Corollary 3.4.1. $\mathcal{K}'_1(\mathbb{R}^n) = \bigcup_\omega \mathcal{B}'_\omega(\mathbb{R}^n) = \bigcup_\omega \dot{\mathcal{B}}'_\omega(\mathbb{R}^n)$. *In particular,* $f \in \mathcal{D}'(\mathbb{R}^n)$ *belongs to* $\mathcal{K}'_1(\mathbb{R}^n)$ *if and only if there is* $s \in \mathbb{R}$ *such that* $\{e^{-s|h|} T_{-h} f : h \in \mathbb{R}^n\}$ *is bounded in* $\mathcal{D}'(\mathbb{R}^n)$.

3.5 Characterizations through modulation spaces

We present here the characterization of the spaces $\mathcal{K}_1(\mathbb{R}^n)$, $\mathcal{K}'_1(\mathbb{R}^n)$, $\mathcal{B}'_\omega(\mathbb{R}^n)$, $\dot{\mathcal{B}}'_\omega(\mathbb{R}^n)$, $\mathcal{U}(\mathbb{C}^n)$, and $\mathcal{U}'(\mathbb{C}^n)$ in terms of modulation spaces.

Let us recall the definition of the modulation spaces. There are several equivalent ways to introduce them [26]. Here we follow the approach from [8, 9] based on Gelfand-Shilov spaces. We are interested in modulation spaces with respect to weights that are exponentially moderate. We denote by $\mathfrak{M}$ the class of all weight functions m on $\mathbb{R}^{2n}$ that satisfy inequalities (for some constants $A > 0$ and $a \geq 0$):

$$\frac{m(x_1 + x_2, \xi_1 + \xi_2)}{m(x_1, \xi_1)} \leq A e^{a(|x_2|+|\xi_2|)}, \quad (x_1,\xi_1),(x_2,\xi_2) \in \mathbb{R}^{2n}.$$

Observe that any so-called v-moderate weight [26] belongs to $\mathfrak{M}$. We also consider the Gelfand-Shilov space $\Sigma_1^1(\mathbb{R}^n)$ of Beurling type (sometimes also denoted as $\mathcal{S}^{(1)}(\mathbb{R}^n)$ or $\mathcal{G}(\mathbb{R}^n)$) and its dual $(\Sigma_1^1)'(\mathbb{R}^n)$. The space $\Sigma_1^1(\mathbb{R}^n)$ consists [7] of all entire functions φ such that

$$\sup_{x\in\mathbb{R}^n} |\varphi(x)| e^{\lambda|x|} < \infty \quad \text{and} \quad \sup_{\xi\in\mathbb{R}^n} |\widehat{\varphi}(\xi)| e^{\lambda|\xi|} d\xi < \infty, \quad \forall \lambda > 0.$$

We refer to [62] for topological properties of $\Sigma_1^1(\mathbb{R}^n)$. The dual space $(\Sigma_1^1)'(\mathbb{R}^n)$ is also known as the space of Silva ultradistributions of exponential type [38, 91] or the space of Fourier ultra-hyperfunctions [60]. If $m \in \mathfrak{M}$, $\psi \in \Sigma_1^1(\mathbb{R}^n) \setminus \{0\}$, and $p, q \in [1, \infty]$, the modulation space $M_m^{p,q}(\mathbb{R}^n)$ is defined as the Banach space

$$M_m^{p,q}(\mathbb{R}^n) = \{f \in (\Sigma_1^1)'(\mathbb{R}^n) : \|f\|_{M_m^{p,q}} := \|V_\psi f\|_{L_m^{p,q}} < \infty\}. \tag{3.27}$$

This definition does not depend on the choice of the window ψ, as different windows lead to equivalent norms. If $p = q$, then we write $M_m^p(\mathbb{R}^n)$ instead of $M_m^{p,q}(\mathbb{R}^n)$. The space $M_m^1(\mathbb{R}^n)$ (for $m = 1$) was original introduced by Feichtinger in [28]. We shall also define $\dot{M}_m^\infty(\mathbb{R}^n)$ as the closed subspace of $M_m^\infty(\mathbb{R}^n)$ given by $\dot{M}_m^\infty(\mathbb{R}^n) = \{f \in (\Sigma_1^1)'(\mathbb{R}^n) : \lim_{|(x,\xi)|\to\infty} m(x,\xi)|V_\psi f(x,\xi)| = 0\}$.

We now connect the space of exponential distributions with the modulation spaces. For it, we consider the weight subclass $\mathfrak{M}_1 \subset \mathfrak{M}$ consisting of all weights m such that (for some $s, a \geq 0$ and $A > 0$)

$$\frac{m(x_1 + x_2, \xi_1 + \xi_2)}{m(x_1, \xi_1)} \leq Ae^{a|x_2|}(1 + |\xi_2|)^s, \quad (x_1, \xi_1), (x_2, \xi_2) \in \mathbb{R}^{2n}. \tag{3.28}$$

Let $m \in \mathfrak{M}_1$. By Proposition 3.3.1, $\mathcal{K}_1(\mathbb{R}^n) \subset M_m^{p,q}(\mathbb{R}^n)$. Since $\Sigma_1^1(\mathbb{R}^n) \hookrightarrow \mathcal{K}_1(\mathbb{R}^n)$, we obtain that $\mathcal{K}_1(\mathbb{R}^n)$ is dense (weakly* dense if $p = \infty$ or $q = \infty$) in $M_m^{p,q}(\mathbb{R}^n)$ and therefore $M_m^{p,q}(\mathbb{R}^n) \subset \mathcal{K}_1'(\mathbb{R}^n)$. It follows from the results of [26] that we may use $\psi \in \mathcal{K}_1(\mathbb{R}^n) \setminus \{0\}$ in (3.27). Also, if $f \in M_m^{p,q}(\mathbb{R}^n)$ and $\psi \in \mathcal{K}_1(\mathbb{R}^n)$ then $V_\psi f$ is an entire function in the second variable (cf. Section 3.3); the next proposition describes the norm behavior of $V_\psi f(x, z)$ in the complex variable $z \in \mathbb{C}^n$.

Proposition 3.5.1. *Let $m \in \mathfrak{M}_1$, $p, q \in [1, \infty]$, and $\psi \in \mathcal{K}_1(\mathbb{R}^n) \setminus \{0\}$. If $f \in M_m^{p,q}(\mathbb{R}^n)$, then* $(\forall \lambda \geq 0)$

$$\sup_{|\eta|\leq\lambda} \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |e^{-2\pi x\cdot\eta} V_\psi f(x, \xi + i\eta) m(x,\xi)|^p dx \right)^{q/p} d\xi \right)^{1/q} < C_\lambda \|f\|_{M_m^{p,q}}. \tag{3.29}$$

(With obvious changes if $p = \infty$ or $q = \infty$.)

Proof. Assume that m satisfies (3.28) and set $v(x, \xi) = (1 + |\xi|)^s e^{a|x|}$. Notice first that $e^{-2\pi x\cdot\eta} V_\psi f(x, \xi + i\eta) = V_{\psi_\eta} f(x, \xi)$, where $\psi_\eta(t) = e^{2\pi\eta\cdot t}\psi(t)$. As in the proof of [26, Prop. 11.3.2, p. 234],

$$||V_{\psi_\eta} f||_{L_m^{p,q}} = \frac{1}{\|\psi\|_{L^2}^2} ||(V_{\psi_\eta} V_\psi^*) V_\psi f||_{L_m^{p,q}} \leq C ||V_{\psi_\eta}\psi||_{L_v^1} ||V_\psi f||_{L_m^{p,q}}.$$

Since $\{\psi_\eta : |\eta| \leq \lambda\}$ is bounded in $\mathcal{K}_1(\mathbb{R}^n)$, we obtain that $\{V_{\psi_\eta}\psi : |\eta| \leq \lambda\}$ is bounded in $\mathcal{K}_1(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}(\mathbb{C}^n)$; hence $\sup_{|\eta|\leq\lambda} ||V_{\psi_\eta}\psi||_{L_v^1} < \infty$.

□

Using the fundamental identity of time-frequency analysis, i.e. [26, p. 40] $V_\psi f(x, \xi) = e^{-2\pi i x\cdot\xi} V_{\widehat{\psi}} \widehat{f}(\xi, -x)$, we can transfer results from $\mathcal{K}_1'(\mathbb{R}^n)$ into $\mathcal{U}'(\mathbb{R}^n)$ by employing the weight class $\mathfrak{M}_2 = \{m \in \mathfrak{M} : \tilde{m}(x, \xi) = m(\xi, x) \in \mathfrak{M}_1\}$.

For $s, a \geq 0$, we employ the following special classes of weights (ω satisfies the conditions imposed in Subsection **??**):

$$v_{s,a}(x,\xi) := e^{a|x|}(1+|\xi|)^s \quad \text{and } \omega_s(x,\xi) := \omega(x)(1+|\xi|)^s.$$

Clearly $v_{s,a}, \omega_s \in \mathfrak{M}_1$. Obviously, for every $m \in \mathfrak{M}_1$ there are $s, a \geq 0$ such that $M^{p,q}_{v_{s,a}}(\mathbb{R}^n) \subseteq M^{p,q}_m(\mathbb{R}^n) \subseteq M^{p,q}_{1/v_{s,a}}(\mathbb{R}^n)$.

Proposition 3.5.2. *Let $p, q \in [1,\infty]$. Then,*

$$\mathcal{K}_1'(\mathbb{R}^n) = \bigcup_{m\in\mathfrak{M}_1} M^{p,q}_m(\mathbb{R}^n), \qquad \mathcal{U}'(\mathbb{C}^n) = \bigcup_{m\in\mathfrak{M}_2} M^{p,q}_m(\mathbb{R}^n), \tag{3.30}$$

$$\mathcal{K}_1(\mathbb{R}^n) = \bigcap_{m\in\mathfrak{M}_1} M^{p,q}_m(\mathbb{R}^n), \qquad \mathcal{U}(\mathbb{C}^n) = \bigcap_{m\in\mathfrak{M}_2} M^{p,q}_m(\mathbb{R}^n), \tag{3.31}$$

$$\mathcal{B}_\omega'(\mathbb{R}^n) = \bigcup_{s>0} M^{\infty}_{1/\omega_s}(\mathbb{R}^n), \quad \textit{and} \quad \dot{\mathcal{B}}_\omega'(\mathbb{R}^n) = \bigcup_{s>0} \dot{M}^{\infty}_{1/\omega_s}(\mathbb{R}^n). \tag{3.32}$$

Proof. The results for $\mathcal{U}(\mathbb{C}^n)$ and $\mathcal{U}'(\mathbb{C}^n)$ follow from those for $\mathcal{K}_1(\mathbb{R}^n)$ and $\mathcal{K}_1'(\mathbb{R}^n)$. The equalities in (3.32) are a reformulation of the equivalences $(a) \Leftrightarrow (c)$ and $(a)' \Leftrightarrow (c)'$ from Theorem 3.4.1. By (3.28) and [26, Cor. 12.1.10, p. 254], given $m \in \mathfrak{M}_1$, there are $s, a > 0$ such that the embeddings $M^{\infty}_{v_{s+n+1,a+\varepsilon}}(\mathbb{R}^n) \subseteq M^{p,q}_m(\mathbb{R}^n) \subseteq M^{\infty}_{1/v_{s,a}}(\mathbb{R}^n)$ hold. Thus, part (a) from Lemma 3.3.1 gives the equality $\mathcal{K}_1'(\mathbb{R}^n) = \bigcup_{s,a>0} M^{\infty}_{1/v_{s,a}}(\mathbb{R}^n) = \bigcup_{m\in\mathfrak{M}_1} M^{p,q}_m(\mathbb{R}^n)$. In view of Proposition 3.3.1, it only remains to show that

$$\bigcap_{m\in\mathfrak{M}_1} M^{p,q}_m(\mathbb{R}^n) = \bigcap_{s,a>0} M^{\infty}_{v_{s,a}}(\mathbb{R}^n) \subseteq \mathcal{K}_1(\mathbb{R}^n).$$

We show the latter inclusion by proving that if $f \in M^{\infty}_{v_{s,a}}(\mathbb{R}^n)$ (with $s, a > 0$), then $\widehat{f}$ is holomorphic in the tube $\mathbb{R}^n + i\{\eta \in \mathbb{R}^n : |\eta| < a/(2\pi)\}$ and satisfies

$$\sup_{|\Im m\, z|\leq\lambda} (1+|z|^2)^{s/2}|\widehat{f}(z)| < \infty, \quad \forall \lambda < \frac{a}{2\pi}. \tag{3.33}$$

In fact, choose a positive window $\psi \in \mathcal{D}(\mathbb{R}^n)$ such that $\sum_{j\in\mathbb{Z}^n}\psi(t-j) = 1$ for all $t \in \mathbb{R}^n$. Since $f = \sum_{j\in\mathbb{Z}^n} fT_j\psi$, we obtain $\widehat{f} = \sum_{j\in\mathbb{Z}^n} V_\psi f(j,\,\cdot\,)$, with convergence in $\mathcal{U}'(\mathbb{C}^n)$. In view of Proposition 3.5.1, each $V_\psi f(j,z)$ is entire in z and satisfies the bounds

$$\sup_{|\Im m\, z|\leq\lambda} |(1+|z|^2)^{s/2}|V_\psi(j,z)| < C_\lambda e^{-(a-2\pi\lambda)|j|}.$$

The Weierstrass theorem implies that $\widehat{f}(z) = \sum_{j\in\mathbb{Z}^n} V_\psi f(j,z)$ is holomorphic in the stated tube domain and we also obtain (3.33). Summing up, if $f \in \bigcap_{s,a>0} M^{\infty}_{v_{s,a}}(\mathbb{R}^n)$, then $\widehat{f} \in \mathcal{U}(\mathbb{C}^n)$, i.e., $f \in \mathcal{K}_1(\mathbb{R}^n)$. □

The following corollary collects what was shown in the proof of Proposition 3.5.2.

Corollary 3.5.1. *Let $s,a>0$. If $f\in M^{\infty}_{v_{s,a}}(\mathbb{R}^n)$, then $\widehat{f}$ is holomorphic in the tube $\mathbb{R}^n+i\{\eta\in\mathbb{R}^n:\ |\eta|<a/(2\pi)\}$ and satisfies the bounds (3.33).*

We make a remark concerning Proposition 3.5.2.

Remark 3.5.1. Employing [102, Thrms. 3.2 and 3.4], Proposition 3.5.2 can be extended for $p,q\in(0,\infty]$.

3.6 Tauberian theorems for S-asymptotics of distributions

In this section we characterize the S-asymptotic behavior of distributions in terms of the STFT. As explained in Section 2.3, the idea of the S-asymptotics is to study the asymptotic properties of the translates $T_{-h}f$ with respect to a locally bounded and measurable comparison function $c:\mathbb{R}^n\to(0,\infty)$. It is said that $f\in\mathcal{K}_1'(\mathbb{R}^n)$ has S-asymptotic behavior with respect to c if there is $g\in\mathcal{D}'(\mathbb{R}^n)$ such that

$$f(t+h)\sim c(h)g(t)\quad \text{in } \mathcal{K}_1'(\mathbb{R}^n)\ \text{ as } |h|\to\infty, \tag{3.34}$$

which of course means that $(f*\check{\varphi})(h)\sim c(h)\int_{\mathbb{R}^n}\varphi(t)g(t)dt$ as $|h|\to\infty$, for each $\varphi\in\mathcal{D}(\mathbb{R}^n)$ (or, equivalently, $\varphi\in\mathcal{K}_1(\mathbb{R}^n)$).

In order to move further, we give an asymptotic representation formula and Potter type estimates [2] for c:

Lemma 3.6.1. *The locally bounded measurable function c satisfies (2.15) if and only if there is $b\in C^\infty(\mathbb{R}^n)$ such that $\lim_{|x|\to\infty}b^{(\alpha)}(x)=0$ for every multi-index $|\alpha|>0$ and*

$$c(x)\sim\exp\left(\beta\cdot x+b(x)\right)\qquad \text{as } |x|\to\infty. \tag{3.35}$$

In particular, for each $\varepsilon>0$ there are constants $a_\varepsilon,A_\varepsilon>0$ such that

$$a_\varepsilon\exp(\beta\cdot t-\varepsilon|t|)\le\frac{c(t+h)}{c(h)}\le A_\varepsilon\exp(\beta\cdot t+\varepsilon|t|),\quad t,h\in\mathbb{R}^n. \tag{3.36}$$

Proof. By considering $e^{-\beta\cdot t}c(t)$, one may assume that $\beta=0$. Let $\varphi\in\mathcal{D}(\mathbb{R}^n)$ be such that $\int_{\mathbb{R}^n}\varphi(t)dt=1$. Set $b(t)=\int_{\mathbb{R}^n}\log c(t+x)\varphi(t)dt$. Clearly, $b\in C^\infty(\mathbb{R}^n)$ and the relation (2.15) implies that $b(x)=\log c(x)+o(1)$ and $b^{(\alpha)}(x)=o(1)$ as $|x|\to\infty$, for each multi-index $|\alpha|>0$. This gives (3.35).

Conversely, since c is locally bounded, we may assume that actually $c(x)=e^{\beta\cdot x+b(x)}$, but $|b(t+h)-b(h)|\le|t|\max_{\xi\in[h,t+h]}|\nabla b(\xi)|$, which gives (2.15). Using the fact that $|\nabla b|$ is bounded, the same argument yields (3.36). □

Observe that Lemma 3.6.1 also tells us that the space $\mathcal{B}_c'(\mathbb{R}^n)$ is well-defined for c. We can now characterize (3.34) in terms of the STFT. The direct part of the following theorem is an Abelian result, while the converse may be regarded as a Tauberian theorem.

Theorem 3.6.1. *Let $f \in \mathcal{K}_1'(\mathbb{R}^n)$ and $\psi \in \mathcal{K}_1(\mathbb{R}^n)\backslash\{0\}$. If $f \in \mathcal{K}_1'(\mathbb{R}^n)$ has the S-asymptotic behavior (3.34) then, for every $\lambda \geq 0$,*

$$\lim_{|h|\to\infty} e^{2\pi i z\cdot h}\frac{V_\psi f(x+h,z)}{c(h)} = V_\psi g(x,z), \tag{3.37}$$

uniformly for $z \in \Pi_\lambda$ and x in compact subsets of $\mathbb{R}^n$.

Conversely, suppose that the limits

$$\lim_{|x|\to\infty} e^{2\pi i\xi\cdot x}\frac{V_\psi f(x,\xi)}{c(x)} = J(\xi) \in \mathbb{C} \tag{3.38}$$

exist for almost every $\xi \in \mathbb{R}^n$. If there is $s \in \mathbb{R}$ such that

$$\sup_{(x,\xi)\in\mathbb{R}^{2n}} (1+|\xi|)^{-s}\frac{|V_\psi f(x,\xi)|}{c(x)} < \infty, \tag{3.39}$$

then f has the S-asymptotic behavior (3.34) with $g(t) = Ce^{\beta\cdot t}$, where the constant is completely determined by the equation $J(\xi) = C\overline{\widehat{\psi}(-\xi - i\beta/(2\pi))}$.

Remark 3.6.1. Assume (3.39). Consider a weight of the form $m_\varepsilon(x,\xi) = e^{\beta\cdot x+\varepsilon|x|}(1+|\xi|)^s$ with $\varepsilon > 0$. It will be shown below that the asymptotics (3.34) holds in the weak* topology of $M^\infty_{1/m_\varepsilon}(\mathbb{R}^n)$, i.e., $(f * \check{\varphi})(h) \sim c(h)\langle g,\varphi\rangle$ as $|h| \to \infty$ for every φ in the modulation space $M^1_{m_\varepsilon}(\mathbb{R}^n)$. Furthermore, one may use in (3.38) and (3.39) a window $\psi \in M^1_{m_\varepsilon}(\mathbb{R}^n)\backslash\{0\}$.

Proof. Fix $\lambda \geq 0$ and a compact $K \subset \mathbb{R}^n$. Note that the set

$$\{\overline{M_z T_x\psi} : (x,z) \in K \times \Pi_\lambda\}$$

is compact in $\mathcal{K}_1(\mathbb{R}^n)$. By the Banach-Steinhaus theorem,

$$\lim_{|h|\to\infty} e^{2\pi i z h}\frac{V_\psi f(x+h,z)}{c(h)} = \lim_{|h|\to\infty}\left\langle \frac{T_{-h}f}{c(h)}, \overline{M_z T_x\psi}\right\rangle = \left\langle g, \overline{M_z T_x\psi}\right\rangle,$$

uniformly with respect to $(x,z) \in K \times \Pi_\lambda$, as asserted in (3.37).

Conversely, assume (3.38) and (3.39). Let $H = \{\xi \in \mathbb{R}^n : \text{(3.38) holds}\}$. In view of Theorem 3.4.1, we have that $f \in \mathcal{B}_c'(\mathbb{R}^n)$ or, equivalently, $\{T_{-h}f/c(h) : h \in \mathbb{R}^n\}$ is bounded in $\mathcal{K}_1'(\mathbb{R}^n)$. By the Banach-Steinhaus theorem and the Montel property of $\mathcal{K}_1'(\mathbb{R}^n)$, $T_{-h}f/c(h)$ converges strongly to a distribution g in $\mathcal{K}_1'(\mathbb{R}^n)$ if and only if $\lim_{|h|\to\infty}\langle T_{-h}f,\varphi\rangle/c(h)$ exists for φ in a dense subspace of $\mathcal{K}_1(\mathbb{R}^n)$. Let D be the linear span of $\{\overline{M_\xi T_x\psi} : (x,\xi) \in \mathbb{R}^n \times H\}$. By the desingularization formula (3.23) and the Hahn-Banach theorem, we have that D is dense in $\mathcal{K}_1(\mathbb{R}^n)$. Thus, it suffices to verify that $\lim_{|h|\to\infty}\langle T_{-h}f, \overline{M_\xi T_x\psi}\rangle/c(h)$ exists for each $(x,\xi) \in \mathbb{R}^n \times H$. But in this case (2.15) and (3.38) yield

$$\begin{aligned}\lim_{|h|\to\infty}\frac{\langle T_{-h}f, \overline{M_\xi T_x\psi}\rangle}{c(h)} &= \lim_{|h|\to\infty} e^{2\pi i\xi\cdot h}\frac{V_\psi f(x+h,\xi)}{c(h)} \\ &= e^{(\beta-2\pi i\xi)\cdot x}\lim_{|h|\to\infty} e^{2\pi i\xi\cdot(x+h)}\frac{V_\psi f(x+h,\xi)}{c(h+x)}\end{aligned}$$

$$= e^{(\beta - 2\pi i\xi)\cdot x} J(\xi),$$

as required. We already know that $g(t) = Ce^{\beta \cdot t}$. Comparison between (3.37) and (3.38) leads to $J(\xi) = V_\psi g(0,\xi) = C\int_{\mathbb{R}^n} \overline{\psi(t)} e^{\beta\cdot t - 2\pi i \xi \cdot t} dt$. To show the assertion from Remark 3.6.1, note first that, by using (3.36), one readily verifies that

$$\sup_{h\in\mathbb{R}^n} \frac{||T_{-h} f||_{M^\infty_{1/m_\varepsilon}}}{c(h)} < \infty.$$

Since we have the dense embedding $\mathcal{K}_1(\mathbb{R}^n) \hookrightarrow M^1_{m_\varepsilon}(\mathbb{R}^n)$, we also have that D is dense in $M^1_{m_\varepsilon}(\mathbb{R}^n)$ and the assertion follows at once. The fact that one may use a window $\psi \in M^1_{m_\varepsilon}(\mathbb{R}^n)\backslash\{0\}$ in (3.38) and (3.39) follows in a similar fashion because in this case the desingularization formula (3.23) still holds.

□

Let us make two addenda to Theorem 3.6.1. The ensuing corollary improves Remark 3.6.1, provided that c satisfies the extended submultiplicative condition (for some $A > 0$):

$$c(t+h) \leq Ac(t)c(h). \tag{3.40}$$

Corollary 3.6.1. *Assume that c satisfies (3.40) and set $c_s(x,\xi) = c(x)(1+|\xi|)^s$, $s \in \mathbb{R}$. If $f \in M^\infty_{1/c_s}(\mathbb{R}^n)$ and there is $\psi \in M^1_{c_s}(\mathbb{R}^n)\setminus\{0\}$ such that the limits (3.38) exist for almost every $\xi \in \mathbb{R}^n$, then, for some g, the S-asymptotic behavior (3.34) holds weakly* in $M^\infty_{1/c_s}(\mathbb{R}^n)$, that is, $(f * \check{\varphi})(h) \sim c(h)\langle g,\varphi\rangle$ as $|h|\to\infty$ for every $\varphi \in M^1_{c_s}(\mathbb{R}^n)$.*

Proof. We retain the notation from the proof of Theorem 3.6.1. The assumption $f \in M^\infty_{1/c_s}(\mathbb{R}^n)$ of course tells us that (3.39) holds. Employing the hypothesis (3.40), one readily sees that $\sup_{h\in\mathbb{R}^n} \|T_{-h} f\|_{M^\infty_{1/c_s}}/c(h) < \infty$. A similar argument to the one used in the proof of Theorem 3.6.1 yields that the set D associated to ψ is dense in $M^1_{c_s}(\mathbb{R}^n)$, which as above yields the result. □

In dimension $n = 1$, the next theorem actually obtains the ordinary asymptotic behavior of f in case it is a regular distribution on $(0,\infty)$ satisfying an additional Tauberian condition. We fix m_ε as in Remark 3.6.1 and c_s as in Corollary 3.6.1.

Theorem 3.6.2. *Let $f \in M^\infty_{1/c_s}(\mathbb{R})$. Suppose that*

$$\lim_{x\to\infty} e^{2\pi i\xi\cdot x} \frac{V_\psi f(x,\xi)}{c(x)} = J(\xi) \in \mathbb{C}, \tag{3.41}$$

for almost every $\xi \in \mathbb{R}$, where $\psi \in M^1_{m_\varepsilon}(\mathbb{R})\setminus\{0\}$ (resp. $\psi \in M^1_{c_s}(\mathbb{R})\setminus\{0\}$ if c satisfies (3.40)). If there is $\alpha \geq 0$ such that $e^{\alpha t} f(t)$ is a positive non-decreasing function on the interval $(0,\infty)$, then

$$\lim_{t\to\infty} \frac{f(t)}{c(t)} = C, \tag{3.42}$$

where C is the constant from Theorem 3.6.1.

Proof. Using (3.41), the same method from Theorem 3.6.1 applies to show that $f(t+h) \sim Cg(t)$ in $\mathcal{K}_1'(\mathbb{R})$ as $h \to \infty$, where $g(t) = Ce^{\beta t}$. We may assume that $\alpha \geq -\beta$. Set $\tilde{f}(t) = e^{\alpha t} f(t)$, $b(t) = e^{\alpha t} c(t)$, and $r = \alpha + \beta \geq 0$. It is enough to show that $\tilde{f}(t) \sim Cb(t)$ as $t \to \infty$, whence (3.42) would follow. By (3.34), we have that

$$\tilde{f}(t+h) \sim b(h) C e^{rt} \quad \text{as } h \to \infty \text{ in } \mathcal{K}_1'(\mathbb{R}),$$

i.e.,

$$\langle \tilde{f}(t+h), \varphi(t) \rangle \sim Cb(h) \int_{-\infty}^{\infty} e^{rt} \varphi(t) dt, \quad \forall \varphi \in \mathcal{K}_1(\mathbb{R}^n). \tag{3.43}$$

Let $\varepsilon > 0$ be arbitrary. Choose a non-negative test function $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\operatorname{supp} \varphi \subseteq (0, \varepsilon)$ and $\int_0^\varepsilon \varphi(t) dt = 1$. Using the fact that $\tilde{f}$ is non-decreasing on $(0, \infty)$ and (3.43), we obtain

$$\begin{aligned} \limsup_{h\to\infty} \frac{\tilde{f}(h)}{b(h)} &= \limsup_{h\to\infty} \frac{\tilde{f}(h)}{b(h)} \int_0^\varepsilon \varphi(t) dt \leq \lim_{h\to\infty} \frac{1}{b(h)} \int_0^\varepsilon \tilde{f}(t+h) \varphi(t) dt \\ &= \lim_{h\to\infty} \frac{\langle \tilde{f}(t+h), \varphi(t) \rangle}{b(h)} = C \int_0^\varepsilon e^{rt} \varphi(t) \mathrm{d}t \leq Ce^{r\varepsilon}, \end{aligned}$$

taking $\varepsilon \to 0^+$, we have shown that $\limsup_{h\to\infty} \tilde{f}(h)/b(h) \leq C$. Similarly, choosing in (3.43) a non-negative φ such that $\operatorname{supp} \varphi \subseteq (-\varepsilon, 0)$ and $\int_{-\varepsilon}^0 \varphi(t) dt = 1$, one obtains $\liminf_{h\to\infty} \tilde{f}(h)/b(h) \geq C$. This shows that $\tilde{f}(t) \sim Cb(t)$ as $t \to \infty$, as claimed. □

We conclude this Section with the following Theorem.

Theorem 3.6.3. *Let f be a positive non-decreasing function on $[0, \infty)$ and let ψ be a positive function such that $\psi'' \in L^1_{loc}(\mathbb{R})$ and $\int_{-\infty}^{\infty} (\psi(t) + |\psi'(t)| + |\psi''(t)|) e^{\beta t + \varepsilon |t|} dt < \infty$, where $\beta \geq 0$ and $\varepsilon > 0$. Suppose that the limits*

$$\lim_{x\to\infty} \frac{e^{2\pi i \xi x}}{e^{\beta x} L(e^x)} \int_0^\infty f(t) \psi(t-x) e^{-2\pi i \xi t} \, dt = J(\xi) \tag{3.44}$$

exist for every $\xi \in \mathbb{R}$, then

$$\lim_{x\to\infty} \frac{f(x)}{e^{\beta x} L(e^x)} = \frac{J(0)}{\int_{-\infty}^{\infty} \psi(t) e^{\beta t} dt} . \tag{3.45}$$

Furthermore, if L satisfies $L(xy) \leq AL(x)L(y)$ for all $x, y > 0$ and some constant A, the requirements over ψ can be relaxed to $\int_{-\infty}^{\infty} (\psi(t) + |\psi'(t)| + |\psi''(t)|) L(e^{|t|}) e^{\beta t} dt < \infty$.

Proof. Set $c(t) = e^{\beta t} L(e^{|t|})$ and, as before (with $s = 0$), $c_0(x, \xi) = c(x)$ and $m_\varepsilon(x, \xi) = e^{\beta x + \varepsilon |x|}$. Note that (3.45) is the same as (3.42). Let us first verify that $\psi \in M^1_{m_\varepsilon}(\mathbb{R})$. In fact, if we take another window $\gamma \in \mathcal{K}_1(\mathbb{R})$, we have

$$\iint_{\mathbb{R}^2} |V_\gamma \psi(x, \xi)| e^{\beta x + \varepsilon |x|} dx d\xi = \iint_{\mathbb{R}^2} (1 + |\xi|^3) |V_\gamma \psi(x, \xi)| e^{\beta x + \varepsilon |x|} dx \frac{d\xi}{1 + |\xi|^3}$$

$$\leq \tilde{C}\left(\iint_{\mathbb{R}^2}\psi(t-x)|\gamma(t)|e^{\beta x+\varepsilon|x|}dtdx+\sum_{j=0}^{3}\iint_{\mathbb{R}^2}|\psi^{(j)}(t-x)\gamma^{(3-j)}(t)|e^{\beta x+\varepsilon|x|}dtdx\right),$$

which is finite (a similar argument shows that $\psi\in M^1_{c_0}(\mathbb{R})$ if $\int_{-\infty}^{\infty}(\psi(t)+|\psi'(t)|+|\psi''(t)|)L(e^{|t|})e^{\beta t}dt<\infty$). In view of Theorem 3.6.2, it is enough to establish $f\in M^{\infty}_{1/c_0}(\mathbb{R})$. Let us first show the crude bound $f(t)=O(c(t))$. Set $A_1=\int_0^{\infty}\psi(t)dt<\infty$. Since f is non-decreasing, we have

$$f(x)\leq\frac{1}{A_1}\int_0^{\infty}f(t+x)\psi(t)dt\leq\frac{1}{A_1}\int_0^{\infty}f(t)\psi(t-x)dt\leq A_2c(x),$$

because of (5.15) with $\xi=0$. Thus

$$|V_{\psi}f(x,\xi)|\leq A_2\int_0^{\infty}c(t)\psi(t-x)dt\leq c(x)\tilde{A}_{\varepsilon}\int_{-\infty}^{\infty}e^{\beta t+\varepsilon|t|}\psi(t)dt<A_3c(x),\quad\forall(x,\xi)\in\mathbb{R}^2$$

(likewise in the other case using $L(xy)\leq AL(x)L(y)$), which completes the proof.

□

Chapter 4

The ridgelet and Radon transforms of distributions

In this Chapter we want to provide a thorough analysis of the ridgelet transform and its transpose, called here the ridgelet synthesis operator, on various test function spaces. Our main results are continuity theorems on such function spaces. We then use our results to develop a distributional framework for the ridgelet transform. In Section 4.5 we apply our continuity theorems for the ridgelet transform to discuss the continuity of the Radon transform on these spaces and their duals. The Radon transform on Lizorkin spaces naturally extends the one considered by Hertle [34] on various distribution spaces. Finally, Section 4.8 deals with Abelian and Tauberian theorems for the ridgelet transform. We should mention that here, we use the constants in the Fourier transform as

$$\widehat{\phi}(\mathbf{w}) = \int_{\mathbb{R}^n} \phi(\mathbf{x}) e^{-i\mathbf{x}\cdot\mathbf{w}} d\mathbf{x}.$$

Moreover, in this Chapter, we use bold letters to denote elements from $\mathbb{R}^n$.

4.1 Preliminaries on the ridgelet and Radon transforms

4.1.1 The ridgelet transform of functions and some distributions

In [4, 5] Candès introduced and studied the continuous ridgelet transform. He developed a harmonic analysis groundwork for this transform and showed that it is possible to obtain constructive and stable approximations of functions by ridgelets. Ridge functions often appear in the literature of approximation theory, statistics, and signal analysis. One of the motivations for the introduction of the "X-let" transforms, such as the ridgelet or curvelet transforms, comes from the search of optimal representations of signals in high-dimensions.

Let $\psi \in \mathcal{S}(\mathbb{R})$. For $(\mathbf{u}, b, a) \in \mathbb{Y}^{n+1}$, where $\mathbf{u}$ is the orientation parameter, b is the location parameter, and a is the scale parameter, we define the function

$\psi_{\mathbf{u},b,a}: \mathbb{R}^n \to \mathbb{C}$, called *ridgelet*, as

$$\psi_{\mathbf{u},b,a}(\mathbf{x}) = \frac{1}{a}\psi\left(\frac{\mathbf{x}\cdot\mathbf{u}-b}{a}\right), \quad \mathbf{x} \in \mathbb{R}^n.$$

This function is constant along hyperplanes $\mathbf{x}\cdot\mathbf{u} = \text{const.}$, called "ridges". In the orthogonal direction it is a wavelet, hence the name ridgelet. The function ψ is often referred in the literature [4, 5] as a *neuronal activation function.* The ridgelet transform $\mathcal{R}_\psi f$ of an integrable function $f \in L^1(\mathbb{R}^n)$ is defined by

$$\mathcal{R}_\psi f\,(\mathbf{u}, b, a) = \int_{\mathbb{R}^n} f(\mathbf{x})\overline{\psi}_{\mathbf{u},b,a}(\mathbf{x})d\mathbf{x} = \left\langle f(\mathbf{x}), \overline{\psi}_{\mathbf{u},b,a}(\mathbf{x})\right\rangle_{\mathbf{x}}. \tag{4.1}$$

where $(\mathbf{u}, b, a) \in \mathbb{Y}^{n+1}$.

The ridgelet transform can also be canonically defined for distributions $f \in \mathcal{D}'_{L^1}(\mathbb{R}^n)$ via (4.1), because the test function $\psi_{\mathbf{u},b,a} \in \mathcal{D}_{L^\infty}(\mathbb{R}^n)$ and thus the integral formula can be still interpreted in the sense of Schwartz integrable distributions [89, p. 203]. In particular, (4.1) makes sense for $f \in \mathcal{E}'(\mathbb{R}^n)$ or $f \in \mathcal{O}'_C(\mathbb{R}^n)$. On the other hand, if one wishes to extend the definition of the ridgelet transform to more general spaces than $\mathcal{D}'_{L^1}(\mathbb{R}^n)$, one must proceed with care. Even in the L^2 case, (4.1) is not directly extendable to $f \in L^2(\mathbb{R}^n)$ because the defining integral might fail to converge. A similar difficulty is faced when trying to extend the ridgelet transform to distributions: the function $\psi_{\mathbf{u},b,a} \notin \mathcal{S}(\mathbb{R}^n)$ and therefore (4.1) is not well defined for $f \in \mathcal{S}'(\mathbb{R}^n)$. We shall overcome this difficulty in Section 4.4 via a duality approach and define the ridgelet transform of Lizorkin distributions for $\psi \in \mathcal{S}_0(\mathbb{R})$.

4.1.2 The continuous wavelet transform

Wavelets have generate significant interest from both theoretical and applied researchers over the last few decade. The concepts for understanding wavelet were provided by [10, 54] and many others. Usually, the wavelet analysis presents two main important features: the wavelet transform as a time-frequency analysis tool, and the wavelet analysis as part of approximation and function space theory. Wavelet transform is now used in a wide variety of applications in the areas of signal processing, image processing, computer science, acoustics, communications, geophysics, medicine, etc.

Given functions f and ψ, the wavelet transform $\mathcal{W}_\psi f(b, a)$ of f is defined by

$$\mathcal{W}_\psi f(b, a) = \int_{\mathbb{R}} f(x)\frac{1}{a}\overline{\psi}\Big(\frac{x-b}{a}\Big)dx, \quad (b, a) \in \mathbb{H}. \tag{4.2}$$

The expression (4.2) is defined, e.g., if $f, \psi \in L^2(\mathbb{R})$, $f \in L^1(\mathbb{R})$ and $\psi \in L^\infty(\mathbb{R})$, or in other circumstances. We will actually work with the wavelet transform of distributions. So if $f \in \mathcal{S}'(\mathbb{R})$ and $\psi \in \mathcal{S}(\mathbb{R})$ (or $f \in \mathcal{S}'_0(\mathbb{R})$ and $\psi \in \mathcal{S}_0(\mathbb{R})$), one replaces (4.2) by

$$\mathcal{W}_\psi f(b, a) = \left\langle f(x), \frac{1}{a}\overline{\psi}\Big(\frac{x-b}{a}\Big)\right\rangle_x, \quad (b, a) \in \mathbb{H}. \tag{4.3}$$

We have that [37, Thrm. 19.0.1], $\mathcal{W}_\psi : \mathcal{S}_0(\mathbb{R}) \mapsto \mathcal{S}(\mathbb{H})$ is a continuous linear map. Given $\Phi \in \mathcal{S}(\mathbb{H})$, we define *wavelet synthesis operator* with respect to the wavelet ψ as

$$\mathcal{M}_\psi \Phi(t) = \int_0^\infty \int_{\mathbb{R}} \Phi(b,a) \frac{1}{a} \psi\Big(\frac{t-b}{a}\Big) dbda, \quad t \in \mathbb{R}. \tag{4.4}$$

One can show that $\mathcal{M}_\psi : \mathcal{S}(\mathbb{H}) \mapsto \mathcal{S}_0(\mathbb{R})$ is continuous.

We shall say that the wavelet $\psi \in \mathcal{S}_0(\mathbb{R})$ admits a *reconstruction wavelet* if there exists $\eta \in \mathcal{S}_0(\mathbb{R})$ such that

$$c_{\psi,\eta}(\omega) = \int_0^\infty \overline{\hat{\psi}}(r\omega) \hat{\eta}(r\omega) \frac{dr}{r}, \quad \omega \in \mathbb{S},$$

is independent of the direction ω; in such case we set $c_{\psi,\eta} := c_{\psi,\eta}(\omega)$. The wavelet η is called a reconstruction wavelet for ψ. It is easy to find explicit examples of wavelet admitting recosntruction wavelets; in fact, any non-trivial rotation invariant element of $\mathcal{S}_0(\mathbb{R})$ is itself own reconstruction wavelet.

If ψ admits the reconstruction wavelet η, one has the reconstruction formula for the wavelet transform on $\mathcal{S}_0(\mathbb{R})$

$$Id_{\mathcal{S}_0(\mathbb{R})} = \frac{1}{c_{\psi,\eta}} \mathcal{M}_\eta \mathcal{W}_\psi.$$

We refer to Holschneider's book [37] for a distribution wavelet transform theory based on the spaces $\mathcal{S}_0(\mathbb{R})$, $\mathcal{S}(\mathbb{H})$, $\mathcal{S}'_0(\mathbb{R})$, and $\mathcal{S}'(\mathbb{H})$. For the wavelet transform of vector-valued distributions, we refer to [71, Sect. 5 and 8]. Let we mention that the quasiasymptotic behavior is a very suitable concept for wavelet analysis [82, 83, 84, 110, 70]. In fact, the wavelet transform can be thought as a sort of mathematical microscope analyzing a distribution on various length scales around any point of the real axis.

4.1.3 The Radon transform

The Radon transform is named after J. Radon in 1917 who showed how to describe a function in terms of its (integral) projections. The mapping from the function onto the projections is the Radon transform. The inverse Radon transform corresponds to the reconstruction of the function from the projections. Within the realm of image analysis, the Radon transform is mostly known for its role in computed tomography. It is used to model the process of acquiring projections of the original object using X-rays. Given the projection data, the inverse Radon transform, in whatever form (e.g. back-projection), can be applied to reconstruct the original object.

Let f be a function that is integrable on hyperplanes of $\mathbb{R}^n$. For $\mathbf{u} \in \mathbb{S}^{n-1}$ and $p \in \mathbb{R}$, the equation $\mathbf{x} \cdot \mathbf{u} = p$ specifies a hyperplane of $\mathbb{R}^n$. Let $\mathbb{P}^n$ denote the space of all hyperplanes in $\mathbb{R}^n$. Each hyperplane $h \in \mathbb{P}^n$ can be written as $h = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \cdot \mathbf{u} = p\}$. Then, the Radon transform of f is defined as

$$Rf(\mathbf{u},p) = Rf_{\mathbf{u}}(p) := \int_{\mathbf{x}\cdot\mathbf{u}=p} f(\mathbf{x}) d\mathbf{x} = \int_{\mathbb{R}^n} f(\mathbf{x}) \delta(p - \mathbf{x}\cdot\mathbf{u}) d\mathbf{x},$$

where δ is the Dirac delta. Fubini's theorem ensures that if $f \in L^1(\mathbb{R}^n)$, then $Rf \in L^1(\mathbb{S}^{n-1} \times \mathbb{R})$.

Remark 4.1.1. Note that the pairs $(\mathbf{u}, p)$ and $(-\mathbf{u}, -p)$ give the same hyperplane, so the mapping $(\mathbf{u}, p) \to h$ is double covering of $\mathbb{S}^{n-1} \times \mathbb{R}$ onto $\mathbb{P}^n$.

The Fourier transform and the Radon transform are connected by the so-called *Fourier slice theorem* [33];

$$\mathcal{F}(f(\mathbf{x}))(\omega\mathbf{u}) = \hat{f}(\omega\mathbf{u}) = \widehat{Rf}(\mathbf{u}, \omega) = \mathcal{F}(Rf(\mathbf{u}, t))(\omega), \quad \omega \in \mathbb{R},\ \mathbf{u} \in \mathbb{R}^n.$$

According to it, the Radon transform can be computed as

$$Rf(\mathbf{u}, p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\omega\mathbf{u}) e^{ip\omega} d\omega, \quad \mathbf{u} \in \mathbb{S}^{n-1},\ p \in \mathbb{R}, \tag{4.5}$$

for sufficiently regular f (e.g., for $f \in L^1(\mathbb{R}^n)$ such that $\widehat{f} \in L^1(\mathbb{R}^n)$).

The dual Radon transform (or back-projection) $R^*\varrho$ of the function $\varrho \in L^\infty(\mathbb{S}^{n-1} \times \mathbb{R})$ is defined as

$$R^*\varrho(\mathbf{x}) = \int_{\mathbb{S}^{n-1}} \varrho(\mathbf{u}, \mathbf{x} \cdot \mathbf{u}) d\mathbf{u}.$$

The transforms R and R^* are then formal transposes, i.e.,

$$\langle Rf, \varrho \rangle = \langle f, R^*\varrho \rangle . \tag{4.6}$$

For instance, for $f \in L^1(\mathbb{R}^n)$ and $\varrho \in L^\infty(\mathbb{S}^{n-1} \times \mathbb{R})$,

$$\int_{\mathbb{R}^n} f(\mathbf{x}) R^*\varrho(\mathbf{x}) d\mathbf{x} = \int_{-\infty}^{\infty} \int_{\mathbb{S}^{n-1}} Rf(\mathbf{u}, p)\varrho(\mathbf{u}, p) d\mathbf{u} dp.$$

More details on the Radon transform can be found in Helgason's book [33]. See also [24, 34, 35, 51, 76]. In particularly, Hertle [34] has exploited the duality relation (4.6) to extend the definition of the Radon transform as a continuous map between various distribution spaces. In fact, the dual Radon transform R^* : $\mathcal{A}(\mathbb{S}^{n-1} \times \mathbb{R}) \to \mathcal{A}(\mathbb{R}^n)$ is continuous for $\mathcal{A} = \mathcal{D}_{L^1}, \mathcal{E}, \mathcal{O}_C$ and the Radon transform can then be defined on their duals by transposition as in (4.6). Namely, [34, Thrm. 1.4] states that the Radon transform defines a continuous operator form $\mathcal{A}'(\mathbb{R}^n)$ into $\mathcal{A}'(\mathbb{S}^{n-1} \times \mathbb{R})$, where $\mathcal{A}' = \mathcal{E}', \mathcal{O}'_c, \mathcal{D}'_{L^1}$. In Section 4.5 we will enlarge the domain of the Radon transform to the Lizorkin distribution space $\mathcal{S}'_0(\mathbb{R}^n)$.

4.1.4 Relation between the Radon, ridgelet and wavelet transforms

The ridgelet transform is intimately connected with the Radon transform. Changing variables in (4.1) to $\mathbf{x} = p\mathbf{u} + \mathbf{y}$, where $p \in \mathbb{R}$ and $\mathbf{y}$ runs over the hyperplane perpendicular to $\mathbf{u}$, one readily obtains

$$\mathcal{R}_\psi f\,(\mathbf{u}, b, a) = \mathcal{W}_\psi(Rf_{\mathbf{u}})(b, a), \tag{4.7}$$

where $\mathcal{W}_\psi$ is a one-dimensional wavelet transform. The relation (4.7) holds if $f \in L^1(\mathbb{R}^n)$. (In fact, we will extend its range of validity in Sections 4.4 and 4.6.) Thus, ridgelet analysis can be seen as a form of wavelet analysis in the Radon domain, i.e., the ridgelet transform is precisely the application of a one-dimensional wavelet transform to the slices of the Radon transform where $\mathbf{u}$ remains fixed and p varies. Furthermore, by the Fourier slice theorem (4.5), the properties of the Fourier transform and the relation (4.7), we get the useful formula

$$\begin{aligned} \mathcal{R}_\psi f(\mathbf{u}, b, a) &= a^{-1} \int_{\mathbb{R}} Rf(\mathbf{u}, p) \bar{\psi}\left(\frac{p-b}{a}\right) dp \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{Rf}_{\mathbf{u}}(\omega) \widehat{\bar{\psi}}(a\omega) e^{ib\omega} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\mathbf{u}\omega) \widehat{\bar{\psi}}(a\omega) e^{ib\omega} d\omega. \end{aligned} \tag{4.8}$$

4.2 Extended reconstruction formulas and Parseval relations

In [5] (see also [4, Chap. 2]), Candès has established reproducing formulas and Parseval's identities for the ridgelet transform under the assumption that $\psi \in \mathcal{S}(\mathbb{R})$ is an *admissible neuronal activation function*, meaning that it satisfies the constrain

$$\int_{-\infty}^{\infty} \frac{|\widehat{\psi}(\omega)|^2}{|\omega|^n} d\omega < \infty. \tag{4.9}$$

We shall establish in this section more general reconstruction and Parseval's formulas employing neuronal activation functions which are not necessarily admissible. The crucial notion involved in our analysis is given in the next definition. As usual, a function is called non-trivial if it is not the zero function.

Definition 4.2.1. Let $\psi \in \mathcal{S}(\mathbb{R})$ be a non-trivial test function. A test function $\eta \in \mathcal{S}(\mathbb{R})$ is said to be a *reconstruction neuronal activation function* for ψ if the constant

$$K_{\psi,\eta} := (2\pi)^{n-1} \int_{-\infty}^{\infty} \overline{\widehat{\psi}}(\omega) \widehat{\eta}(\omega) \frac{d\omega}{|\omega|^n} \tag{4.10}$$

is non-zero and finite.

It is then easy to show that *any* ψ admits a reconstruction neuronal activation function η, as long as ψ is non-trivial, and, in such a case, one may take $\eta \in \mathcal{S}_0(\mathbb{R})$, if needed. Our first result states that it is always possible to do ridgelet reconstruction for non-trivial neuronal activation functions.

Proposition 4.2.1 (Reconstruction formula). *Let $\psi \in \mathcal{S}(\mathbb{R})$ be non-trivial and let $\eta \in \mathcal{S}(\mathbb{R})$ be a reconstruction neuronal activation function for it. If $f \in L^1(\mathbb{R}^n)$ is such that $\widehat{f} \in L^1(\mathbb{R}^n)$, then the following reconstruction formula holds pointwisely,*

$$f(\mathbf{x}) = \frac{1}{K_{\psi,\eta}} \int_{\mathbb{S}^{n-1}} \int_0^\infty \int_{-\infty}^\infty \mathcal{R}_\psi f(\mathbf{u}, b, a)\, \eta_{\mathbf{u},b,a}(\mathbf{x}) \frac{dbdad\mathbf{u}}{a^n}. \tag{4.11}$$

Remark 4.2.1. Proposition 4.2.1 shows that ridgelet reconstruction is possible for non-oscillatory neuronal activation functions; indeed, for test functions that might not satisfy the admissibility condition (4.9) (e.g., the Gaussian $\psi(x) = e^{-x^2}$). Nevertheless, if ψ is not oscillatory, then the reconstruction function η should compensate this fact by having its first $n+1$ moments equal to 0.

Proof. Indeed, (4.8) yields

$$\begin{aligned}
&\int_{\mathbb{S}^{n-1}} \int_0^\infty \int_{-\infty}^\infty \mathcal{R}_\psi f(\mathbf{u}, b, a)\, \eta_{\mathbf{u},b,a}(\mathbf{x}) \frac{dbdad\mathbf{u}}{a^n} \\
&= \frac{1}{2\pi} \int_{-\infty}^\infty \int_{\mathbb{S}^{n-1}} \int_0^\infty e^{i\omega\mathbf{u}\cdot\mathbf{x}} \overline{\widehat{\psi}}(\omega a) \widehat{\eta}(\omega a) \widehat{f}(\omega\mathbf{u}) \frac{dad\mathbf{u}d\omega}{a^n} \\
&= \frac{K_{\psi,\eta}}{(2\pi)^n} \int_0^\infty \int_{\mathbb{S}^{n-1}} e^{i\omega\mathbf{u}\cdot\mathbf{x}} \omega^{n-1} \widehat{f}(\omega\mathbf{u}) d\mathbf{u}d\omega.
\end{aligned}$$

□

A similar calculation leads to the ensuing result.

Proposition 4.2.2 (Extended Parseval's relation). *Let $\psi \in \mathcal{S}(\mathbb{R})$ be non-trivial and let $\eta \in \mathcal{S}(\mathbb{R})$ be a reconstruction neuronal activation function for it. Then,*

$$\int_{\mathbb{R}^n} f(\mathbf{x}) g(\mathbf{x}) d\mathbf{x} = \frac{1}{K_{\psi,\eta}} \int_{\mathbb{S}^{n-1}} \int_0^\infty \int_{-\infty}^\infty \mathcal{R}_\psi f(\mathbf{u}, b, a) \mathcal{R}_{\overline{\eta}} g(\mathbf{u}, b, a) \frac{dbdad\mathbf{u}}{a^n}, \tag{4.12}$$

for any $f, g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

According to our choice of the standard measure on $\mathbb{Y}^{n+1}$ (cf. Subsection **??**), we denote by $L^2(\mathbb{Y}^{n+1}) := L^2(\mathbb{Y}^{n+1}, a^{-n} d\mathbf{u}dbda)$ so that the inner product on this space is

$$(F, G)_{L^2(\mathbb{Y}^{n+1})} := \int_0^\infty \int_{-\infty}^\infty \int_{\mathbb{S}^{n-1}} F(\mathbf{u}, b, a)\, \overline{G}(\mathbf{u}, b, a) \frac{d\mathbf{u}dbda}{a^n}.$$

As already observed by Candès [5], the transform $\sqrt{K_{\psi,\psi}^{-1}}\mathcal{R}_\psi$ is L^2-norm preserving whenever ψ is an admissible function. In such a case $||\mathcal{R}_\psi||_{L^2(\mathbb{Y}^{n+1})} = K_{\psi,\psi} ||f||_{L^2(\mathbb{R}^n)}$ on a dense subspace of $L^2(\mathbb{R}^n)$, as follows from (4.12). Consequently, $\mathcal{R}_\psi$ extends to a constant multiple of an isometric embedding $L^2(\mathbb{R}^n) \to L^2(\mathbb{Y}^{n+1})$.

The reconstruction formula (4.11) suggests to define an operator that maps functions on $\mathbb{Y}^{n+1}$ to functions on $\mathbb{R}^n$ as superposition of ridgelets. Given $\psi \in \mathcal{S}(\mathbb{R}^n)$, we introduce the *ridgelet synthesis operator* as

$$\mathcal{R}^t_\psi \Phi(\mathbf{x}) := \int_{\mathbb{S}^{n-1}} \int_0^\infty \int_{-\infty}^\infty \Phi(\mathbf{u}, b, a)\psi_{\mathbf{u},b,a}(\mathbf{x})\frac{dbdad\mathbf{u}}{a^n}, \quad \mathbf{x} \in \mathbb{R}^n. \tag{4.13}$$

The integral (4.13) is absolutely convergent, for instance, if $\Phi \in \mathcal{S}(\mathbb{Y}^{n+1})$. In Section 4.3 we will show that if $\psi \in \mathcal{S}_0(\mathbb{R})$, then $\mathcal{R}^t_\psi$ maps continuously $\mathcal{S}(\mathbb{Y}^{n+1}) \to \mathcal{S}_0(\mathbb{R}^n)$. It will then be shown in Section 4.4 that $\mathcal{R}^t_\psi$ can be even extended to act on the distribution space $\mathcal{S}'(\mathbb{Y}^{n+1})$. Observe that the relation (4.11) takes the form $(\mathcal{R}^t_\eta \circ \mathcal{R}_\psi)f = K_{\psi,\eta}f$.

We remark that $\mathcal{R}^t_{\bar{\psi}}$ and $\mathcal{R}_\psi$ are actually formal transposes. The proof of the next proposition is left to the reader, it is a simple consequence of Fubini's theorem.

Proposition 4.2.3. *Let $\psi \in \mathcal{S}(\mathbb{R})$. If $f \in L^1(\mathbb{R}^n)$ and $\Phi \in \mathcal{S}(\mathbb{Y}^{n+1})$, then*

$$\int_{\mathbb{R}^n} f(\mathbf{x})\mathcal{R}^t_\psi \Phi(\mathbf{x})d\mathbf{x} = \int_{\mathbb{S}^{n-1}} \int_0^\infty \int_{-\infty}^\infty \mathcal{R}_{\bar{\psi}} f(\mathbf{u}, b, a)\Phi(\mathbf{u}, b, a)\frac{dbdad\mathbf{u}}{a^n}. \tag{4.14}$$

Following our convention for regular distributions on $\mathbb{Y}^{n+1}$ (cf. (1.10)), we may write (4.14) as $\langle f, \mathcal{R}^t_{\bar{\psi}}\Phi\rangle = \langle \mathcal{R}_\psi f, \Phi\rangle$. This dual relation will be the model for our definition of the distributional ridgelet transform.

4.3 Continuity of the ridgelet transform on test function spaces

The aim of the section is to prove that the ridgelet mappings

$$\mathcal{R}_\psi : \mathcal{S}_0(\mathbb{R}^n) \to \mathcal{S}(\mathbb{Y}^{n+1}) \qquad \text{and} \qquad \mathcal{R}^t_\psi : \mathcal{S}(\mathbb{Y}^{n+1}) \to \mathcal{S}_0(\mathbb{R}^n)$$

are continuous when $\psi \in \mathcal{S}_0(\mathbb{R})$. For non-trivial ψ, the ridgelet transform $\mathcal{R}_\psi$ is injective and $\mathcal{R}^t_\psi$ is surjective, due to the reconstruction formula (cf. Proposition 4.2.1). Recall that we endow $\mathcal{S}(\mathbb{Y}^{n+1})$ with the system of seminorms (1.9).

Notice that we can extend the definition of the ridgelet transform as a sesquilinear mapping $\mathcal{R} : (f, \psi) \mapsto \mathcal{R}_\psi f$, whereas the ridgelet synthesis operator extends to the bilinear form $\mathcal{R}^t : (\Phi, \psi) \mapsto \mathcal{R}^t_\psi \Phi$.

Theorem 4.3.1. *The ridgelet mapping $\mathcal{R} : \mathcal{S}_0(\mathbb{R}^n) \times \mathcal{S}_0(\mathbb{R}) \to \mathcal{S}(\mathbb{Y}^{n+1})$ is continuous.*

Proof. For the seminorms on $\mathcal{S}_0(\mathbb{R}^n)$, we make the choice

$$\rho_\nu(\phi) = \sup_{\mathbf{x}\in\mathbb{R}^n, |m|\leq\nu} (1 + |\mathbf{x}|)^\nu \left|\phi^{(m)}(\mathbf{x})\right|, \quad \nu \in \mathbb{N}_0. \tag{4.15}$$

We will show that, given $s, r, m, l, k \in \mathbb{N}_0$, there exist $\nu, \tau \in \mathbb{N}$ and $C > 0$ such that

$$\rho^{l,m,k}_{s,r}(\mathcal{R}_\psi \phi) \leq C\rho_\nu(\phi)\rho_\tau(\psi), \quad \phi \in \mathcal{S}_0(\mathbb{R}^n),\ \psi \in \mathcal{S}_0(\mathbb{R}). \tag{4.16}$$

We may assume that r is even and $s \geq 1$. We divide the proof into six steps.

1. Using the definition of the ridgelet transform and the Leibniz formula, we have

$$\begin{aligned}
&\left|\frac{\partial^l}{\partial a^l}\frac{\partial^m}{\partial b^m}\mathcal{R}_\psi\phi(\mathbf{u},b,a)\right| \\
&= \left|\frac{\partial^l}{\partial a^l}\int_{\mathbb{R}^n}\phi(\mathbf{x})\frac{1}{a^{m+1}}\psi^{(m)}\left(\frac{\mathbf{x}\cdot\mathbf{u}-b}{a}\right)d\mathbf{x}\right| \\
&= \sum_{j=0}^{l}\binom{l}{j}\left|\int_{\mathbb{R}^n}\phi(\mathbf{x})\left(\frac{1}{a^{m+1}}\right)^{(l-j)}\psi^{(m+i)}\left(\frac{\mathbf{x}\cdot\mathbf{u}-b}{a}\right)\left(\frac{\mathbf{x}\cdot\mathbf{u}-b}{a}\right)^{(j)}d\mathbf{x}\right| \\
&= \sum_{j=0}^{l}\frac{C_{m,l,j}}{a^{m+l-j}}\sum_{\substack{i,q\leq j\\ d\leq 2j}}a^{-d-1}B^{i,q,d}_{m,l,j}\left|\int_{\mathbb{R}^n}\phi(\mathbf{x})\psi^{(m+i)}\left(\frac{\mathbf{x}\cdot\mathbf{u}-b}{a}\right)(\mathbf{x}\cdot\mathbf{u}-b)^q d\mathbf{x}\right| \\
&\leq C\left(a^{m+2l}+\frac{1}{a^{m+2l}}\right)(1+b^2)^{l/2}\sum_{|\alpha|,i\leq l}\left|\frac{1}{a}\int_{\mathbb{R}^n}\mathbf{x}^\alpha\phi(\mathbf{x})\psi^{(m+i)}\left(\frac{\mathbf{x}\cdot\mathbf{u}-b}{a}\right)d\mathbf{x}\right|.
\end{aligned}$$

Setting $\phi_\alpha(\mathbf{x}) = \mathbf{x}^\alpha\phi(\mathbf{x})$, this yields

$$\rho^{l,m,k}_{s,r}(\mathcal{R}_\psi\phi) \leq C\sum_{\substack{j\leq m+l\\ |\alpha|\leq l}}\rho^{0,0,k}_{s+m+2l,r+l}(\mathcal{R}_{\psi^{(j)}}(\phi_\alpha)).$$

So we can assume that $m = l = 0$ because multiplication by $\mathbf{x}^\alpha$ and differentiation are continuous operators on $\mathcal{S}_0$.

2. We now show that we may assume that $k = 0$. Notice that

$$\begin{aligned}
\triangle^k_{\mathbf{u}}\mathcal{R}_\psi\phi(\mathbf{u},b,a) &= \triangle^k_{\mathbf{u}}\int_{\mathbb{R}^n}\phi(\mathbf{x})a^{-1}\psi\left(\frac{\mathbf{x}\cdot\mathbf{u}-b}{a}\right)d\mathbf{x} \\
&= \sum_{|\alpha|,j,d\leq 2k}a^{-d}P_{\alpha,j,d}(\mathbf{u})\frac{1}{a}\int_{\mathbb{R}^n}\mathbf{x}^\alpha\phi(\mathbf{x})\psi^{(j)}\left(\frac{\mathbf{x}\cdot\mathbf{u}-b}{a}\right)d\mathbf{x},
\end{aligned}$$

where the $P_{\alpha,j,d}(\mathbf{u})$ are certain polynomials. The $P_{\alpha,j,d}$ are bounded, thus

$$\left|\triangle^k_{\mathbf{u}}\mathcal{R}_\psi\phi(\mathbf{u},b,a)\right| \leq C\left(a^{2k}+\frac{1}{a^{2k}}\right)\sum_{|\alpha|,j\leq 2k}\left|\frac{1}{a}\int_{\mathbb{R}^n}\mathbf{x}^\alpha\phi(\mathbf{x})\psi^{(j)}\left(\frac{\mathbf{x}\cdot\mathbf{u}-b}{a}\right)d\mathbf{x}\right|.$$

This gives (with ϕ_α as before)

$$\rho^{0,0,k}_{s,r}(\mathcal{R}_\psi\phi) \leq C\sum_{|\alpha|,j\leq 2k}\rho^{0,0,0}_{2k+s,r}(\mathcal{R}_{\psi^{(j)}}(\phi_\alpha)).$$

Reasoning as above, we can assume that $k = 0$.

3. Observe that, by (4.8),

$$\left(1+b^2\right)^{r/2}\mathcal{R}_\psi\phi(\mathbf{u},b,a) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{\phi}(\omega\mathbf{u})\overline{\widehat{\psi}}(a\omega)\left(1-\frac{\partial^2}{\partial\omega^2}\right)^{r/2}e^{ib\omega}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ib\omega}\left(1-\frac{\partial^2}{\partial\omega^2}\right)^{r/2}(\widehat{\phi}(\omega\mathbf{u})\overline{\widehat{\psi}}(a\omega))d\omega$$
$$= \sum_{|\alpha|,j\le r} a^j Q_{\alpha,j}(\mathbf{u})\int_{-\infty}^{\infty} e^{ib\omega}\widehat{\phi}^{(\alpha)}(\omega\mathbf{u})\overline{\widehat{\psi}^{(j)}}(a\omega)d\omega,$$

for some polynomials $Q_{\alpha,j}$. Taking (4.8) into account, and writing $\psi_j(x) = x^j\psi(x)$ and again $\phi_\alpha(\mathbf{x}) = \mathbf{x}^\alpha\phi(\mathbf{x})$, we conclude that

$$\rho_{s,r}^{0,0,0}(\mathcal{R}_\psi\phi) \le C\sum_{|\alpha|,j\le r}\rho_{s+r,0}^{0,0,0}(\mathcal{R}_{\psi_j}(\phi_\alpha)).$$

Consequently, we can assume $r = 0$.

4. We consider the part involving multiplication by a^s in $\rho_{s,0}^{0,0,0}$. Using the Taylor expansion of $\widehat{\phi}$, we obtain

$$\begin{aligned} a^s\left|\mathcal{R}_\psi\phi\left(\mathbf{u},b,a\right)\right| &= \frac{a^s}{2\pi}\left|\int_{-\infty}^{\infty}\widehat{\phi}(\omega\mathbf{u})\overline{\widehat{\psi}}(a\omega)e^{ib\omega}d\omega\right| \\ &\le \sum_{|\alpha|=s-1}\frac{a^s}{2\pi}\left|\int_{-\infty}^{\infty}\frac{(\omega\mathbf{u})^\alpha}{\alpha!}\widehat{\phi}^{(\alpha)}(\omega_0\mathbf{u})\overline{\widehat{\psi}}(a\omega)e^{ib\omega}d\omega\right| \\ &\le \left(\sum_{|\alpha|=s-1}\frac{1}{2\pi\alpha!}\int_{\mathbb{R}^n}|\mathbf{x}^\alpha\phi(\mathbf{x})|d\mathbf{x}\right)\int_{-\infty}^{\infty}\left|\omega^{s-1}\widehat{\psi}(\omega)\right|d\omega \\ &\le C\rho_{s+n}(\phi)\rho_{s+1}(\psi). \end{aligned}$$

5. For the multiplication by a^{-s}, we develop $\widehat{\psi}$ into its Taylor expansion of order s. Then,

$$\begin{aligned} a^{-s}\left|\mathcal{R}_\psi\phi\left(\mathbf{u},b,a\right)\right| &= \frac{1}{2\pi a^s}\left|\int_{-\infty}^{\infty}\widehat{\phi}(\omega\mathbf{u})\overline{\widehat{\psi}}(a\omega)e^{ib\omega}d\omega\right| \\ &\le \frac{1}{2\pi s!}\int_{-\infty}^{\infty}|\omega^s\widehat{\phi}(\omega\mathbf{u})\overline{\widehat{\psi}^{(s)}}(a\omega_0)|d\omega. \end{aligned}$$

It is easy to see that last integral is less than $C\rho_{s+n+1}(\phi)\rho_{s+2}(\psi)$. Combining this fact with the bound from step 4, we obtain

$$\rho_{s,0}^{0,0,0}(\mathcal{R}_\psi\phi) \le C\rho_{s+n+1}(\phi)\rho_{s+2}(\psi).$$

6. Summing up all the estimates, we find that (4.16) holds with $\nu = s + 2r + 4l + 4k + m + n + 1$ and $\tau = s + 2r + 4l + 4k + 2m + 2$. This completes the proof. □

We now study the ridgelet synthesis operator.

Theorem 4.3.2. *The bilinear mapping $\mathcal{R}^t : \mathcal{S}(\mathbb{Y}^{n+1})\times\mathcal{S}_0(\mathbb{R})\to\mathcal{S}_0(\mathbb{R}^n)$ is continuous.*

Proof. Let us first verify that the ridgelet synthesis operator has the claimed range, that is, we show that if $\psi \in \mathcal{S}_0(\mathbb{R})$ and $\Phi \in \mathcal{S}(\mathbb{Y}^{n+1})$, then $\phi(\mathbf{x}) := \mathcal{R}^t_\psi \Phi \in \mathcal{S}_0(\mathbb{R}^n)$. In other words, we have to prove that

$$\lim_{\mathbf{w}\to 0} \frac{\widehat{\phi}(\mathbf{w})}{|\mathbf{w}|^k} = 0, \quad \forall k \in \mathbb{N}_0. \tag{4.17}$$

Observe that

$$\phi(\mathbf{x}) = \frac{1}{2\pi} \int_{\mathbb{S}^{n-1}} \int_{-\infty}^{\infty} \omega^{n-1} e^{i\omega \mathbf{u}\cdot\mathbf{x}} \left(\int_0^\infty \widehat{\Phi}(\mathbf{u},\omega,a) \frac{\widehat{\psi}(\omega a)}{(\omega a)^{n-1}} \frac{da}{a} \right) d\omega d\mathbf{u};$$

hence, by Fourier inversion in polar coordinates,

$$\widehat{\phi}(\omega\mathbf{u}) = (2\pi)^{n-1} \int_0^\infty \left(\widehat{\Phi}(\mathbf{u},\omega,a) \frac{\widehat{\psi}(\omega a)}{(\omega a)^{n-1}} + \widehat{\Phi}(\mathbf{u},-\omega,a) \frac{\widehat{\psi}(-\omega a)}{(-\omega a)^{n-1}} \right) \frac{da}{a}, \tag{4.18}$$

$\omega \in \mathbb{R}_+$, $\mathbf{u} \in \mathbb{S}^{n-1}$. (Here $\widehat{\Phi}$ stands for the Fourier transform of $\Phi(\mathbf{u},b,a)$ with respect to the variable b.) Since Φ belongs to $\mathcal{S}(\mathbb{Y}^{n+1})$, we have that for any $k \in \mathbb{N}$ we can find a constant $C_k > 0$ such that $|\widehat{\Phi}(\mathbf{u},\omega,a)| \leq C_k a^{-k-1}$, uniformly for $\omega \in \mathbb{R}$ and $\mathbf{u} \in \mathbb{S}^{n-1}$. Thus,

$$\left|\widehat{\phi}(\omega\mathbf{u})\right| \leq C_k \int_{-\infty}^{\infty} \frac{|\widehat{\psi}(\omega a)|}{|\omega a|^{n-1}} \frac{da}{|a|^{k+2}} = C_k \omega^{k+1} \int_{-\infty}^{\infty} \left| \frac{\widehat{\psi}(a)}{a^{n+k+1}} \right| da, \quad \omega \in \mathbb{R},\ \mathbf{u} \in \mathbb{S}^{n-1},$$

whence (4.17) follows.

We now prove the continuity of the bilinear ridgelet synthesis mapping. Since the Fourier transforms $\psi \mapsto \widehat{\psi}$ and $\Phi \mapsto \widehat{\Phi}$ are continuous automorphisms on the $\mathcal{S}$ spaces, the families (cf. (4.15) and (1.9)) $\hat{\rho}_\nu(\psi) = \rho_\nu(\widehat{\psi})$, $\psi \in \mathcal{S}_0(\mathbb{R})$, $\nu \in \mathbb{N}_0$, and $\hat{\rho}^{l,m,k}_{s,r}(\Phi) = \rho^{l,m,k}_{s,r}(\widehat{\Phi})$, $\Phi \in \mathcal{S}(\mathbb{Y}^{n+1})$, $l,m,k,s,r \in \mathbb{N}_0$, are bases of seminorms for the topologies of $\mathcal{S}_0(\mathbb{R})$ and $\mathcal{S}(\mathbb{Y}^{n+1})$, respectively. We shall need a different family of seminorms on $\mathcal{S}_0(\mathbb{R}^n)$. Observe first that the Fourier transform provides a Fréchet space isomorphism from $\mathcal{S}_0(\mathbb{R}^n)$ onto $\mathcal{S}_*(\mathbb{R}^n)$, the closed subspace of $\mathcal{S}(\mathbb{R}^n)$ consisting of all those test functions that vanish at the origin together with all their partial derivatives. On the other hand, polar coordinates $\varphi(\omega\mathbf{u})$ provide a continuous mapping $\mathcal{S}_*(\mathbb{R}^n) \to \mathcal{S}_*(\mathbb{S}^{n-1} \times \mathbb{R})$; the range of this mapping is closed (it consists of even test functions, i.e., $\varrho(-\mathbf{u},-\omega) = \varrho(\mathbf{u},\omega)$ [13, 33]), and therefore the open mapping theorem implies that it is an isomorphism into its image. Summarizing, the seminorms $\dot{\rho}_{N,q,k}$, given by

$$\dot{\rho}_{N,q,k}(\phi) := \sup_{(\mathbf{u},\omega)\in\mathbb{S}^{n-1}\times\mathbb{R}} \left| \omega^N \frac{\partial^q}{\partial\omega^q} \Delta^k_{\mathbf{u}} \widehat{\phi}(\omega\mathbf{u}) \right|, \quad N,q,k \in \mathbb{N}_0,$$

are a base of continuous seminorms for the topology of $\mathcal{S}_0(\mathbb{R}^n)$. We show that given $N,q,k \in \mathbb{N}_0$ there are $C > 0$ and $\nu \in \mathbb{N}$ such that

$$\dot{\rho}_{N,q,k}\left(\mathcal{R}^t_\psi \Phi\right) \leq C \hat{\rho}_{n-1+q}(\psi) \sum_{m,s\leq\nu} \hat{\rho}^{0,m,k}_{s,N}(\Phi).$$

Now, setting again $\phi(\mathbf{x}) := \mathcal{R}^t_\psi \Phi \in \mathcal{S}_0(\mathbb{R}^n)$, using the expression (4.18), the Leibniz formula, and the Taylor expansion for ψ, we get

$$\begin{aligned}\left|\omega^N \frac{\partial^q}{\partial \omega^q} \Delta^k_{\mathbf{u}} \widehat{\phi}(\omega \mathbf{u})\right| &\leq C \sum_{j=0}^{q} \sum_{d=0}^{j} \int_{-\infty}^{\infty} \left| a^{-j-1} \omega^N \frac{\partial^{q-j}}{\partial \omega^{q-j}} \Delta^k_{\mathbf{u}} \widehat{\Phi}(\mathbf{u}, \omega, a) \frac{\widehat{\psi}^{(j-d)}(\omega a)}{(\omega a)^{n-1+d}} \right| da \\ &= C \sum_{j=0}^{q} \sum_{d=0}^{j} \int_{-\infty}^{\infty} \left| a^{-j-1} \omega^N \frac{\partial^{q-j}}{\partial \omega^{q-j}} \Delta^k_{\mathbf{u}} \widehat{\Phi}(\mathbf{u}, \omega, a) \widehat{\psi}^{(j+n-1)}(\omega_0 a) \right| da \\ &\leq C \hat{\rho}_{n-1+q}(\psi) \sum_{j=0}^{q} (j+1) \hat{\rho}^{0,q-j,k}_{j+3,N}(\Phi) \int_{-\infty}^{\infty} \frac{a^2 da}{a^4+1},\end{aligned}$$

as claimed. □

For future use, it is convenient to introduce wavelet analysis on $\mathcal{S}(\mathbb{S}^{n-1} \times \mathbb{R})$. Given $\psi \in \mathcal{S}(\mathbb{R})$, we let $\mathcal{W}_\psi$ act on the real variable p of functions $g(\mathbf{u}, p)$ (or distributions), that is,

$$\mathcal{W}_\psi g(\mathbf{u}, b, a) := \int_{-\infty}^{\infty} \frac{1}{a} \overline{\psi} \left(\frac{p-b}{a} \right) g(\mathbf{u}, p) dp = \left\langle g(\mathbf{u}, p), \frac{1}{a} \overline{\psi} \Big(\frac{p-b}{a} \Big) \right\rangle_p, \tag{4.19}$$

$(\mathbf{u}, b, a) \in \mathbb{Y}^{n+1}$. Similarly, we define the wavelet synthesis operator on $\mathcal{S}(\mathbb{Y}^{n+1})$ as

$$\mathcal{M}_\psi \Phi(\mathbf{u}, p) = \int_0^{\infty} \int_{-\infty}^{\infty} \frac{1}{a} \psi \left(\frac{p-b}{a} \right) \Phi(\mathbf{u}, b, a) \frac{dbda}{a}. \tag{4.20}$$

A straightforward variant of the method employed in the proofs of Theorem 4.3.1 and Theorem 4.3.2 applies to show the following continuity result. Alternatively, since $\mathcal{S}(\mathbb{S}^{n-1} \times \mathbb{R}) = \mathcal{D}(\mathbb{S}^{n-1}) \hat{\otimes} \mathcal{S}(\mathbb{R})$, $\mathcal{S}_0(\mathbb{S}^{n-1} \times \mathbb{R}) = \mathcal{D}(\mathbb{S}^{n-1}) \hat{\otimes} \mathcal{S}_0(\mathbb{R})$ and $\mathcal{S}(\mathbb{Y}^{n+1}) = \mathcal{D}(\mathbb{S}^{n-1}) \hat{\otimes} \mathcal{S}(\mathbb{H})$, the result may also be deduced from a tensor product argument and the continuity of the corresponding mappings on $\mathcal{S}(\mathbb{R})$, $\mathcal{S}_0(\mathbb{R})$, and $\mathcal{S}(\mathbb{H})$ (cf. [37] or [74]).

Corollary 4.3.1. *The mappings*

(i) $\mathcal{W} : \mathcal{S}_0(\mathbb{S}^{n-1} \times \mathbb{R}) \times \mathcal{S}_0(\mathbb{R}) \to \mathcal{S}(\mathbb{Y}^{n+1})$

(ii) $\mathcal{M} : \mathcal{S}(\mathbb{Y}^{n+1}) \times \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{S}^{n-1} \times \mathbb{R})$

(iii) $\mathcal{M} : \mathcal{S}(\mathbb{Y}^{n+1}) \times \mathcal{S}_0(\mathbb{R}) \to \mathcal{S}_0(\mathbb{S}^{n-1} \times \mathbb{R})$

are continuous.

We end this section with a remark concerning reference [78].

Remark 4.3.1. In dimension $n = 2$, Roopkumar has considered [78] the analogs of our Theorem 4.3.1 and Theorem 4.3.2 for the space $\mathcal{S}_\#(\mathbb{Y}^{n+1})$, where $\mathcal{S}_\#(\mathbb{Y}^{n+1})$ consists of all those smooth functions Φ on $\mathbb{Y}^{n+1}$ satisfying

$$\gamma^{l,m,k}_{s,r}(\Phi) := \sup_{(\mathbf{u},b,a) \in \mathbb{Y}^{n+1}} \left| a^s b^r \frac{\partial^l}{\partial a^l} \frac{\partial^m}{\partial b^m} \triangle^k_{\mathbf{u}} \Phi\left(\mathbf{u}, b, a\right) \right| < \infty, \qquad l, m, k, s, r \in \mathbb{N}_0.$$

Observe that his system of seminorms $\{\gamma_{s,r}^{l,m,k}\}$ does not take decay into account for small values of the scaling variable a (the term a^{-s} does not occur in his considerations). He claims [78, Thrm. 3.1 and 3.3] to have shown that $\mathcal{R}_\psi : \mathcal{S}(\mathbb{R}^2) \to \mathcal{S}_\#(\mathbb{Y}^3)$ and $\mathcal{R}_\psi^t : \mathcal{S}_\#(\mathbb{Y}^3) \to \mathcal{S}(\mathbb{R}^2)$ are continuous when $\psi \in \mathcal{S}(\mathbb{R})$ satisfies the admissibility condition (4.9). His proof of the continuity of $\mathcal{R}_\psi^t : \mathcal{S}_\#(\mathbb{Y}^3) \to \mathcal{S}(\mathbb{R}^2)$ appears to be incorrect because it seems to make use of the erroneous relation $x_1 \cos\theta + x_2 \sin\theta = (x_1 + ix_2)e^{i\theta}$ [78, p. 436]. Furthermore, his result on the continuity of $\mathcal{R}_\psi : \mathcal{S}(\mathbb{R}^2) \to \mathcal{S}_\#(\mathbb{Y}^3)$ turns out to be false because the ridgelet transform $\mathcal{R}_\psi$ does not even map $\mathcal{S}(\mathbb{R}^n)$ into Roopkumar's space $\mathcal{S}_\#(\mathbb{Y}^{n+1})$. We show the latter fact with the following example. Choose the admissible function $\widehat{\psi}(\omega) = 2\pi^{-n/2+1}\omega^{2n}e^{-\omega^2/4}$, $\omega \in \mathbb{R}$, and $\phi(\mathbf{w}) = e^{-|\mathbf{w}|^2}$, $\mathbf{w} \in \mathbb{R}^n$. Then, by (4.8),

$$\begin{aligned}\mathcal{R}_\psi\phi(\mathbf{u},0,a) &= \int_{-\infty}^{\infty} e^{-\omega^2/4}(a\omega)^{2n}e^{-(a\omega)^2/4}d\omega = \frac{1}{a}\int_{-\infty}^{\infty} e^{-\omega^2/(4a^2)}\omega^{2n}e^{-\omega^2/4}d\omega \\ &\sim \frac{1}{a}\int_{-\infty}^{\infty}\omega^{2n}e^{-\omega^2/4}d\omega = \frac{c}{a}, \quad a \to \infty,\end{aligned}$$

where $c \neq 0$. This shows that $\gamma_{2,0}^{0,0,0}(\mathcal{R}_\psi\phi) = \infty$. Therefore, $\mathcal{R}_\psi\phi \notin \mathcal{S}_\#(\mathbb{Y}^{n+1})$.

4.4 The ridgelet transform on $\mathcal{S}_0'(\mathbb{R}^n)$

We are ready to define the ridgelet transform of Lizorkin distributions.

Definition 4.4.1. Let $\psi \in \mathcal{S}_0(\mathbb{R})$. We define the ridgelet transform of $f \in \mathcal{S}_0'(\mathbb{R}^n)$ with respect to ψ as the element $\mathcal{R}_\psi f \in \mathcal{S}'(\mathbb{Y}^{n+1})$ whose action on test functions is given by

$$\langle \mathcal{R}_\psi f, \Phi\rangle := \langle f, \mathcal{R}_{\overline{\psi}}^t \Phi\rangle, \quad \Phi \in \mathcal{S}(\mathbb{Y}^{n+1}). \tag{4.21}$$

The consistence of Definition 4.4.1 is guaranteed by Theorem 4.3.2. Likewise, Theorem 4.3.1 allows us to define the ridgelet synthesis operator $\mathcal{R}_\psi^t$ for $\psi \in \mathcal{S}_0(\mathbb{R})$ as a linear mapping from $\mathcal{S}'(\mathbb{Y}^{n+1})$ to $\mathcal{S}_0'(\mathbb{R}^n)$ (and not to $\mathcal{S}'(\mathbb{R}^n)$).

Definition 4.4.2. Let $\psi \in \mathcal{S}_0(\mathbb{R})$. The ridgelet synthesis operator $\mathcal{R}_\psi^t : \mathcal{S}'(\mathbb{Y}^{n+1}) \to \mathcal{S}_0'(\mathbb{R}^n)$ is defined as

$$\langle \mathcal{R}_\psi^t F, \phi\rangle := \langle F, \mathcal{R}_{\overline{\psi}}\phi\rangle, \quad F \in \mathcal{S}'(\mathbb{Y}^{n+1}), \quad \phi \in \mathcal{S}(\mathbb{R}^n). \tag{4.22}$$

Taking transposes in Theorems 4.3.1 and 4.3.2, we immediately obtain the ensuing continuity result.

Proposition 4.4.1. *Let $\psi \in \mathcal{S}_0(\mathbb{R})$. The ridgelet transform $\mathcal{R}_\psi : \mathcal{S}_0'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{Y}^{n+1})$ and the ridgelet synthesis operator $\mathcal{R}_\psi^t : \mathcal{S}'(\mathbb{Y}^{n+1}) \to \mathcal{S}_0'(\mathbb{R}^n)$ are continuous linear maps.*

We can generalize the reconstruction formula (4.11) to distributions.

Theorem 4.4.1 (Inversion formula)**.** *Let $\psi \in \mathcal{S}_0(\mathbb{R})$ be non-trivial. If $\eta \in \mathcal{S}_0(\mathbb{R})$ is a reconstruction neuronal activation function for ψ, then*

$$\mathrm{id}_{\mathcal{S}_0'(\mathbb{R}^n)} = \frac{1}{K_{\psi,\eta}}(\mathcal{R}_\eta^t \circ \mathcal{R}_\psi). \tag{4.23}$$

Proof. Applying Definition 4.4.1, Definition 4.4.2, and Proposition 4.2.1, we obtain at once $\langle \mathcal{R}_\eta^t(\mathcal{R}_\psi f), \phi\rangle = \langle f, \mathcal{R}_{\overline{\psi}}^t(\mathcal{R}_{\overline{\eta}}\,\phi)\rangle = K_{\overline{\eta},\overline{\psi}}\langle f,\phi\rangle = K_{\psi,\eta}\langle f,\phi\rangle$. □

In Subsection 4.1.1 we have given a different definition of the ridgelet transform of distributions $f \in \mathcal{D}'_{L^1}(\mathbb{R}^n)$ via the formula (4.1). We now show that Definition 4.4.1 is consistent with (4.1) (under our convention (1.10) for identifying functions with distributions on $\mathbb{Y}^{n+1}$). In particular, our definition of the ridgelet transform for distributions is consistent with that for test functions.

Theorem 4.4.2. *Let $f \in \mathcal{D}'_{L^1}(\mathbb{R}^n)$. The ridgelet transform of f is given by the function (4.1), that is,*

$$\langle \mathcal{R}_\psi f, \Phi\rangle = \int_0^\infty \int_{-\infty}^\infty \int_{\mathbb{S}^{n-1}} \mathcal{R}_\psi f(\mathbf{u},b,a)\Phi(\mathbf{u},b,a)\frac{d\mathbf{u}dbda}{a^n}, \quad \Phi \in \mathcal{S}(\mathbb{Y}^{n+1}). \quad (4.24)$$

Proof. By Schwartz' structural theorem [89], we can write $f = \sum_{j=1}^N f_j^{(m_j)}$, where each $f_j \in L^1(\mathbb{R}^n)$. Observe first that $\langle f_j^{(m_j)}, \psi_{\mathbf{u},b,a}\rangle = (-a^{-1}\mathbf{u})^{m_j}\langle f_j, (\psi^{(m_j)})_{\mathbf{u},b,a}\rangle$. On the other hand, since

$$(-1)^{|m_j|}\frac{\partial^{|m|_j}}{\partial x^{m_j}}\mathcal{R}_{\overline{\psi}}^t\Phi = \mathcal{R}^t_{\overline{\psi}^{(m_j)}}\left((-a^{-1}\mathbf{u})^{m_j}\,\Phi\right),$$

the ridgelet transform $\mathcal{R}_\psi f$, defined via (4.21), satisfies $\mathcal{R}_\psi(f_j^{(m_j)}) = (-a^{-1}\mathbf{u})^{m_j}\,\mathcal{R}_{\psi^{(m_j)}} f_j$. Therefore, we may assume that $f \in L^1(\mathbb{R}^n)$. But in the latter case, the result is a consequence of Proposition 4.2.3. □

Remark 4.4.1. Let us point out that (4.24) holds in particular for compactly supported distributions $f \in \mathcal{E}'(\mathbb{R}^n)$ or, more generally, for convolutors $f \in \mathcal{O}'_C(\mathbb{R}^n)$. Furthermore, when $f \in \mathcal{O}'_C(\mathbb{R}^n)$, one can easily check that $\mathcal{R}_\psi f \in C^\infty(\mathbb{Y}^{n+1})$.

4.5 On the Radon transform on $\mathcal{S}_0'(\mathbb{R}^n)$

In this section we explain how one can define the Radon transform of Lizorkin distributions. Its connection with the ridgelet and wavelet transforms will be discussed in Section 4.6.

We begin with test functions. Hertle [35] has made nice discussions about the range of the Radon transform on $\mathcal{D}(\mathbb{R}^n)$ and $\mathcal{E}(\mathbb{R}^n)$ and manage to prove that $R : \mathcal{E}'(\mathbb{R}^n) \to \mathcal{E}'(\mathbb{S}^{n-1}\times\mathbb{R})$ is a topological isomorphism. Helgason [33] and Gelfand et al. [24] gave the range theorem for the Radon transform on $\mathcal{S}(\mathbb{R}^n)$. Indeed, its range $R(\mathcal{S}(\mathbb{R}^n))$ consists of the closed subspace of all those $\varrho \in \mathcal{S}(\mathbb{S}^{n-1}\times\mathbb{R})$ such that ϱ is even on $\mathbb{S}^{n-1}\times\mathbb{R}$, i.e., $\varrho(-\mathbf{u},-p) = \varrho(\mathbf{u},p)$, and $\int_{-\infty}^\infty p^k\varrho(\mathbf{u},p)$ is a k-th degree homogeneous polynomial in $\mathbf{u}$ for all $k \in \mathbb{N}_0$. The situation is not so satisfactory for the dual Radon transform R^*, because it does not map $\mathcal{S}(\mathbb{S}^{n-1}\times\mathbb{R})$ to $\mathcal{S}(\mathbb{R}^n)$. Consequently, the duality relation (4.6) fails to produce a definition for the Radon transform on $\mathcal{S}'(\mathbb{R}^n)$. The Radon transform on $\mathcal{S}'(\mathbb{R}^n)$ can be defined [24, 51, 76], but it does not take values in $\mathcal{S}'(\mathbb{S}^{n-1}\times\mathbb{R})$. The range $R(\mathcal{S}'(\mathbb{R}^n))$ is particularly complicated to describe in even dimensions n.

As Helgason points out [33], a more satisfactory situation is obtained if we restrict our attention to the smaller test function spaces $\mathcal{S}_0(\mathbb{R}^n)$ and $\mathcal{S}_0(\mathbb{S}^{n-1}\times\mathbb{R})$. In such a case,

$$R : \mathcal{S}_0(\mathbb{R}^n) \to \mathcal{S}_0(\mathbb{S}^{n-1} \times \mathbb{R}) \tag{4.25}$$

and

$$R^* : \mathcal{S}_0(\mathbb{S}^{n-1} \times \mathbb{R}) \to \mathcal{S}_0(\mathbb{R}^n). \tag{4.26}$$

We apply our results from Section 4.3 to deduce the following continuity result for R and R^*.

Corollary 4.5.1. *The mappings (4.25) and (4.26) are continuous.*

Proof. Let $\psi \in \mathcal{S}_0(\mathbb{R})$ have a reconstruction wavelet [37] $\eta \in \mathcal{S}_0(\mathbb{R})$, that is, one that satisfies

$$c_{\psi,\eta} = \int_0^\infty \overline{\widehat{\psi}}(\omega)\widehat{\eta}(\omega)\frac{d\omega}{\omega} = \int_{-\infty}^0 \overline{\widehat{\psi}}(\omega)\widehat{\eta}(\omega)\frac{d\omega}{|\omega|} \neq 0. \tag{4.27}$$

From the one-dimensional reconstruction formula [37], we obtain $c_{\psi,\eta}\mathrm{id}_{\mathcal{S}_0(\mathbb{S}^{n-1}\times\mathbb{R})} = \mathcal{M}_\eta\mathcal{W}_\psi$. By (4.7), $R = c_{\psi,\eta}^{-1}(\mathcal{M}_\eta\mathcal{R}_\psi)$, and so the continuity of R follows from Theorem 4.3.1 and Corollary 4.3.1. Next, define the (continuous) multiplier operators

$$J_s : \mathcal{S}(\mathbb{Y}^{n+1}) \to \mathcal{S}(\mathbb{Y}^{n+1}), \quad (J_s\Phi)(\mathbf{u}, b, a) = a^s\Phi(\mathbf{u}, b, a), \quad s \in \mathbb{R}. \tag{4.28}$$

We have that $c_{\psi,\eta}R^* = R^*\mathcal{M}_\eta J_{1-n}J_{n-1}\mathcal{W}_\psi = \mathcal{R}^t_\eta J_{n-1}\mathcal{W}_\psi$ is continuous in view of Theorem 4.3.2 and Corollary 4.3.1. □

The mapping (4.26) allows one to extend the definition of the Radon transform to $\mathcal{S}'_0(\mathbb{R}^n)$.

Definition 4.5.1. The Radon transform

$$R : \mathcal{S}'_0(\mathbb{R}^n) \to \mathcal{S}'_0(\mathbb{S}^{n-1} \times \mathbb{R}) \tag{4.29}$$

is defined via (4.6).

Since (4.29) is the transpose of (4.26), we obtain,

Corollary 4.5.2. *The Radon transform is continuous on $\mathcal{S}'_0(\mathbb{R}^n)$.*

Notice that the dual Radon transform (4.26) is surjective [33]. Therefore, the Radon transform is injective on $\mathcal{S}'_0(\mathbb{R}^n)$. The restriction of (4.29) to the subspaces $\mathcal{D}'_{L^1}(\mathbb{R}^n)$, $\mathcal{E}'(\mathbb{R}^n)$, $\mathcal{O}'_C(\mathbb{R}^n)$, clearly coincides with the Radon transform treated by Hertle in [34].

4.6 Ridgelet desingularization in $\mathcal{S}_0'(\mathbb{R}^n)$

The ridgelet transform of $f \in \mathcal{S}_0'(\mathbb{R}^n)$ is in turn highly regular in "the variables" b and a. This last section is devoted to prove this fact. We also give a ridgelet desingularization formula and establish the connection between the ridgelet, wavelet, and Radon transforms.

As mentioned in Subsection 1.2.5, we have $\mathcal{S}(\mathbb{Y}^{n+1}) = \mathcal{D}(\mathbb{S}^{n-1})\hat{\otimes}\mathcal{S}(\mathbb{H})$. The nuclearity of the Schwartz spaces leads to the isomorphisms $\mathcal{S}'(\mathbb{Y}^{n+1}) \cong \mathcal{S}'(\mathbb{H}, \mathcal{D}'(\mathbb{S}^{n-1}))$ $\cong \mathcal{D}'(\mathbb{S}^{n-1}, \mathcal{S}'(\mathbb{H}))$, the very last two spaces being spaces of vector-valued distributions [92, 103]. We shall identify these three spaces and write

$$\mathcal{S}'(\mathbb{Y}^{n+1}) = \mathcal{S}'(\mathbb{H}, \mathcal{D}'(\mathbb{S}^{n-1})) = \mathcal{D}'(\mathbb{S}^{n-1}, \mathcal{S}'(\mathbb{H})). \tag{4.30}$$

The equality (4.30) being realized via the standard identification

$$\langle F, \varphi \otimes \Psi \rangle = \langle \langle F, \Psi \rangle, \varphi \rangle = \langle \langle F, \varphi \rangle, \Psi \rangle, \quad \Psi \in \mathcal{S}(\mathbb{H}),\ \varphi \in \mathcal{D}(\mathbb{S}^{n-1}), \tag{4.31}$$

Thus, given $F \in \mathcal{S}'(\mathbb{Y}^{n+1})$, the statement F is smooth in (b, a) has the clear interpretation $F \in C^\infty(\mathbb{H}, \mathcal{D}'(\mathbb{S}^{n-1})) = \mathcal{D}'(\mathbb{S}^{n-1}, C^\infty(\mathbb{H}))$. Moreover, we shall say that $F \in \mathcal{S}'(\mathbb{Y}^{n+1})$ is a function of slow growth in the variables $(b, a) \in \mathbb{H}$ if $\langle F(\mathbf{u}, b, a), \varphi(\mathbf{u})\rangle_{\mathbf{u}}$ is such for every $\varphi \in \mathcal{D}(\mathbb{S}^{n-1})$, namely, it is a function that satisfies the bound

$$|\langle F(\mathbf{u}, b, a), \varphi(\mathbf{u})\rangle_{\mathbf{u}}| \le C\left(a^s + \frac{1}{a^s}\right)(1 + |b|)^s, \quad (b, a) \in \mathbb{H},$$

for some positive constants $C = C_\varphi$ and $s = s_\varphi$.

Notice also that $\mathcal{S}_0'(\mathbb{S}^{n-1} \times \mathbb{R}) = \mathcal{S}_0'(\mathbb{R}, \mathcal{D}'(\mathbb{S}^{n-1}))$ (again under the standard identification). This allows us to define the wavelet transform ($\psi \in \mathcal{S}_0(\mathbb{R})$),

$$\mathcal{W}_\psi : \mathcal{S}_0'(\mathbb{S}^{n-1} \times \mathbb{R}) = \mathcal{S}_0'(\mathbb{R}, \mathcal{D}'(\mathbb{S}^{n-1})) \to \mathcal{S}'(\mathbb{H}, \mathcal{D}'(\mathbb{S}^{n-1})) = \mathcal{S}'(\mathbb{Y}^{n+1}),$$

by direct application of the formula (4.3) as a smooth vector-valued function $\mathcal{W}_\psi g : \mathbb{H} \to \mathcal{D}'(\mathbb{S}^{n-1})$, for $g \in \mathcal{S}_0'(\mathbb{S}^{n-1} \times \mathbb{R})$. One can also check that this wavelet transform satisfies

$$\langle g, \mathcal{M}_{\overline{\psi}}\Phi \rangle = \int_0^\infty \int_{-\infty}^\infty \langle \mathcal{W}_\psi g(\mathbf{u}, b, a), \Phi(\mathbf{u}, b, a)\rangle_{\mathbf{u}} \frac{dbda}{a}, \tag{4.32}$$

for $g \in \mathcal{S}_0'(\mathbb{S}^{n-1} \times \mathbb{R})$ and $\Phi \in \mathcal{S}(\mathbb{Y}^{n+1})$, where $\mathcal{M}_{\overline{\psi}}$ is as in (4.20). Implicit in (4.32) is the fact that we are using the measure $a^{-1}dbda$ as the *standard measure* on $\mathbb{H}$ for the identification of functions of slow growth with distributions on $\mathbb{H}$. This choice is the natural one for wavelet analysis, in the sense that one can check that the following duality relation holds:

$$\langle \mathcal{W}_\psi g, \Phi \rangle = \langle g, \mathcal{M}_{\overline{\psi}}\Phi \rangle,$$

for all for $g \in \mathcal{S}_0'(\mathbb{S}^{n-1} \times \mathbb{R})$ and $\Phi \in \mathcal{S}(\mathbb{Y}^{n+1})$. (See [71, Sect. 5 and 8] for additional comments on the vector-valued wavelet transform.)

The relation between the Radon transform, the wavelet transform, and the ridgelet transform is stated in the following theorem, which also tells us that the ridgelet transform is regular in the location and scale parameters.

Theorem 4.6.1. *Let $f \in \mathcal{S}_0'(\mathbb{R}^n)$ and $\psi \in \mathcal{S}_0(\mathbb{R})$. Then,*

$$\langle \mathcal{R}_\psi f, \Phi \rangle = \int_0^\infty \int_{-\infty}^\infty \langle \mathcal{W}_\psi (Rf)(\mathbf{u}, b, a), \Phi(\mathbf{u}, b, a) \rangle_{\mathbf{u}} \frac{dbda}{a^n}, \quad \Phi \in \mathcal{S}(\mathbb{Y}^{n+1}). \tag{4.33}$$

Furthermore, $\mathcal{R}_\psi f \in C^\infty(\mathbb{H}, \mathcal{D}'(\mathbb{S}^{n-1}))$ and it is of slow growth on $\mathbb{H}$.

Proof. That $\mathcal{R}_\psi$ is smooth and of slow growth in the variables b, a follows from (4.33) and the corresponding property for the wavelet transform. Let us show (4.33). The multiplier operator J_s was introduced in (4.28). By (**??**),

$$\int_0^\infty \int_{-\infty}^\infty \langle \mathcal{W}_\psi (Rf)(\mathbf{u}, b, a), \Phi(\mathbf{u}, b, a) \rangle_{\mathbf{u}} \frac{dbda}{a^n} = \left\langle Rf, \mathcal{M}_{\overline{\psi}} J_{1-n} \Phi \right\rangle = \left\langle f, R^* \mathcal{M}_{\overline{\psi}} J_{1-n} \Phi \right\rangle$$
$$= \langle f, \mathcal{R}^t_{\overline{\psi}} \Phi \rangle = \langle \mathcal{R}_\psi f, \Phi \rangle .$$

□

It should be emphasized that the relation (4.33) is consistent with the ridgelet transform of test functions, as follows from Theorem 4.4.2 and (4.7).

We end this article with a desingularization formula, a corollary of Theorem 4.6.1. The next result generalizes the extended Parseval's relation obtained in Proposition 4.2.2.

Corollary 4.6.1 (Ridgelet desingularization)**.** *Let $f \in \mathcal{S}_0'(\mathbb{R}^n)$ and let $\psi \in \mathcal{S}_0(\mathbb{R})$ be non-trivial. If $\eta \in \mathcal{S}_0(\mathbb{R})$ is a reconstruction neuronal activation function for ψ, then*

$$\langle f, \phi \rangle = \frac{1}{K_{\psi,\eta}} \int_0^\infty \int_{\mathbb{R}} \langle \mathcal{W}_\psi (Rf)(\mathbf{u}, b, a), \mathcal{R}_{\overline{\eta}}\, \phi(\mathbf{u}, b, a) \rangle_{\mathbf{u}} \frac{dbda}{a^n}, \quad \phi \in \mathcal{S}_0(\mathbb{R}^n). \tag{4.34}$$

Proof. By Theorem 4.4.1, $K_{\psi,\eta} \langle f, \phi \rangle = \left\langle f, \mathcal{R}^t_{\overline{\psi}} \mathcal{R}_{\overline{\eta}}\, \phi \right\rangle = \langle \mathcal{R}_\psi f, \mathcal{R}_{\overline{\eta}}\, \phi \rangle$. The desingularization formula (4.34) follows then from (4.33). □

According to (4.32), the relation (4.33) for distributions might be rewritten as

$$\mathcal{R}_\psi = J_{1-n} \circ \mathcal{W}_\psi \circ R. \tag{4.35}$$

Observe that (4.35) is not in contradiction with (4.7). Indeed, if $f \in L^1(\mathbb{R}^n)$ (or more generally $f \in \mathcal{D}'_{L^1}(\mathbb{R}^n)$), then (4.7) expresses an equality between functions, (4.33) is then in agreement with (4.24), whereas (4.35) simply responds to our convention (4.32) of using the measure $a^{-1}dbda$ for identifying wavelet transforms with vector-valued distributions on $\mathbb{H}$. We also have to warn the reader that under this convention, the smooth function $F_\varphi(b, a) = \langle \mathcal{R}_\psi f(\mathbf{u}, b, a), \varphi(\mathbf{u}) \rangle_{\mathbf{u}}$ from the standard identification (4.31), where $\varphi \in \mathcal{D}(\mathbb{S}^{n-1})$, is the one that satisfies

$$\langle \mathcal{R}_\psi f(\mathbf{u}, b, a), \varphi(\mathbf{u}) \Psi(b, a) \rangle_{\mathbf{u}} = \int_0^\infty \int_{-\infty}^\infty F_\varphi(b, a) \Psi(b, a) \frac{dbda}{a}, \quad \Psi \in \mathcal{S}(\mathbb{H}); \tag{4.36}$$

so that if $f \in \mathcal{D}'_{L^1}(\mathbb{R}^n)$, we have, as pointwise equality between functions,

$$\langle \mathcal{R}_\psi f(\mathbf{u}, b, a), \varphi(\mathbf{u}) \rangle_{\mathbf{u}} = a^{-(n-1)} \int_{\mathbb{S}^{n-1}} \mathcal{R}_\psi f(\mathbf{u}, b, a) \varphi(\mathbf{u}) d\mathbf{u}. \tag{4.37}$$

4.7 Ridgelet characterization of bounded subsets of $\mathcal{S}_0'(\mathbb{R}^n)$

This section is dedicated to prove a characterization of bounded subsets of $\mathcal{S}_0'(\mathbb{R}^n)$ via the ridgelet transform. We begin the ensuing useful proposition. Note that [33] $R(\mathcal{S}_0(\mathbb{R}^n))$ is a closed subspace of $\mathcal{S}_0(\mathbb{S}^{n-1}\times\mathbb{R})$. The open mapping theorem implies that $R:\mathcal{S}_0(\mathbb{R}^n)\to R(\mathcal{S}_0(\mathbb{R}^n))$ is an isomorphism of topological vector spaces. We prove a similar result for the distributional Radon transform.

Proposition 4.7.1. *The Radon transform $R:\mathcal{S}_0'(\mathbb{R}^n)\to R(\mathcal{S}_0'(\mathbb{R}^n))$ is an isomorphism of topological vector spaces.*

Proof. Since $R^*:\mathcal{S}_0(\mathbb{S}^{n-1}\times\mathbb{R})\to\mathcal{S}_0(\mathbb{R}^n)$ is a continuous surjection between Fréchet spaces, its transpose $R:\mathcal{S}_0'(\mathbb{R}^n)\to\mathcal{S}_0'(\mathbb{S}^{n-1}\times\mathbb{R})$ must be continuous, injective, and must have weakly closed range [103, Chap. 37]. The subspace $R(\mathcal{S}_0'(\mathbb{R}^n))$ is thus strongly closed because $\mathcal{S}_0'(\mathbb{S}^{n-1}\times\mathbb{R})$ is reflexive. Pták's theory [48, 77] applies to show that $R:\mathcal{S}_0'(\mathbb{R}^n)\to R(\mathcal{S}_0'(\mathbb{R}^n))$ is open if we verify that $\mathcal{S}_0'(\mathbb{R}^n)$ is fully complete (B-complete in the sense of Pták) and that $R(\mathcal{S}_0'(\mathbb{R}^n))$ is barrelled. It is well known [77, p. 123] that the strong dual of a reflexive Fréchet space is fully complete, so $\mathcal{S}_0'(\mathbb{R}^n)$, as a DFS space, is fully complete. Now, a closed subspace of a DFS space must itself be a DFS-space. Since $\mathcal{S}_0'(\mathbb{S}^{n-1}\times\mathbb{R})$ is a DFS space, we obtain that $R(\mathcal{S}_0'(\mathbb{R}^n))$ is a DFS space and hence barrelled. □

We then have,

Theorem 4.7.1. *Let $\psi\in\mathcal{S}_0(\mathbb{R})\setminus\{0\}$ and let $\mathfrak{B}\subset\mathcal{S}_0'(\mathbb{R}^n)$. The following three statements are equivalent:*

(i) $\mathfrak{B}$ is bounded in $\mathcal{S}_0'(\mathbb{R}^n)$.

(ii) There are positive constants $l=l_{\mathfrak{B}}$ and $m=m_{\mathfrak{B}}$ such that for every $\varphi\in\mathcal{D}(\mathbb{S}^{n-1})$ one can find $C=C_{\varphi,\mathfrak{B}}>0$ with

$$|\langle\mathcal{R}_\psi f(\mathbf{u},b,a),\varphi(\mathbf{u})\rangle_{\boldsymbol{u}}|\leq C\left(a+\frac{1}{a}\right)^l(1+|b|)^m,\ \text{for all }\ (b,a)\in\mathbb{H}\text{ and }f\in\mathfrak{B}. \tag{4.38}$$

(iii) $\mathcal{R}_\psi(\mathfrak{B})$ is bounded in $\mathcal{S}'(\mathbb{Y}^{n+1})$.

Proof. By Proposition 4.7.1, $\mathfrak{B}$ is bounded if and only if $\mathfrak{B}_1:=R(\mathfrak{B})$ is bounded in $\mathcal{S}_0'(\mathbb{S}^{n-1}\times\mathbb{R})=\mathcal{S}_0'(\mathbb{R},\mathcal{D}'(\mathbb{S}^{n-1}))$. On the other hand, in view of (4.35), the estimate (4.38) is equivalent to one of the form

$$|\langle\mathcal{W}_\psi h(\mathbf{u},b,a),\varphi(\mathbf{u})\rangle_{\mathbf{u}}|\leq C\left(a+\frac{1}{a}\right)^s(1+|b|)^m,\ \text{ for all }\ h\in\mathfrak{B}_1. \tag{4.39}$$

$(i)\Rightarrow(ii)$. Assume that $\mathfrak{B}_1$ is bounded. As a DFS space, $\mathcal{D}'(\mathbb{S}^{n-1})$ is the regular inductive limit of an inductive sequence of Banach spaces, [71, Prop. 3.2] then implies the existence of $s=s_{\mathfrak{B}}$ and $m=m_{\mathfrak{B}}$ such that $(a+1/a)^{-s}(1+|b|)^{-m}\mathcal{W}_\psi(\mathfrak{B}_1)$ is bounded in $\mathcal{D}'(\mathbb{S}^{n-1})$, which implies (4.39).

(ii) $\Rightarrow$ *(iii)*. If the estimates (4.38) hold, we clearly have that for fixed $\varphi \in \mathcal{D}(\mathbb{S})$ and $\Psi \in \mathcal{S}(\mathbb{H})$ the quantity $\langle \mathcal{R}_\psi f(\mathbf{u}, b, a), \varphi(\mathbf{u})\Psi(b, a)\rangle$ (see (5.28)) remains uniformly bounded for $f \in \mathfrak{B}$. A double application of the Banach-Steinhaus theorem shows that $\mathcal{R}_\psi(\mathfrak{B})$ is a bounded subset of $L_b(\mathcal{S}(\mathbb{H}), \mathcal{D}'(\mathbb{S}^{n-1})) =: \mathcal{S}'(\mathbb{H}, \mathcal{D}'(\mathbb{S}^{n-1}))$ $(= \mathcal{S}'(\mathbb{Y}^{n+1}))$.

(iii) $\Rightarrow$ *(i)*. Let $\eta \in \mathcal{S}_0(\mathbb{R})$. Since $\mathcal{R}^t_\eta$ is continuous, it maps $\mathcal{R}_\psi(\mathfrak{B})$ into a bounded subset of $\mathcal{S}'_0(\mathbb{R}^n)$. That $\mathfrak{B}$ is bounded follows at once from the inversion formula (4.23).

□

4.8 Abelian and Tauberian theorems

In this last section we characterize the quasiasymptotic behavior of elements of $\mathcal{S}'_0(\mathbb{R}^n)$ in terms of Abelian and Tauberian theorems for the ridgelet transform.

4.8.1 An Abelian result

We provide here an Abelian proposition for the ridgelet transform. The following simple but useful lemma connects the quasiasymptotic properties of a distribution with those of its Radon transform.

Lemma 4.8.1. *$f \in \mathcal{S}'_0(\mathbb{R})$.*

(i) f has the quasiasymptotic behavior (2.7)(resp. (2.9)) if and only if its Radon transform has the quasiasymptotic behavior

$$Rf\,(\mathbf{u}, \lambda p) \sim \lambda^{\alpha+n-1} L(\lambda)\ Rg\,(\mathbf{u}, p) \quad \text{as } \lambda \to 0^+ \ (\text{resp. } \lambda \to \infty) \quad \text{in } \mathcal{S}'_0(\mathbb{R}, \mathcal{D}'(\mathbb{S}^{n-1})).$$

(ii) f satisfies (2.17) if and only if its Radon transform satisfies

$$Rf\,(\mathbf{u}, \lambda p) = O(\lambda^{\alpha+n-1} L(\lambda)) \quad \text{as } \lambda \to 0^+ \ (\text{resp. } \lambda \to \infty) \quad \text{in } \mathcal{S}'_0(\mathbb{R}, \mathcal{D}'(\mathbb{S}^{n-1})).$$

Proof. Set $f_\lambda(\mathbf{x}) = f(\lambda \mathbf{x})$. If $\varrho \in \mathcal{S}_0(\mathbb{S}^{n-1} \times \mathbb{R})$, we have,

$$\begin{aligned}
\langle Rf_\lambda(\mathbf{u}, p), \varrho(\mathbf{u}, p)\rangle &= \frac{1}{\lambda^n}\langle f(\mathbf{x}), R^*\varrho(\mathbf{x}/\lambda)\rangle \\
&= \frac{1}{\lambda^{n-1}}\left\langle f(\mathbf{x}), \frac{1}{\lambda}\int_{\mathbb{S}^{n-1}} \varrho\left(\mathbf{u}, \frac{\mathbf{x}\cdot\mathbf{u}}{\lambda}\right) d\mathbf{u}\right\rangle \\
&= \frac{1}{\lambda^{n-1}}\langle Rf(\mathbf{u}, \lambda p), \varrho(\mathbf{u}, p)\rangle,
\end{aligned}$$

namely, $Rf_\lambda(\mathbf{u}, p) = \lambda^{-(n-1)} Rf(\mathbf{u}, \lambda p)$. The result is then a consequence of Proposition 4.7.1.

□

Proposition 4.8.1. *Suppose that $f \in \mathcal{S}_0'(\mathbb{R})$ has the quasiasymptotic behavior (2.7)(resp. (2.9)). Then, given any $\varphi \in \mathcal{D}(\mathbb{S}^{n-1})$ and $(b,a) \in \mathbb{H}$, we have*

$$\lim_{\lambda\to 0^+} \frac{\langle \mathcal{R}_\psi f\left(\mathbf{u}, \lambda b, \lambda a\right), \varphi(\mathbf{u})\rangle_{\mathbf{u}}}{\lambda^\alpha L(\lambda)} = \langle \mathcal{R}_\psi g\left(\mathbf{u}, b, a\right), \varphi(\mathbf{u})\rangle_{\mathbf{u}} \quad \left(\text{resp.} \lim_{\lambda\to\infty}\right) \tag{4.40}$$

Proof. This proposition follows by combining Lemma 4.8.1 and the relation (4.35) with the DFS-space-valued version of [71, Prop. 3.1] for the wavelet transform (see comments in [71, Sect. 8]) □

Remark 4.8.1. The limit (4.40) holds uniformly for (b,a) in compact subsets of $\mathbb{H}$.

Remark 4.8.2. If $f \in \mathcal{D}'_{L^1}(\mathbb{R}^n)$, then (4.40) reads

$$\int_{\mathbb{S}^{n-1}} \mathcal{R}_\psi f(\mathbf{u}, \lambda b, \lambda a)\varphi(\mathbf{u})d\mathbf{u} \sim \lambda^{\alpha+n-1}L(\lambda)\int_{\mathbb{S}^{n-1}} \mathcal{R}_\psi g(\mathbf{u}, b, a)\varphi(\mathbf{u})d\mathbf{u},$$

as follows from (4.37).

4.8.2 Tauberian theorem

Our next goal is to provide a Tauberian converse for Proposition 4.8.1. The next theorem characterizes the quasiasymtotic behavior in terms of the ridgelet transform.

Theorem 4.8.1. *Let $\psi \in \mathcal{S}_0(\mathbb{R}) \setminus \{0\}$ and $f \in \mathcal{S}_0'(\mathbb{R}^n)$. The following two conditions:*

$$\lim_{\lambda\to 0^+} \frac{1}{\lambda^\alpha L(\lambda)} \langle \mathcal{R}_\psi f\left(\mathbf{u}, \lambda b, \lambda a\right), \varphi(\mathbf{u})\rangle = \mathrm{M}_{b,a}(\varphi) \quad \left(\textit{resp.} \lim_{\lambda\to\infty}\right) \tag{4.41}$$

exists (and is finite) for every $\varphi \in \mathcal{D}(\mathbb{S}^{n-1})$ and $(b,a) \in \mathbb{H} \cap \mathbb{S}$, and there exist $m, l > 0$ such that for every $\varphi \in \mathcal{D}(\mathbb{S}^{n-1})$

$$|\langle \mathcal{R}_\psi f\left(\mathbf{u}, \lambda b, \lambda a\right), \varphi(\mathbf{u})\rangle_{\mathbf{u}}| \leq C_\varphi \lambda^\alpha L(\lambda)\left(a + \frac{1}{a}\right)^l (1+|b|)^m \tag{4.42}$$

for all $(b,a) \in \mathbb{H} \cap \mathbb{S}$ and $0 < \lambda < 1$ (resp. $\lambda > 1$) are necessary and sufficient for the existence of a distribution g such that f has the quasiasymptotic behavior (2.7)(resp. (2.9)).

Proof. Assume first that f has the quasiasymptotic behavior (2.7)(resp. (2.9)). Proposition 4.8.1 implies that (4.41) holds with $M_{b,a}(\varphi) = \langle \mathcal{R}_\psi g\left(\mathbf{u}, b, a\right), \varphi(\mathbf{u})\rangle_{\mathbf{u}}$. Set $f_\lambda(\mathbf{x}) = f(\lambda \mathbf{x})$. Using (4.33), one readily verifies the relation

$$\mathcal{R}_\psi f_\lambda(\mathbf{u}, b, a) = \mathcal{R}_\psi f(\mathbf{u}, \lambda b, \lambda a). \tag{4.43}$$

On the other hand, f satisfies (2.17). That (4.42) must necessarily hold follows from Theorem 4.7.1.

Conversely, assume (4.41) and (4.42). Applying the same argument as in the proof of [71, Lem. 6.1], one may assume that they hold *for all* $(b,a) \in \mathbb{H}$ (in the case of (4.42), one may need to replace l and m by bigger exponents). We will show that there is $G \in \mathcal{S}'(\mathbb{Y}^{n+1})$ such that

$$\lim_{\lambda \to 0^+} \left\langle \frac{\mathcal{R}_\psi f\left(\mathbf{u}, \lambda b, \lambda a\right)}{\lambda^\alpha L(\lambda)}, \Phi(\mathbf{u},b,a) \right\rangle = \left\langle G\left(\mathbf{u},b,a\right), \Phi(\mathbf{u},b,a) \right\rangle \quad \left(\text{resp. } \lim_{\lambda\to\infty}\right) \tag{4.44}$$

for each $\Phi \in \mathcal{S}(\mathbb{Y}^{n+1})$. Once (4.44) had been established, the inversion formula (4.23) would imply that (2.7)(resp. (2.9)) holds with $g = (1/K_{\psi,\eta})\mathcal{R}^t_\eta G$. Using Theorem 4.7.1 and (4.43) again, the estimates (4.42) are equivalent to the quasi-asymptotic boundedness (2.17), but also to the boundedness in $\mathcal{S}'(\mathbb{Y}^{n+1})$ of the set

$$\left\{ \frac{\mathcal{R}_\psi f\left(\mathbf{u}, \lambda b, \lambda a\right)}{\lambda^\alpha L(\lambda)} : \, 0 < \lambda < 1 \right\} \quad (\text{resp. } \lambda > 1)\,. \tag{4.45}$$

By the Banach-Steinhaus theorem, the set (4.45) is equicontinuous. It is thus enough to show that the limit in the left-hand side of (4.44) exists for Φ in the dense subspace $\mathcal{D}(\mathbb{S}^{n-1})\otimes\mathcal{S}(\mathbb{H})$ of $\mathcal{S}(\mathbb{Y}^{n+1})$. So, we check this for $\Phi(\mathbf{u},b,a) = \varphi(\mathbf{u})\Psi(b,a)$ with $\varphi \in \mathcal{D}(\mathbb{S}^{n-1})$ and $\Psi \in \mathcal{S}(\mathbb{H})$. The function $M_{b,a}(\varphi)$ occurring in (4.41) is measurable in $(b,a) \in \mathbb{H}$ and, in view of (4.42), is of slow growth, i.e., it satisfies

$$|M_{b,a}(\varphi)| \le C_\varphi \left(a + \frac{1}{a}\right)^l (1+|b|)^m, \quad \text{for all}\,(b,a) \in \mathbb{H}.$$

So, employing (5.28) and the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned} &\lim_{\lambda\to 0^+} \left\langle \frac{\mathcal{R}_\psi f\left(\mathbf{u}, \lambda b, \lambda a\right)}{\lambda^\alpha L(\lambda)}, \varphi(\mathbf{u})\Psi(b,a) \right\rangle \\ &= \lim_{\lambda\to 0^+} \int_0^\infty \int_{-\infty}^\infty \left\langle \frac{\mathcal{R}_\psi f\left(\mathbf{u}, \lambda b, \lambda a\right)}{\lambda^\alpha L(\lambda)}, \varphi(\mathbf{u}) \right\rangle \Psi(b,a) \frac{dbda}{a} \\ &= \int_0^\infty \int_{-\infty}^\infty M_{b,a}(\varphi)\Psi(b,a)\frac{dbda}{a} \end{aligned}$$

(resp. $\lim_{\lambda\to\infty}$). This completes the proof. □

The following fact was already shown within the proof of Theorem 4.8.1.

Corollary 4.8.1. *Let* $\psi \in \mathcal{S}_0(\mathbb{R}) \setminus \{0\}$ *and* $f \in \mathcal{S}'_0(\mathbb{R}^n)$. *Then,* f *satisfies (2.17) if and only if there are* $m, l > 0$ *such that for every* $\varphi \in \mathcal{D}(\mathbb{S}^{n-1})$ *the estimate (4.42) holds for all* $0 < \lambda < 1$ *(resp.* $\lambda > 1$*) and* $(b,a) \in \mathbb{H} \cap \mathbb{S}$ *(or, equivalently,* $(b,a) \in \mathbb{H}$*).*

Chapter 5

Multiresolution expansions and quasiasymptotic behavior of distributions

The notion of multiresolution analysis (MRA) was introduced by Mallat and Meyer as a natural approach to the construction of orthogonal wavelets [53, 56]. Approximation properties of multiresolution expansions in function and distribution spaces have been extensively investigated, see e.g. [56]. The problem of pointwise convergence of multiresolution expansions is very important from a computational point of view and has also been studied by many authors. Our purpose here is to study the pointwise behavior of Schwartz distributions, in several variables, via multiresolution expansions. In particular, we shall extend and improve results from [70, 95, 97, 114]. The second aim is to study the quasiasymptotic behavior of a distribution at a point through multiresolution expansions.

In this Chapter, we use bold letters to denote elements from $\mathbb{R}^n$.

5.1 Multiresolution analysis in $L^2(\mathbb{R}^n)$

Research workers in the various specialities were hoping to find practical algorithms for decomposing arbitrary function into sums of special functions which combine the advantages of the trigonometric and the Haar systems. These systems stand at two extremes, in the following sense: the functions of the trigonometric systems (see [117, Chapter 1.2.1]) are exactly localized by frequency, that is in the Fourier variable, but have no precise localization in space. On the other hand, the functions of the Haar system (see [117, Chapter 1.2.2]) are perfectly localized in space but are badly localized in the Fourier variable. The idea of a multiresolution analysis enables us to combine analysis in the space variable with analysis in the Fourier variable while satisfying the Heisenberg's uncertainty principle:

$$\int_{\mathbb{R}^n} |\mathbf{x} f(\mathbf{x})|^2 dx \cdot \int_{\mathbb{R}^n} |\omega \hat{f}(\omega)| d\omega \geq \frac{n^2}{4(2\pi)^{n-1}},$$

for any $f \in L^2(\mathbb{R}^n)$ with $\|f\|_2 = 1$.

Definition 5.1.1. Let $\{V_j\}_{j\in\mathbb{Z}}$ be a sequence of closed linear subspaces of $L^2(\mathbb{R}^n)$. The collection of spaces $\{V_j\}_{j\in\mathbb{Z}}$ is called a *multiresolution analysis* (MRA) if they satisfy the following four conditions:

(i) (scaling) $f \in V_j \Leftrightarrow f(2\,\cdot\,) \in V_{j+1}$,

(ii) $f \in V_0 \Leftrightarrow f(\,\cdot\, - \mathbf{m}) \in V_0$, $\mathbf{m} \in \mathbb{Z}^n$,

(iii) (separation) $\bigcap_j V_j = \{\mathbf{0}\}$,

(density) $\overline{\bigcup_j V_j} = L^2(\mathbb{R}^n)$,

(iv) (orthonormal basis) there is $\phi \in L^2(\mathbb{R}^n)$ such that $\{\phi(\,\cdot\, - \mathbf{m})\}_{\mathbf{m}\in\mathbb{Z}^n}$ is an orthonormal basis of V_0.

The function ϕ from (iv) is called a *scaling function* of the given MRA. Moreover, from the properties (i) and (iv) we have that $\{2^{j/2}\phi(2^j\cdot - \mathbf{m})\}_{\mathbf{m}\in\mathbb{Z}^n}$ is an orthonormal basis of V_j.

There may be several choices of ϕ corresponding to a system of approximation spaces. Different choices for ϕ yield different MRA. Although we require the translates of $\phi(\mathbf{x})$ to be orthonormal, we don't have to. All that is needed is a ϕ for which the set $\{\phi(\mathbf{x} - \mathbf{k}) : \mathbf{k} \in \mathbb{Z}^n\}$ is basis. We can then use ϕ to obtain a new scaling function $\widetilde{\phi}$ for which $\{\widetilde{\phi}(\mathbf{x} - \mathbf{k}) : \mathbf{k} \in \mathbb{Z}^n\}$ is orthonormal (this othonormalization procedure can be done in several ways [10, 56]).

Example 5.1.1. *Multiresolution analysis of Littlewood-Paley type.* Here we start with a function $\theta(\xi)$, of the real variable ξ belonging to $\mathcal{D}(\mathbb{R})$, which is even, equals to 1 on $[-2\pi/3, 2\pi/3]$ and is 0 outside $[-4\pi/3, 4\pi/3]$. We suppose in addition that $\theta(\xi) \in [0,1]$, for all $\xi \in \mathbb{R}$, and that $\theta^2(\xi) + \theta^2(2\pi - \xi) = 1$ when $0 \le \xi \le 2\pi$. Let ϕ denote the function in $\mathcal{S}(\mathbb{R})$ whose Fourier transform is $\theta(\xi)$. Then, we can verify that the sequence $\phi(x - k), k \in \mathbb{Z}$ is the orthonormal basis of a closed subspace of $L^2(\mathbb{R})$ which we call V_0. In fact, applying the Fourier transform, $\mathcal{F}V_0$ is the vector space of products $m(\xi)\theta(\xi)$, where $m(\xi)$ is 2π-periodic, and $m(\xi)$ restricted to $[0, 2\pi]$ belongs to $L^2[0, 2\pi]$. V_1 is then defined with (i) and $\mathcal{F}V_1$ is the set of function $m_1(\xi)\theta(\xi/2)$, where $m_1(\xi)$ is 4π-periodic. The other properties of the MRA can be verified without difficulties.

Theorem 5.1.1. *[3, Thrm. 5.6] Let $\{V_j\}_{j\in\mathbb{Z}}$ is a MRA with scaling function ϕ. Then, the following scaling relation holds:*

$$\phi(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^n} p_{\mathbf{k}}\phi(2\mathbf{x} - \mathbf{k}), \quad \text{where} \quad p_{\mathbf{k}} = 2\int_{\mathbb{R}^n} \phi(\mathbf{x})\overline{\phi(2\mathbf{x} - \mathbf{k})}d\mathbf{x}. \tag{5.1}$$

Moreover, we also have

$$\phi(2^{j-1}\mathbf{x} - \mathbf{l}) = \sum_{\mathbf{k}\in\mathbb{Z}^n} p_{\mathbf{k}-2\mathbf{l}}\phi(2^j\mathbf{x} - \mathbf{k}). \tag{5.2}$$

Recall that V_j is a subset of V_{j+1}. In order to carry out the decomposition algorithm in the general case, we need to decompose V_{j+1} into an orthogonal direct sum of V_j and its orthogonal complement, which we denote by W_j. This means that

$$V_j = V_{j+1} \oplus W_{j+1}$$

and

$$W_j \perp W_l \quad \text{if} \quad j \neq l.$$

In addition, we need to construct a function ψ whose translates generate the space W_j. Once ϕ is specified, the scaling relation can be used to construct ψ. This is shown with the following theorem.

Theorem 5.1.2. *[3, Thrm. 5.10] Let $\{V_j\}_{j\in\mathbb{Z}}$ is a MRA with scaling function ϕ that satisfies* (5.1)*. Let W_j be the span of $\{\psi(2^j\mathbf{x}-\mathbf{k}) : \mathbf{k} \in \mathbb{Z}^n\}$, where*

$$\psi(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^n} (-1)^{|\mathbf{k}|}\overline{p_{1-\mathbf{k}}}\phi(2\mathbf{x}-\mathbf{k}). \tag{5.3}$$

Then, $W_j \subset V_{j+1}$ is the orthogonal complement of V_j in V_{j+1}. Furthermore, $\{\psi_{j\mathbf{k}}(\mathbf{x}) := 2^{j/2}\psi(2^j\mathbf{x}-\mathbf{k}) : \mathbf{k} \in \mathbb{Z}^n\}$ is an orthonormal basis for the W_j.

It is required $\psi(\mathbf{x})$ to be orthogonal to $\phi(\mathbf{x}-\mathbf{k})$. Hence, the two conditions

1. $\sum_{\mathbf{k}\in\mathbb{Z}^n} \hat{\psi}(\mathbf{w}+\pi\mathbf{k})\overline{\hat{\psi}(\mathbf{w}+2\pi\mathbf{k})} = 0$,
2. $\sum_{\mathbf{k}\in\mathbb{Z}^n} |\overline{\hat{\psi}(\mathbf{w}+2\pi\mathbf{k})}|^2 = 1$,

must be satisfied.

By virtue of (iii) we have

$$L^2(\mathbb{R}^n) = \bigoplus_{j\in\mathbb{Z}} W_j,$$

a decomposition of $L^2(\mathbb{R}^n)$ into mutually orthogonal spaces. Because of this and again (iii) we have that $\{\psi_{j\mathbf{k}}(\mathbf{x}) = 2^{j/2}\psi(2^j\mathbf{x}-\mathbf{k}) : \mathbf{k} \in \mathbb{Z}^n, j \in \mathbb{Z}\}$ is an orthonormal basis for L^2.

It can be shown that for $\hat{\phi}$ and $\hat{\psi}$, the Fourier transforms of a scaling function ϕ and its corresponding wavelet ψ, respectively, the following relation holds

$$\hat{\psi}(\mathbf{w}) = \left(\left(\hat{\phi}\left(\frac{\mathbf{w}}{2}\right)\right)^2 - \left(\hat{\phi}\left(\mathbf{w}\right)\right)^2\right)^{1/2} e^{-i\frac{\mathbf{w}}{2}}, \ \mathbf{w} \in \mathbb{R}^n.$$

Example 5.1.2. Most simple and oldest example of scaling function for which $\{2^{j/2}\phi(2^j\ \cdot\ -m)\}_{m\in\mathbb{Z}}$ is an orthonormal basis of V_j is the Haar scaling function

$$\phi(x) = \begin{cases} 1, & 0 \le x < 1 \\ 0, & \text{elsewhere} \end{cases}.$$

The choices of the coefficients p_k in (5.2) are $p_0 = p_1 = 1$ and the other p_k are zero. The appropriate ψ from Theorem (5.1.2) is the Haar wavelet

$$\psi(x) = \phi(2x) - \phi(2x-1) = \begin{cases} 1, & 0 \le x < 1/2 \\ -1, & 1/2 \le x < 1 \\ 0, & \text{elsewhere} \end{cases}.$$

Example 5.1.3. The B -spline of order n is the function $\phi(x)$ obtained by convolving the Haar scaling function with itself n-times. Its Fourier transform is $\hat{\phi}(\omega) = (2\pi)^{-1/2} e^{-iK\xi/2} \left(\frac{\sin \xi/2}{\xi/2}\right)^{N+1}$, where $K = 0$ if N is odd, and $K = 1$ if N is even. The mother wavelet obtained with (5.3) is called Battle-Lemarié wavelet.

If we take $N = 2$ we get the piecewise quadratic B-spline

$$\phi(x) = \begin{cases} \frac{1}{2}(x+1)^2, & -1 \leq x < 0 \\ \frac{3}{4} - \left(x - \frac{1}{2}\right)^2, & 0 \leq x < 1 \\ \frac{1}{2}(x-2)^2, & 1 \leq x < 2 \\ 0, & \text{elsewhere} \end{cases}.$$

Now, ϕ satisfies $\phi(x) = \frac{1}{4}\phi(2x+1) + \frac{3}{4}\phi(2x) + \frac{3}{4}\phi(2x-1) + \frac{1}{4}\phi(2x-2)$, and we have that

$$\hat{\psi}(\omega) = (2\pi)^{-1/2} e^{-i\xi/2} \left(\frac{\sin \xi/2}{\xi/2}\right)^3.$$

In [10, Cor. 5.4.2] it is proven that all the Battle-Lemarié wavelets and the corresponding scaling function can be chosen from $C^r(\mathbb{R})$ and have exponential decay (the decay rate decrease as r increase).

Remark 5.1.1. Note that it is possible to find a function $\psi \in L^2$ such that the family $\{\psi_{j\mathbf{k}}(\,\cdot\,) = 2^{j/2}\psi(2^j\,\cdot\, - \mathbf{k}) : \mathbf{k} \in \mathbb{Z}^n, j \in \mathbb{Z}\}$ is an orthonormal basis of L^2 but the corresponding scaling function ϕ does not exists.

Each of the spaces V_j in the MRA is a reproducing kernel Hilbert space. Such a space consists of a Hilbert space H of functions f on an interval T in which all evaluation functions $\xi_{\mathbf{t}}(f) := f(\mathbf{t}), f \in H, \mathbf{t} \in T$ are continuous on H. Then, by the Riesz representation theorem, for each $\mathbf{t} \in T$ there is unique $k_{\mathbf{t}} \in H$ such that for each $f \in H$, $f(\mathbf{t}) = \langle f, k_{\mathbf{t}} \rangle$. The function defined by $k(\mathbf{t}, \mathbf{u}) = \langle k_{\mathbf{t}}, k_{\mathbf{u}} \rangle$ for $\mathbf{t}, \mathbf{u} \in T$ is the reproducing kernel. Note that $L^2(\mathbb{R}^n)$ is not a reproducing kernel Hilbert space but has subspaces that are. The reproducing kernel of V_0 is given by

$$q_0(\mathbf{x}, \mathbf{y}) = \sum_{\mathbf{m} \in \mathbb{Z}^n} \phi(\mathbf{x} - \mathbf{m}) \overline{\phi(\mathbf{y} - \mathbf{m})}. \tag{5.4}$$

where ϕ is the scaling function. It holds

$$|q_0(\mathbf{x}, \mathbf{y})| \leq \frac{1}{(1 + |\mathbf{x} - \mathbf{y}|)^n}, n \in \mathbb{N}. \tag{5.5}$$

The reproducing kernel of V_j is given by

$$q_j(\mathbf{x}, \mathbf{y}) = 2^j q_0(2^j \mathbf{x}, 2^j \mathbf{y}). \tag{5.6}$$

5.2 Multiresolution analysis in distribution spaces

We now explain how one can study multiresolution expansions of tempered distributions and distributions of M-exponential growth. We show below that, under certain regularity assumptions on an MRA, multiresolution expansions converge in $\mathcal{K}'_{M,r}(\mathbb{R}^n)$ or $\mathcal{S}'_r(\mathbb{R}^n)$. Observe that (1.13) (resp. (1.14)) allows us to analyze also elements of $\mathcal{K}'_M(\mathbb{R}^n)$ (resp. $\mathcal{S}'(\mathbb{R}^n)$) by reduction to one of the spaces $\mathcal{K}'_{M,r}(\mathbb{R}^n)$ (resp. $\mathcal{S}'_r(\mathbb{R}^n)$). We mention the references [75, 72, 96, 101], where related results have been discussed. The difference here is that we give emphasis to uniform convergence over bounded subsets of test functions and other parameters, which will be crucial for our arguments in the subsequent sections.

In order to be able to analyze various classes of distributions with the MRA, we shall impose some regularity assumptions on the scaling function ϕ. One says that the MRA is *r-regular* [56, 117], $r \in \mathbb{N}$, if the scaling function from (iv) can be chosen in such a way that:

(v) $\phi \in \mathcal{S}_r(\mathbb{R}^n)$.

The r-regular MRA are well-suited for the analysis of tempered distributions [**?**, 101, 117]. For distributions of M-growth, we need to impose stronger regularity conditions on the scaling function. We say that the MRA is (M, r)*-regular* [97, 96] if the scaling function from (iv) can be chosen such that ϕ fulfills the requirement:

$(v)'$ $\phi \in \mathcal{K}_{M,r}(\mathbb{R}^n)$.

Example 5.2.1. *Multiresolution analysis by splines of order* r. This example will be given in dimension 1, by nested spaces of splines of order r: the nodes of the functions $f \in V_j$ being precisely the points $k2^{-j}, k \in \mathbb{Z}$. We start with an integer $r \in \mathbb{N}$ and denote by V_0 the subspace of $L^2(\mathbb{R})$ consisting of the functions in C^{r-1} whose restriction to each interval $[k, k+1), k \in \mathbb{Z}$, coincide with polynomial of degree less than or equal to r. V_j is the defined by (i) and the other properties $(ii) - (iv)$ can be verified immediately.

Throughout the rest of this Chapter, whenever we speak about an r-regular MRA (resp. (M, r)-regular MRA) we fix the scaling function ϕ satisfying (v) (resp. $(v)'$). We remark that it is possible to find MRA with scaling functions $\phi \in \mathcal{S}(\mathbb{R}^n)$ [56], therefore satisfying (v) for all r. In contrast, it is worth mentioning that the condition $(v)'$ cannot be replaced by $\phi \in \mathcal{K}_M(\mathbb{R}^n)$; in fact [10, Corol. 5.5.3], there cannot be an exponentially decreasing scaling function $\phi \in C^\infty(\mathbb{R}^n)$ with all bounded derivatives. On the other hand, Daubechies [10] has shown that given an arbitrary r, there exists always an (M, r)-regular MRA of $L^2(\mathbb{R}^n)$ where the scaling function can even be taken to be compactly supported. By tensorizing, this leads to the existence of (M, r)-regular MRA of $L^2(\mathbb{R}^n)$ with compactly supported scaling functions.

Theorem 5.2.1. *[117, Thrm. 3.2] Let* $\phi \in \mathcal{S}_r$ *be a scaling function with MRA* $\{V_j\}_{j\in\mathbb{Z}}$. *Then, if* $\hat{\phi}(\mathbf{0}) \geq 0$

(i) $\hat{\phi}(2\mathbf{k}\pi) = \delta_{0k}, \ \mathbf{k} \in \mathbb{Z}^n,$

(ii) $\sum_n \phi(\mathbf{x} - \mathbf{n}) = 1, \ \mathbf{x} \in \mathbb{R}^n.$

The reproducing kernel of the Hilbert space V_0 is given by

$$q_0(\mathbf{x},\mathbf{y}) = \sum_{\mathbf{m}\in\mathbb{Z}^n} \phi(\mathbf{x}-\mathbf{m})\overline{\phi(\mathbf{y}-\mathbf{m})}. \tag{5.7}$$

If the MRA is (M,r)-regular (resp. r-regular), the series (5.7) and its partial derivatives with respect to $\mathbf{x}$ and $\mathbf{y}$ of order less or equal to r are convergent because of the regularity of ϕ. Furthermore, for fixed $\mathbf{x}$, $q_0(\mathbf{x},\,\cdot\,) \in \mathcal{K}_{M,r}(\mathbb{R}^n)$ (resp. $q_0(\mathbf{x},\,\cdot\,) \in \mathcal{S}_r(\mathbb{R}^n)$). Using the assumptions (1) and (2) on M, one verifies [96, 97] that for every $l \in \mathbb{N}$ and $|\alpha|, |\beta| \leq r$, there exists $C_l > 0$ such that

$$\left|\partial^\alpha_{\mathbf{x}}\partial^\beta_{\mathbf{y}} q_0(\mathbf{x},\mathbf{y})\right| \leq C_l e^{-M(l(\mathbf{x}-\mathbf{y}))} \tag{5.8}$$

$$(\text{resp. } \left|\partial^\alpha_{\mathbf{x}}\partial^\beta_{\mathbf{y}} q_0(\mathbf{x},\mathbf{y})\right| \leq C_l(1+|\mathbf{x}-\mathbf{y}|)^{-l}),$$

and that $q_0(\mathbf{x}+\mathbf{k},\mathbf{y}+\mathbf{k}) = q_0(\mathbf{x},\mathbf{y})$ for $\mathbf{k} \in \mathbb{Z}^n$. One can also show [56] that

$$\int_{\mathbb{R}^n} q_0(\mathbf{x},\mathbf{y})P(\mathbf{y})d\mathbf{y} = P(\mathbf{x}), \quad \text{for each polynomial } P \text{ of degree } \leq r. \tag{5.9}$$

Note that the reproducing kernel of the projection operator onto V_j is

$$q_j(\mathbf{x},\mathbf{y}) = 2^{nj} q_0(2^j\mathbf{x}, 2^j\mathbf{y}), \quad \mathbf{x},\mathbf{y} \in \mathbb{R}^n,$$

so that the projection of $f \in L^2(\mathbb{R}^n)$ onto V_j is explicitly given by

$$(q_j f)(\mathbf{x}) := \langle f(\mathbf{y}), q_j(\mathbf{x},\mathbf{y})\rangle = \int_{\mathbb{R}^n} f(\mathbf{y})q_j(\mathbf{x},\mathbf{y})d\mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^n. \tag{5.10}$$

The sequence $\{q_j f\}_{j\in\mathbb{Z}}$ given in (5.10) is called the multiresolution expansion of $f \in L^2(\mathbb{R}^n)$. Since for an (M,r)-regular (resp. r-regular) MRA $q_j(\mathbf{x},\cdot) \in \mathcal{K}_{M,r}(\mathbb{R}^n)$ (resp. $q_j(\mathbf{x},\,\cdot\,) \in \mathcal{S}_r(\mathbb{R}^n)$), the formula (5.10) also makes sense for $f \in \mathcal{K}'_{M,r}(\mathbb{R}^n)$ (resp. $f \in \mathcal{S}'_r(\mathbb{R}^n)$) and it is not hard to verify that $(q_j f)(\mathbf{x})$ turns out to be a continuous function in $\mathbf{x}$. It is convenient for our future purposes to extend the definition of the operators (5.10) by allowing j to be a continuous variable and also by allowing a translation term.

Walter [117] states that the sequence $\{q_j((\mathbf{x},\mathbf{y}))\}$ is a delta sequence in $\mathcal{S}'_r(\mathbb{R}^n)$, i.e. $q_j(\mathbf{x},\mathbf{y}) \to \delta((\mathbf{x}-\mathbf{y}))$ as $j \to \infty$. In [117], Chapter 5.1. is devoted to MRA of tempered distribution.

Definition 5.2.1. Let $\{V_j\}_{j\in\mathbb{Z}}$ be an (M,r)-regular (resp. r-regular) MRA. Given $\mathbf{z} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, the operator $q_{\lambda,\mathbf{z}}$ is defined on elements $f \in \mathcal{K}'_{M,r}(\mathbb{R}^n)$ (resp. $f \in \mathcal{S}'_r(\mathbb{R}^n)$) as

$$(q_{\lambda,\mathbf{z}}f)(\mathbf{x}) := \langle f(\mathbf{y}), q_{\lambda,\mathbf{z}}(\mathbf{x},\mathbf{y})\rangle_{\mathbf{y}}, \quad \mathbf{x} \in \mathbb{R}^n,$$

by means of the kernel $q_{\lambda,\mathbf{z}}(\mathbf{x},\mathbf{y}) = 2^{n\lambda}q_0(2^\lambda\mathbf{x}+\mathbf{z}, 2^\lambda\mathbf{y}+\mathbf{z})$, $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$. The net $\{q_{\lambda,\mathbf{z}}f\}_{\lambda\in\mathbb{R}}$ is called the generalized *multiresolution expansion* of f.

Clearly, when restricted to $L^2(\mathbb{R}^n)$, $q_{\lambda,\mathbf{z}}$ is the orthogonal projection onto the Hilbert space $V_{\lambda,\mathbf{z}} = \{f(2^\lambda\cdot\,+\mathbf{z}) : f \in V_0\} \subset L^2(\mathbb{R}^n)$. When $\mathbf{z} = 0$, we simply write $q_\lambda := q_{\lambda,0}$. The consideration of the parameter $\mathbf{z}$ will play an important role in Section 5.4. Note also that $\langle q_{\lambda,\mathbf{z}} f, \varphi \rangle = \langle f, q_{\lambda,\mathbf{z}} \varphi \rangle$, for any $f \in \mathcal{K}'_{M,r}(\mathbb{R}^n)$ and $\varphi \in \mathcal{K}_{M,r}(\mathbb{R}^n)$ (resp. $f \in \mathcal{S}'_r(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}_r(\mathbb{R}^n)$).

We now study the convergence of the generalized multiresolution expansions of distributions. We need a preparatory result. In dimension $n = 1$, Pilipović and Teofanov [72] have shown that if $f \in C^r(\mathbb{R})$ and all of its derivatives up to order r are of at most polynomial growth, then its multiresolution expansion $q_j f$ with respect to an r-regular MRA converges to f uniformly over compact intervals. Sohn has considered in [96] the analog result for functions of growth $O(e^{M(kx)})$, but his arguments contain various inaccuracies (compare, e.g., his formulas (17) and (21) with our (5.13) below). We extend those results here to the multidimensional case and for the generalized multiresolution projections $q_{\lambda,\mathbf{z}}$ with uniformity in the parameter $\mathbf{z}$.

Let $\psi \in \mathcal{D}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \psi(\mathbf{x}) d\mathbf{x} = 1$. In the case of an r-regular MRA, it is shown in [56, p. 39] that given any multi-index $|\alpha| \leq r$, there are functions $R^{\alpha,\beta} \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$|R^{\alpha,\beta}(\mathbf{x},\mathbf{y})| \leq \tilde{C}_l (1 + |\mathbf{x} - \mathbf{y}|)^{-l}, \quad \forall l \in \mathbb{N}, \tag{5.11}$$

$$\int_{\mathbb{R}^n} R^{\alpha,\beta}(\mathbf{x},\mathbf{y}) d\mathbf{y} = 0, \quad \forall \mathbf{x} \in \mathbb{R}^n, \tag{5.12}$$

and for any $f \in C^r(\mathbb{R}^n)$, with partial derivatives of at most polynomial growth,

$$\partial^\alpha (q_0 f) = \psi * (\partial^\alpha f) + \sum_{|\beta|=|\alpha|} R^{\alpha,\beta}(\partial^\alpha f).$$

Denoting as $R^{\alpha,\beta}_{\lambda,\mathbf{z}}$ the integral operator with kernel $R^{\alpha,\beta}_{\lambda,\mathbf{z}}(\mathbf{x},\mathbf{y}) = 2^{n\lambda} R^{\alpha,\beta}(2^\lambda \mathbf{x} + \mathbf{z}, 2^\lambda \mathbf{y} + \mathbf{z})$, we obtain the formulas

$$\partial^\alpha_{\mathbf{x}} q_{\lambda,\mathbf{z}} f(\mathbf{x}) = 2^{n\lambda} \int_{\mathbb{R}^n} \psi(2^\lambda(\mathbf{x} - \mathbf{y})) \partial^\alpha_{\mathbf{y}} f(\mathbf{y}) d\mathbf{y} + \sum_{|\beta|=|\alpha|} \int_{\mathbb{R}^n} R^{\alpha,\beta}_{\lambda,\mathbf{z}}(\mathbf{x},\mathbf{y}) \partial^\alpha_{\mathbf{y}} f(\mathbf{y}) d\mathbf{y}. \tag{5.13}$$

Likewise for an (M,r)-regular MRA, one can modify the arguments from [56] in such a way that one chooses the $R^{\alpha,\beta}$ with decay

$$|R^{\alpha,\beta}(\mathbf{x},\mathbf{y})| \leq \tilde{C}_l e^{-M(l(\mathbf{x}-\mathbf{y}))}, \quad \forall l \in \mathbb{N}. \tag{5.14}$$

Proposition 5.2.1. *Assume that the MRA is (M,r)-regular (resp. r-regular).*

(a) If $f \in C^r(\mathbb{R}^n)$ and there is $k \in \mathbb{N}$ such that $f^{(\alpha)}(\mathbf{x}) = O(e^{M(k\mathbf{x})})$ (resp. $f^{(\alpha)}(\mathbf{x}) = O((1 + |\mathbf{x}|)^k)$) for each $|\alpha| \leq r$, then $\lim_{\lambda\to\infty} q_{\lambda,\mathbf{z}} f = f$ in $C^r(\mathbb{R}^n)$.

(b) Suppose that the subset $\mathfrak{B} \subset C^r(\mathbb{R}^n)$ is such that for each $|\alpha| \leq r$ one has $f^{(\alpha)}(\mathbf{x}) = O(e^{M(k\mathbf{x})})$ (resp. $f^{(\alpha)}(\mathbf{x}) = O((1 + |\mathbf{x}|)^k)$) uniformly with respect to $f \in \mathfrak{B}$, then $\lim_{\lambda\to\infty} q_{\lambda,\mathbf{z}} f = f$ in $C^{r-1}(\mathbb{R}^n)$ uniformly for $f \in \mathfrak{B}$.

All the limits hold uniformly with respect to the parameter $\mathbf{z} \in \mathbb{R}^n$.

Proof. We only show the statement for an (M,r)-regular MRA, the case of an r-regular MRA is analogous. We first give the proof of part (a). In view of the decomposition (5.13) and the condition (5.12), it suffices to show that for each $|\alpha| = |\beta| \leq r$ one has

$$\lim_{\lambda\to\infty} 2^{n\lambda} \int_{\mathbb{R}^n} R^{\alpha,\beta}(2^\lambda \mathbf{x}+\mathbf{z}, 2^\lambda \mathbf{y}+\mathbf{z})[f^{(\alpha)}(\mathbf{y}) - f^{(\alpha)}(\mathbf{x})]d\mathbf{y} = 0. \tag{5.15}$$

Note that if $\mathbf{x}$ remains in a compact subset of $\mathbb{R}^n$, there is a non-increasing function E_α such that $|f^{(\alpha)}(\mathbf{y}) - f^{(\alpha)}(\mathbf{x})| \leq E_\alpha(|\mathbf{x}-\mathbf{y}|)$, where $E_\alpha(t) \to 0$ as $t \to 0^+$ and $E_\alpha(t) = O(e^{M(2kt)})$. Since

$$\begin{aligned}&\lim_{\lambda\to\infty} 2^{n\lambda} \int_{\mathbb{R}^n} |R^{\alpha,\beta}(2^\lambda \mathbf{x}+\mathbf{z}, 2^\lambda \mathbf{y}+\mathbf{z})| E_\alpha(|\mathbf{x}-\mathbf{y}|)d\mathbf{y}\\ &\leq \lim_{\lambda\to\infty} C_{2k+1} \int_{\mathbb{R}^n} e^{-M((2k+1)(\mathbf{x}-\mathbf{y}))} E_\alpha(2^{-\lambda}|\mathbf{x}-\mathbf{y}|)d\mathbf{y} = 0,\end{aligned}$$

we obtain (5.15). For part (b), it is enough to observe that, as the the mean value theorem shows, the functions E_α from above can be taken to be the same for all $f \in \mathfrak{B}$ and $|\alpha| \leq r-1$. □

We then have,

Theorem 5.2.2. *Suppose that the MRA is (M,r)-regular (resp. r-regular). Let $\varphi \in \mathcal{K}_{M,r}(\mathbb{R}^n)$ and $f \in \mathcal{K}'_{M,r}(\mathbb{R}^n)$ (resp. $\varphi \in \mathcal{S}_r(\mathbb{R}^n)$ and $f \in \mathcal{S}'_r(\mathbb{R}^n)$). Then,*

$$\lim_{\lambda\to\infty} q_{\lambda,\mathbf{z}}\varphi = \varphi \quad \text{in } \mathcal{K}_{M,r}(\mathbb{R}^n) \ (\text{resp. in } \mathcal{S}_r(\mathbb{R}^n)) \tag{5.16}$$

and

$$\lim_{\lambda\to\infty} q_{\lambda,\mathbf{z}} f = f \quad \text{weakly}^* \text{ in } \mathcal{K}'_{M,r}(\mathbb{R}^n) \ (\text{resp. in } \mathcal{S}'_r(\mathbb{R}^n)). \tag{5.17}$$

Furthermore, if $f \in \mathcal{K}'_{M,r-1}(\mathbb{R}^n)$ (resp. $f \in \mathcal{S}'_{r-1}(\mathbb{R}^n)$), then the limit (5.17) holds strongly in $\mathcal{K}'_{M,r}(\mathbb{R}^n)$ (resp. in $\mathcal{S}'_r(\mathbb{R}^n)$). All the limits hold uniformly in the parameter $\mathbf{z} \in \mathbb{R}^n$.

Proof. By Lemma 1.2.1 and part (a) from Proposition 5.2.1, the limit (5.16) would follow once we establish the following claim:

Claim 5.2.1. *Let $\mathfrak{B} \subset \mathcal{K}_{M,r}(\mathbb{R}^n)$ be a bounded set. Then the set*

$$\{q_{\lambda,\mathbf{z}}\varphi : \ \varphi \in \mathfrak{B}, \ \lambda \geq 1, \mathbf{z} \in \mathbb{R}^n\}$$

is bounded in $\mathcal{K}_{M,r}(\mathbb{R}^n)$ (resp. in $\mathcal{S}_r(\mathbb{R}^n)$).

Let us show Claim 5.2.1 for $\mathcal{K}_{M,r}(\mathbb{R}^n)$. Using (5.13), (5.14), and the assumptions (1) and (2) on M, we have

$$\begin{aligned}v_{r,l}(q_{\lambda,\mathbf{z}}\varphi) &= \sup_{|\alpha|\leq r,\, \mathbf{x}\in\mathbb{R}^n} e^{M(l\mathbf{x})} 2^{n\lambda} \left| \int_{\mathbb{R}^n} \phi(\mathbf{y}) q_0^{(\alpha)}(2^\alpha \mathbf{x}+\mathbf{z}, 2^\alpha \mathbf{y}+\mathbf{z})d\mathbf{y}\right|\\ &\leq A_l v_{r,2l}(\varphi) \sup_{|\alpha|\leq r,\, \mathbf{x}\in\mathbb{R}^n} 2^{n\lambda} \left| \int_{\mathbb{R}^n} q_0^{(\alpha)}(2^\alpha \mathbf{x}+\mathbf{z}, 2^\alpha \mathbf{y}+\mathbf{z}) e^{M(l\mathbf{x})-M(2l\mathbf{y})}d\mathbf{y}\right|\end{aligned}$$

$$\leq A_l v_{r,2l}(\varphi) \sup_{\mathbf{x}\in\mathbb{R}^n} 2^{n\lambda} \int_{\mathbb{R}^n} e^{-M(2^{\lambda+1}(l+1)(\mathbf{x}-\mathbf{y}))} e^{M(l\mathbf{x})-M(2l\mathbf{y})} d\mathbf{y}$$

$$\leq A_l v_{r,2l}(\varphi) \sup_{\mathbf{x}\in\mathbb{R}^n} 2^{n\lambda} \int_{\mathbb{R}^n} e^{-M(2^{\lambda+1}(l+1)(\mathbf{x}-\mathbf{y}))} e^{M(2l(\mathbf{x}-\mathbf{y}))} d\mathbf{y}$$

$$\leq \frac{A_l}{2^n} v_{r,2l}(\varphi) \int_{\mathbb{R}^n} e^{-M((l+1)\mathbf{y})} e^{M(l\mathbf{y})} d\mathbf{y} \leq \frac{A_l}{2^n} v_{r,2l}(\varphi) \int_{\mathbb{R}^n} e^{-M(\mathbf{y})} d\mathbf{y}.$$

For $\mathcal{S}_r(\mathbb{R}^n)$ we make use of (5.11),

$$\rho_{r,l}(q_{\lambda,\mathbf{z}}\varphi) \leq \tilde{A}_l \rho_{r,l}(\varphi) \sup_{\mathbf{x}\in\mathbb{R}^n} 2^{n\lambda} \int_{\mathbb{R}^n} (1+2^{\lambda}|\mathbf{x}-\mathbf{y}|)^{-l-n-1}(1+|\mathbf{x}-\mathbf{y}|)^l d\mathbf{y}$$

$$\leq \frac{A_l}{2^n} \rho_{r,l}(\varphi) \int_{\mathbb{R}^n} \frac{d\mathbf{y}}{(1+|\mathbf{y}|)^{n+1}} .$$

The limit (5.17) is an immediate consequence of (5.16) and the relation $\langle q_{\lambda,\mathbf{z}}f,\varphi\rangle = \langle f, q_{\lambda,\mathbf{z}}\varphi\rangle$. Assume now that $f \in \mathcal{K}'_{M,r-1}(\mathbb{R}^n)$ (resp. $f \in \mathcal{S}'_{r-1}(\mathbb{R}^n)$) and let $\mathfrak{B}$ be a bounded set in $\mathcal{K}_{M,r}(\mathbb{R}^n)$ (resp. in $\mathcal{S}_r(\mathbb{R}^n)$). From Claim 5.2.1, part (*b*) from Proposition 5.2.1, and again Lemma 1.2.1, we get that $\lim_{\lambda\to\infty} q_{\lambda,\mathbf{z}}\varphi = \varphi$ in $\mathcal{K}_{M,r-1}(\mathbb{R}^n)$ (resp. in $\mathcal{S}_{r-1}(\mathbb{R}^n)$) uniformly for $\varphi \in \mathfrak{B}$ and $\mathbf{z} \in \mathbb{R}^n$. Hence,

$$\lim_{\lambda\to\infty} \sup_{\varphi\in\mathfrak{B}} |\langle q_{\lambda,\mathbf{z}}f - f, \varphi\rangle| = \lim_{\lambda\to\infty} \sup_{\varphi\in\mathfrak{B}} |\langle f, q_{\lambda,\mathbf{z}}\varphi - \varphi\rangle| = 0.$$

□

For the spaces $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$, we have:

Corollary 5.2.1. *Suppose that the MRA admits a scaling function $\phi \in \mathcal{S}(\mathbb{R}^n)$. Then, $\lim_{\lambda\to\infty} q_{\lambda,\mathbf{z}}\varphi = \varphi$ in $\mathcal{S}(\mathbb{R}^n)$ and $\lim_{\lambda\to\infty} q_{\lambda,\mathbf{z}}f = f$ in $\mathcal{S}'(\mathbb{R}^n)$ uniformly in $\mathbf{z} \in \mathbb{R}^n$, for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$.*

Remark 5.2.1. The proof of Theorem 5.2.2 also applies to show that $\lim_{\lambda\to\infty} q_{\lambda,\mathbf{z}}\varphi = \varphi$ in the Banach space $\mathcal{K}_{M,r,l}(\mathbb{R}^n)$ (resp. in $\mathcal{S}_{r,l}(\mathbb{R}^n)$) for each $\varphi \in \mathcal{K}_{M,r,2(l+1)}(\mathbb{R}^n)$ (resp. $\mathcal{S}_{r,l+1}(\mathbb{R}^n)$).

5.3 Pointwise convergence of multiresolution expansions

Walter was the first to study the pointwise convergence of multiresolution expansions for tempered distributions. Under mild conditions, he proved [114] (cf. [117]) in dimension 1 that the multiresolution expansion of a tempered distribution is convergent at every point where $f \in \mathcal{S}'(\mathbb{R})$ possesses a distributional point value. The notion of distributional point value for generalized functions was introduced by Łojasiewicz [50, 49]. Not only is this concept applicable to distributions that might not even be locally integrable, but also includes the Lebesgue points of locally integrable functions as particular instances. Interestingly, the distributional point values of tempered distributions can be characterized by the pointwise

Fourier inversion formula in a very precise fashion [104, 109], but in contrast to multiresolution expansions, one should employ summability methods in the case of Fourier transforms and Fourier series. The problem of pointwise summability of distribution expansions with respect to various orthogonal systems has been considered by Walter in [115].

We shall use the notion of distributional point value of generalized functions introduced by Łojasiewicz [50, 49]. Let $f \in \mathcal{D}'(\mathbb{R}^n)$ and let $\mathbf{x}_0 \in \mathbb{R}^n$. Recall (cf. Section 2.2.1), we say that f has the distributional point value γ at the point $\mathbf{x}_0$, and we write

$$f(\mathbf{x}_0) = \gamma \quad \text{distributionally,} \tag{5.18}$$

if

$$\lim_{\varepsilon \to 0} f(\mathbf{x}_0 + \varepsilon \mathbf{x}) = \gamma \quad \text{in the space } \mathcal{D}'(\mathbb{R}^n), \tag{5.19}$$

that is, if

$$\lim_{\varepsilon \to 0} \langle f(\mathbf{x}_0 + \varepsilon \mathbf{x}), \varphi(\mathbf{x}) \rangle = \gamma \int_{\mathbb{R}^n} \varphi(\mathbf{x}) d\mathbf{x}, \tag{5.20}$$

for all test functions $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Naturally, the evaluation in (5.20) is with respect to the variable $\mathbf{x}$. Due to the Banach-Steinhaus theorem, it is evident that there exists $r \in \mathbb{N}$ such that (5.20) holds uniformly for φ in bounded subsets of $\mathcal{D}^r(\mathbb{R}^n)$. In such a case, we shall say[1] that the distributional point value is of *order* $\leq r$. Here, $B(\mathbf{x}_0, A)$ stands for the Euclidean ball with center $\mathbf{x}_0$ and radius $A > 0$ and $|\mu|$ stands for the total variation measure associated to a measure μ. One can show [49, Sect. 8.3] that (5.18) holds and the distributional point value is of order $\leq r$ if and only if there is a neighborhood of $\mathbf{x}_0$ where f can be written as

$$f = \gamma + \sum_{|\alpha| \leq r} \mu_\alpha^{(\alpha)}, \tag{5.21}$$

where each μ_α is a (complex) Radon measure such that

$$|\mu_\alpha|(B(\mathbf{x}_0, \varepsilon)) = o(\varepsilon^{n+|\alpha|}) \quad \text{as } \varepsilon \to 0^+. \tag{5.22}$$

Note that (5.22) implies that each μ_α is a continuous measure at $\mathbf{x}_0$ in the sense that $\mu_\alpha(\{\mathbf{x}_0\}) = 0$. The decomposition (5.21) and the conditions (5.22) yield [49, Sect. 4] the existence of a multi-index $\beta \in \mathbb{N}^n$, with $|\beta| \leq r + n$, and a β primitive of f, say, F with $F^{(\beta)} = f$, that is a continuous function in a neighborhood of the point $\mathbf{x} = \mathbf{x}_0$ and that satisfies

$$F(\mathbf{x}) = \frac{\gamma(\mathbf{x} - \mathbf{x}_0)^\beta}{\beta!} + o(|\mathbf{x} - \mathbf{x}_0|^{|\beta|}) \quad \text{as } \mathbf{x} \to \mathbf{x}_0. \tag{5.23}$$

On the other hand, the existence of a β primitive F of f satisfying (5.23) clearly suffices to conclude (5.18) of order $\leq |\beta|$. Before going any further, we would like to discuss the connection between distributional point values and pointwise notions for measures.

[1] This definition of the order of a distributional point value is due to Łojasiewicz [49, Sect. 8]. It is more general than those used in [20, 97, 109, 114], which are rather based on (5.23).

Let us define the notion of Lebesgue density points. We denote by m the Lebesgue measure on $\mathbb{R}^n$ and $B(x_0,\varepsilon)$ stands for the Euclidean ball with center $\mathbf{x}_0 \in \mathbb{R}^n$ and radius $\varepsilon > 0$. A sequence $\{B_\nu\}_{\nu=0}^{\infty}$ of Borel subsets of $\mathbb{R}^n$ is said to *shrink regularly* to a point $\mathbf{x}_0 \in \mathbb{R}^n$ if there is a sequence of radii $\{\varepsilon_\nu\}_{\nu=0}^{\infty}$ such that $\lim_{\nu\to\infty}\varepsilon_\nu = 0$, $B_\nu \subseteq B(\mathbf{x}_0,\varepsilon_\nu)$ for all ν, and there is a constant $a > 0$ such that $m(B_\nu) \geq a\varepsilon_\nu^n$ for all ν. We write $B_\nu \to \mathbf{x}_0$ regularly.

Definition 5.3.1. We call $\mathbf{x}_0$ a *Lebesgue density point* of a (complex) Radon measure μ if there is $\gamma_{\mathbf{x}_0}$ such that

$$\lim_{\nu\to\infty}\frac{\mu(B_\nu)}{m(B_\nu)} = \gamma_{\mathbf{x}_0}, \tag{5.24}$$

for every sequence of Borel sets $\{B_\nu\}_{\nu=0}^{\infty}$ such that $B_\nu \to \mathbf{x}_0$ regularly.

It is well known that almost every point $\mathbf{x}_0$ (with respect to the Lebesgue measure) is a Lebesgue density point of μ. If $d\mu = f dm + d\mu_s$ is the Lebesgue decomposition of μ, namely, $f \in L^1_{loc}(\mathbb{R}^n)$ and μ_s is a singular measure, then $f(\mathbf{x}_0) = \gamma_{\mathbf{x}_0}$ a.e. with respect to m [80, Chap. 7]. If μ is absolutely continuous with respect to the Lebesgue measure, then a density point of μ amounts to the same as a Lebesgue point of its Radon-Nikodym derivative $d\mu/dm$.

Example 5.3.1. If $f \in L^1_{loc}(\mathbb{R}^n)$ has a Lebesgue point at $\mathbf{x}_0$, then it has a distributional point value of order 0 at $\mathbf{x}_0$ and (5.23) holds with $\beta = (1,1,\ldots,1)$. More generally if $f = \mu$ is a (complex) Radon measure, then it has distributional point value of order 0 at a point $\mathbf{x}_0$ if and only if $\mathbf{x}_0$ is a Lebesgue density point of the measure (cf. Definition 5.3.1).

Let us verify this fact. Clearly, upon considering the measure $\mu - \gamma_{\mathbf{x}_0}(= \mu - \gamma_{\mathbf{x}_0}m)$, one may assume that $\gamma_{\mathbf{x}_0} = 0$. By decomposing into real and imaginary parts, we can also assume that μ is real-valued. We should show that

$$|\mu|(B(\mathbf{x}_0,\varepsilon)) = o(\varepsilon^n) \quad \text{as } \varepsilon \to 0^+ \tag{5.25}$$

if and only if

$$\mu(B_\mu) = o(m(B_\mu)) \quad \text{as } \nu \to \infty, \tag{5.26}$$

for every sequence of Borel sets $\{B_\nu\}_{\nu=0}^{\infty}$ such that $B_\nu \to \mathbf{x}_0$. Assume (5.25), if $B_\nu \to \mathbf{x}_0$, there is a sequence of radii $\{\varepsilon_\nu\}_{\nu=0}^{\infty}$ such that $\lim_{\nu\to\infty}\varepsilon_\nu = 0$, $B_\nu \subseteq B(\mathbf{x}_0,\varepsilon_\nu)$ for all ν, and there is a constant $a > 0$ such that $m(B_\nu) \geq a\varepsilon_\nu^n$ for all ν. Then,

$$|\mu(B_\mu)| \leq |\mu|(B_\mu) \leq |\mu|(B(\mathbf{x}_0,\varepsilon_\nu)) = o(\varepsilon_\nu^n) \leq o(a^{-1}m(B_\nu)) = o(m(B_\nu)).$$

Conversely, assume that (5.26) holds for every $B_\nu \to \mathbf{x}_0$ regularly. For (5.25), it is enough to show that if a sequence $\varepsilon_\nu \to 0^+$, there is a subsequence such that $|\mu|(B(\mathbf{x}_0,\varepsilon_{\nu_k})) = o(\varepsilon_{\nu_k}^n)$. Write $\mu = \mu_+ - \mu_-$ in Hahn-Jordan decomposition form [80] so that $|\mu| = \mu_+ + \mu_-$. Find disjoint sets S^+ and S^- such that $\mu_\pm$ are respectively concentrated at $S^\pm$ and $\mathbb{R}^n = S^- \cup S^+$. Set $B_\nu^\pm = S^\pm \cap B(\mathbf{x}_0,\varepsilon_\nu)$. There are indices $\{\nu_k\}_{k=0}^{\infty}$ such that at least one of the subsequences $\{B_{\nu_k}^-\}_{k=0}^{\infty}$ or

$\{B^+_{\nu_k}\}_{k=0}^\infty$ shrinks regularly to $\mathbf{x}_0$. Say $B^+_{\nu_k} \to \mathbf{x}_0$ regularly, then $\mu_+(B(\mathbf{x}_0, \varepsilon_{\nu_k})) = \mu(B^+_{\nu_k}) = o(m(B^+_{\nu_k})) = o(\varepsilon^n_{\nu_k})$. But we also have $\mu(B(\mathbf{x}_0, \varepsilon_{\nu_k})) = o(B(\mathbf{x}_0, \varepsilon_{\nu_k})) = o(\varepsilon^n_{\nu_k})$ by hypothesis. Therefore, $|\mu|(B(\mathbf{x}_0, \varepsilon_{\nu_k})) = o(\varepsilon^n_{\nu_k})$, since $|\mu| = \mu_+ + \mu_- = \mu_+ + \mu_+ - \mu$.

Example 5.3.2. The notion of distributional point values applies to distributions that are not necessarily locally integrable nor measures, but if $f \in L^1_{loc}(\mathbb{R}^n)$, then (5.23) reads as

$$\int_{x_{0,1}}^{x_1}\int_{x_{0,2}}^{x_2}\cdots\int_{x_{0,n}}^{x_n} f(\mathbf{y})(\mathbf{x}-\mathbf{y})^{\beta-\mathbf{1}}d\mathbf{y} = \frac{\gamma(\mathbf{x}-\mathbf{x}_0)^\beta}{\beta^{\mathbf{1}}} + o(|\mathbf{x}-\mathbf{x}_0|^{|\beta|}) \quad \text{as } \mathbf{x}\to\mathbf{x}_0, \tag{5.27}$$

where $\mathbf{x} = (x_1, x_2, \dots, x_n)$, $\mathbf{x}_0 = (x_{0,1}, x_{0,2}, \dots, x_{0,n})$ and $\mathbf{1} = (1, 1, \dots, 1)$. Observe that (5.27) also makes sense for a measure μ, one simply has to replace $f(\mathbf{y})d\mathbf{y}$ by $d\mu(\mathbf{y})$. In particular, $\mu(\mathbf{x}_0) = \gamma$ distributionally at every *density point* of μ, namely, at points where we merely assume that

$$\lim_{\nu\to\infty} \frac{\mu(I_\nu)}{\operatorname{vol}(I_\nu)} = \gamma, \tag{5.28}$$

for every sequence of hyperrectangles $\{I_\nu\}_\nu^\infty$ such that $\mathbf{x}_0 \in I_\nu$ for all $\nu \in \mathbb{N}$ and $I_\nu \to \mathbf{x}_0$ regularly. In the latter case, the distributional point value of μ will not be, in general, of order 0 but of order $\leq n$ and (5.23) holds with $\beta = (2, 2, \dots, 2)$. Notice that (5.28) for balls instead of hyperrectangles does not guarantee the existence of the distributional point value at $\mathbf{x}_0$; in one variable, an simple example is provided by the absolutely continuous measure with density $d\mu(x) = \operatorname{sgn} x dx$ at the point $x_0 = 0$. Naturally, the distributional point value of μ exists under much weaker assumptions than having a density point in the sense explained here, but if the measure μ is positive, then the notion of distributional point values coincides with that of density points, as shown by Łojasiewicz in [49, Sect. 4.6].

Example 5.3.3. Let $a \in \mathbb{C}$ and $b > 0$. One can show that the function $|\mathbf{x}|^a \sin(1/|\mathbf{x}|^b)$ has a regularization $f_{a,b} \in \mathcal{S}'(\mathbb{R}^n)$ that satisfies $f_{a,b}(\mathbf{x}) = |\mathbf{x}|^a \sin(1/|\mathbf{x}|^b)$ for $\mathbf{x} \neq \mathbf{0}$ and $f_a(\mathbf{0}) = 0$ distributionally [50]. Observe that if $\Re e\ a < 0$ the function $|\mathbf{x}|^a \sin(1/|\mathbf{x}|^b)$ is unbounded and if $\Re e\ a \leq -n$ it is not even Lebesgue integrable near $\mathbf{x} = \mathbf{0}$. If $\Re e\ a < -n$ is fixed and $b > 0$ is small, the order of the point value of $f_{a,b}$ at $\mathbf{x} = \mathbf{0}$ can be very large.

Example 5.3.4. In one variable, it is possible to characterize the distributional point values of a periodic distribution in terms of a certain summability of its Fourier series [16]. Indeed, let $f(x) = \sum_{\nu=-\infty}^\infty c_\nu e^{i\nu x} \in \mathcal{S}'(\mathbb{R})$; then $f(x_0) = \gamma$ distributionally, if and only if there exists $\kappa \geq 0$ such that

$$\lim_{x\to\infty} \sum_{-x<\nu\leq ax} c_\nu e^{i\nu x_0} = \gamma \quad (\mathrm{C}, \kappa)\ , \quad \text{for each } a > 0\ ,$$

where (C, κ) stands for Cesàro summability. Remarkably, an analog result is true for Fourier transforms in one variable [104, 109], but no such characterizations are known in the multidimensional case.

The result of Walter on pointwise convergence of multiresolution expansions was generalized by Sohn and Pahk [97] to distributions of superexponential growth, that is, elements of $\mathcal{K}'_M(\mathbb{R})$. The important case $M(\mathbf{x}) = |\mathbf{x}|^p$, $p > 1$, of $\mathcal{K}'_M(\mathbb{R}^n)$ was introduced by Sznajder and Zieleźny in connection with solvability questions for convolution equations [99].

Our goal here is to extend the results from [114, 97] to the multidimensional case. In particular, we shall show the following result. Given an MRA $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R}^n)$, we denote by q_j the orthogonal projection onto V_j. If the MRA admits a scaling function from $\mathcal{S}(\mathbb{R}^n)$, then $q_j f$ makes sense for $f \in \mathcal{S}'(\mathbb{R}^n)$.

Theorem 5.3.1. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Suppose that the MRA $\{V_j\}_{j\in\mathbb{Z}}$ admits a scaling function in $\mathcal{S}(\mathbb{R}^n)$, then*

$$\lim_{j\to\infty} (q_j f)(\mathbf{x}_0) = f(\mathbf{x}_0)$$

at every point $\mathbf{x}_0$ where the distributional point value of f exists.

Our approach differs from that of Walter and Sohn and Pahk. The distributional point values are defined by distributional limits, involving certain local averages with respect to test functions from the Schwartz class of compactly supported smooth functions. We will show a general result that allows us to employ test functions in wider classes for such averages (Theorem 5.3.3). This will lead to quick proofs of various pointwise convergence results for multiresolution expansions of distributions. Actually, our results improve those from [114, 97], even in the one-dimensional case, because our hypotheses on the order of distributional point values are much weaker. For instance, the next theorem on convergence of multiresolution expansions to Lebesgue density points of measures appears to be new and is not covered by the results from [39, 114].

Theorem 5.3.2. *Suppose that the MRA $\{V_j\}_{j\in\mathbb{Z}}$ has a continuous scaling function ϕ such that $\lim_{|\mathbf{x}|\to\infty} |\mathbf{x}|^l \phi(\mathbf{x}) = 0$, $\forall l \in \mathbb{N}$. Let μ be a tempered Radon measure on $\mathbb{R}^n$, that is, one that satisfies*

$$\int_{\mathbb{R}^n} \frac{d|\mu|(\mathbf{x})}{(1+|\mathbf{x}|)^k} < \infty \tag{5.29}$$

for some $k \geq 0$. Then

$$\lim_{j\to\infty} (q_j \mu)(\mathbf{x}_0) = \gamma_{\mathbf{x}_0} \tag{5.30}$$

at every Lebesgue density point $\mathbf{x}_0$ of μ, i.e., at every point where (5.24) holds for every $B_\nu \to \mathbf{x}_0$ regularly. In particular, the limit (5.30) exists and $\gamma_{\mathbf{x}_0} = f(\mathbf{x}_0)$ almost everywhere (with respect to the Lebesgue measure), where $d\mu = f dm + d\mu_s$ is the Lebesgue decomposition of μ.

It is worth comparing Theorem 5.3.1 with Theorem 5.3.2. On the one hand Theorem 5.3.1 requires more regularity from the MRA, but on the other hand, when applied to a tempered measure, it gives in turn a bigger set for the pointwise convergence (5.30) of the multiresolution expansion of μ, because the set where μ possesses distributional point values is larger than that of its Lebesgue density points.

In order to study pointwise convergence of multiresolution expansions, we will first establish two results about distributional point values of tempered distributions and distributions of M-exponential growth. A priori, $f(\mathbf{x}_0) = \gamma$ distributionally gives us only the right to consider test functions from $\mathcal{D}(\mathbb{R}^n)$ in (5.20); however, it has been shown in [108] that if $f \in \mathcal{S}'(\mathbb{R}^n)$ then the limit (5.19) holds in the space $\mathcal{S}'(\mathbb{R}^n)$, namely, (5.20) remains valid for $\varphi \in \mathcal{S}(\mathbb{R}^n)$ (see also [71, 106, 119]). Theorem 5.3.3 below goes in this direction, it gives conditions under which the functions φ in (5.20) can be taken from larger spaces than $\mathcal{D}(\mathbb{R}^n)$. The next useful proposition treats the case of distributions that vanish in a neighborhood of the point.

Proposition 5.3.1. *Let $f \in \mathcal{K}'_{M,r}(\mathbb{R}^n)$ (resp. $f \in \mathcal{S}'_r(\mathbb{R}^n)$) be such that $\mathbf{x}_0 \notin \operatorname{supp} f$ and let $\mathfrak{B}$ be a bounded subset of $\mathcal{K}_{M,r}(\mathbb{R}^n)$ (resp. $\mathcal{S}_r(\mathbb{R}^n)$). Then, for any $k \in \mathbb{N}$, there is $C_k > 0$ such that*

$$|\langle f(\mathbf{x}_0 + \varepsilon \mathbf{x}), \varphi(x)\rangle| \leq C_k \varepsilon^k, \quad \forall \varepsilon \in (0,1],\ \forall \varphi \in \mathfrak{B}. \tag{5.31}$$

Proof. There are $A, C > 0$ and $l \in \mathbb{N}$ such that

$$|\langle f, \psi\rangle| \leq C \sup_{|\alpha|\leq r,\ |\mathbf{x}-\mathbf{x}_0|\geq A} e^{M(l\mathbf{x})} |\psi^{(\alpha)}(\mathbf{x})| \tag{5.32}$$

$$\left(\text{resp. } |\langle f, \psi\rangle| \leq C \sup_{|\alpha|\leq r,\ |\mathbf{x}-\mathbf{x}_0|\geq A} (1+|\mathbf{x}|)^l |\psi^{(\alpha)}(\mathbf{x})| \right),$$

for all $\psi \in \mathcal{K}_{M,r}(\mathbb{R}^n)$ (resp. $\psi \in \mathcal{S}_r(\mathbb{R}^n)$). Let us consider first the case of $f \in \mathcal{K}'_{M,r}(\mathbb{R}^n)$. Substituting $\psi(\mathbf{y}) = \varepsilon^{-n}\varphi(\varepsilon^{-1}(\mathbf{x} - \mathbf{x}_0))$ in (5.32), we get

$$\begin{aligned}
|\langle f(\mathbf{x}_0 + \varepsilon \mathbf{x}), \psi(x)\rangle| &\leq C e^{M(2l\mathbf{x}_0)} \varepsilon^{-n-r} \sup_{|\alpha|\leq r,\ |\mathbf{y}|\geq A} e^{M(2l\mathbf{y})} \left|\varphi^{(\alpha)}\left(\frac{\mathbf{y}}{\varepsilon}\right)\right| \\
&\leq C \nu_{r,2l+1}(\varphi) e^{M(2l\mathbf{x}_0)} \varepsilon^{-n-r} \sup_{|\mathbf{y}|\geq A/\varepsilon} e^{M(2l\mathbf{y}) - M((2l+1)\mathbf{y})} \\
&\leq C \nu_{r,2l+1}(\varphi) e^{M(2l\mathbf{x}_0)} \varepsilon^{-n-r} e^{-M(A/\varepsilon)},
\end{aligned}$$

which yields (5.31). The tempered case is similar. In this case (5.32) gives the estimate

$$\begin{aligned}
|\langle f(\mathbf{x}_0 + \varepsilon \mathbf{x}), \varphi(x)\rangle| &\leq C(1+|\mathbf{x}_0|)^l \varepsilon^{-n-r} \sup_{|\alpha|\leq r,\ |\mathbf{y}|\geq A/\varepsilon} (1+|\mathbf{y}|)^l \left|\varphi^{(\alpha)}(\mathbf{y})\right| \\
&\leq C \rho_{r,n+r+k+l}(\varphi)(1+|\mathbf{x}_0|)^l A^{-n-r-k} \varepsilon^k.
\end{aligned}$$

□

Theorem 5.3.3. *Let $f \in \mathcal{K}'_{M,r}(\mathbb{R}^n)$ (resp. $f \in \mathcal{S}'_r(\mathbb{R}^n)$). If $f(\mathbf{x}_0) = \gamma$ distributionally of order $\leq r$, then (5.19) holds strongly in $\mathcal{K}'_{M,r}(\mathbb{R}^n)$ (resp. f in $\mathcal{S}'_r(\mathbb{R}^n)$), that is, the limit (5.20) holds uniformly for φ in bounded subsets of $\mathcal{K}_{M,r}(\mathbb{R}^n)$ (resp. $\mathcal{S}_r(\mathbb{R}^n)$).*

Proof. We can decompose f as $f = f_1 + \gamma\chi_{B(x_0,1)} + \sum_{|\alpha|\leq r} \mu_\alpha^{(\alpha)}$, where $\mathbf{x}_0 \notin \operatorname{supp} f_1$, $\chi_{B(\mathbf{x}_0,1)}$ is the characteristic function of the ball $B(\mathbf{x}_0, 1)$, and each μ_α is a Radon measure with support in the ball $B(\mathbf{x}_0, 1)$ and satisfies (5.22). Proposition 5.3.1 applies to f_1, we may therefore assume that

$$f = \gamma\chi_{B(\mathbf{x}_0,1)} + \sum_{|\alpha|\leq r} \mu_\alpha^{(\alpha)}.$$

Let $\mathfrak{B}$ be a bounded set in $\mathcal{K}_{M,r}(\mathbb{R}^n)$ (resp. $\mathcal{S}_r(\mathbb{R}^n)$). Note that (5.22) implies that

$$\sum_{|\alpha|\leq r} \int_{\mathbb{R}^n} \frac{d|\mu_\alpha|(\mathbf{x})}{|\mathbf{x}-\mathbf{x}_0|^{n+|\alpha|}} = C < \infty. \tag{5.33}$$

We have,

$$\begin{aligned}
&\limsup_{\varepsilon\to0^+} \sup_{\varphi\in\mathfrak{B}} \left| \langle f(\mathbf{x}_0+\varepsilon\mathbf{x}), \varphi(\mathbf{x})\rangle - \gamma \int_{\mathbb{R}^n} \varphi(\mathbf{x})d\mathbf{x} \right| \\
&\leq \limsup_{\varepsilon\to0^+} \sup_{\varphi\in\mathfrak{B}} \left(\int_{|\mathbf{x}|\geq 1/\varepsilon} |\varphi(\mathbf{x})|d\mathbf{x} + \sum_{|\alpha|\leq r} \varepsilon^{-n-|\alpha|} \int_{\mathbb{R}^n} \left| \varphi^{(\alpha)}\left(\frac{\mathbf{x}-\mathbf{x}_0}{\varepsilon}\right)\right| d|\mu_\alpha|(\mathbf{x}) \right) \\
&= \limsup_{\varepsilon\to0^+} \sup_{\varphi\in\mathfrak{B}} \sum_{|\alpha|\leq r} \varepsilon^{-n-|\alpha|} \int_{\mathbb{R}^n} \left| \varphi^{(\alpha)}\left(\frac{\mathbf{x}-\mathbf{x}_0}{\varepsilon}\right)\right| d|\mu_\alpha|(\mathbf{x}),
\end{aligned}$$

The boundedness of $\mathfrak{B}$ implies that there is a positive and continuous function G on $[0,\infty)$ such that $t^{n+r}G(t)$ is decreasing on $(1,\infty)$, $\lim_{t\to\infty} t^{n+r}G(t) = 0$, and $|\varphi^{(\alpha)}(\mathbf{x})| \leq G(|\mathbf{x}|)$ for all $\mathbf{x} \in \mathbb{R}^n$, $|\alpha| \leq r$, and $\varphi \in \mathfrak{B}$. Fix $A > 1$. By (5.22), (5.33), and the previous inequalities,

$$\begin{aligned}
&\limsup_{\varepsilon\to0^+} \sup_{\varphi\in\mathfrak{B}} \left| \langle f(\mathbf{x}_0+\varepsilon\mathbf{x}), \varphi(\mathbf{x})\rangle - \gamma \int_{\mathbb{R}^n} \varphi(\mathbf{x})d\mathbf{x} \right| \\
&\leq \lim_{\varepsilon\to0^+} \sum_{|\alpha|\leq r} \varepsilon^{-n-|\alpha|} \int_{\mathbb{R}^n} G\left(\frac{|\mathbf{x}-\mathbf{x}_0|}{\varepsilon}\right) d|\mu_\alpha|(\mathbf{x}) \\
&\leq \lim_{\varepsilon\to0^+} \sum_{|\alpha|\leq r} ||G||_\infty \frac{|\mu_\alpha|(B(\mathbf{x}_0,\varepsilon A))}{\varepsilon^{n+|\alpha|}} + \lim_{\varepsilon\to0} \sum_{|\alpha|\leq r} \varepsilon^{-n-|\alpha|} \int_{\varepsilon A\leq|\mathbf{x}-\mathbf{x}_0|} G\left(\frac{|\mathbf{x}-\mathbf{x}_0|}{\varepsilon}\right) d|\mu_\alpha|(\mathbf{x}) \\
&\leq CA^{n+r}G(A).
\end{aligned}$$

Since the above estimate is valid for all $A > 1$ and $A^{n+r}G(A) \to 0$ as $A \to \infty$, we obtain

$$\lim_{\varepsilon\to0^+} \sup_{\varphi\in\mathfrak{B}} \left| \langle f(\mathbf{x}_0+\varepsilon\mathbf{x}), \varphi(\mathbf{x})\rangle - \gamma \int_{\mathbb{R}^n} \varphi(\mathbf{x})d\mathbf{x} \right| = 0,$$

as claimed. □

We obtain the ensuing corollary.

Corollary 5.3.1. *Let $f \in \mathcal{K}'_M(\mathbb{R}^n)$ (resp. $f \in \mathcal{S}'(\mathbb{R}^n)$). If $f(\mathbf{x}_0) = \gamma$ distributionally, then the limit (5.19) holds in the space $\mathcal{K}'_M(\mathbb{R}^n)$ (resp. $\mathcal{S}'(\mathbb{R}^n)$).*

Proof. In fact, there is r such that $f \in \mathcal{K}'_{M,r}(\mathbb{R}^n)$ (resp. $f \in \mathcal{S}'_r(\mathbb{R}^n)$) and $f(\mathbf{x}_0) = \gamma$ distributionally of order $\leq r$. □

We end this section with the announced result on pointwise convergence of multiresolution expansions for distributional point values. We give a quick proof based on Theorem 5.3.3.

Theorem 5.3.4. *Let $f \in \mathcal{K}'_{M,r}(\mathbb{R}^n)$ (resp. $f \in \mathcal{S}'_r(\mathbb{R}^n)$). If $\{q_\lambda f\}_{\lambda\in\mathbb{R}}$ is the (generalized) multiresolution expansion of f in an (M,r)-regular (resp. r-regular) MRA, then*

$$\lim_{\lambda\to\infty} (q_\lambda f)(\mathbf{x}_0) = f(\mathbf{x}_0) \tag{5.34}$$

at every point $\mathbf{x}_0$ where the distributional point value of f exists and is of order $\leq r$.

Proof. Assume that $f(\mathbf{x}_0) = \gamma$ distributionally of order r. Note first that

$$(q_\lambda f)(\mathbf{x}_0) = \langle f(\mathbf{y}), q_\lambda(\mathbf{x}_0, \mathbf{y})\rangle = \left\langle f(\mathbf{x}_0 + 2^{-\lambda}\mathbf{y}), \varphi_\lambda(\mathbf{y})\right\rangle, \tag{5.35}$$

where $\varphi_\lambda(\mathbf{y}) = q_0(2^\lambda \mathbf{x}_0, 2^\lambda \mathbf{x}_0 + \mathbf{y})$. The relation (5.9) implies that $\int_{\mathbb{R}^n} \varphi_\lambda(\mathbf{y})d\mathbf{y} = 1$. Using the estimates (5.8), one concludes that $\{\varphi_\lambda : \lambda \geq 0\}$ is a bounded subset of $\mathcal{K}_{M,r}(\mathbb{R}^n)$ (resp. $\mathcal{S}_r(\mathbb{R}^n)$). Finally, invoking Theorem 5.3.3, we get at once

$$\lim_{\lambda\to\infty} (q_\lambda f)(\mathbf{x}_0) = \gamma + \lim_{\lambda\to\infty}\left(\left\langle f(\mathbf{x}_0 + 2^{-\lambda}\mathbf{y}), \varphi_\lambda(\mathbf{y})\right\rangle - \gamma \int_{\mathbb{R}^n} \varphi_\lambda(\mathbf{y})d\mathbf{y}\right) = \gamma.$$

□

Note that Theorems 5.3.1 and 5.3.2 are immediate consequences of Theorem 5.3.4. Moreover, we obtain the following corollary:

Corollary 5.3.2. *Suppose that the MRA $\{V_j\}_{j\in\mathbb{Z}}$ has a continuous scaling function ϕ such that $\lim_{|\mathbf{x}|\to\infty} e^{M(l\mathbf{x})}\phi(\mathbf{x}) = 0$, $\forall l \in \mathbb{N}$. Let μ be a measure on $\mathbb{R}^n$ such that*

$$\int_{\mathbb{R}^n} e^{-M(k\mathbf{x})} d|\mu|(\mathbf{x}) < \infty, \tag{5.36}$$

for some $k \geq 0$. Then

$$\lim_{\lambda\to\infty} (q_\lambda \mu)(\mathbf{x}_0) = \gamma_{\mathbf{x}_0} \tag{5.37}$$

at every Lebesgue density point $\mathbf{x}_0$ of μ, i.e., at every point where (5.24) holds whenever $I_\nu \to \mathbf{x}_0$ regularly. In particular, the limit (5.36) exists and $\gamma_{\mathbf{x}_0} = f(\mathbf{x}_0)$ almost everywhere (with respect to the Lebesgue measure), where $d\mu = f dm + d\mu_s$ is the Lebesgue decomposition of μ.

Let us also remark that if the MRA in Corollary 5.3.2 (resp. in Theorem 5.3.2) is (M,n)-regular (resp. n-regular), then (5.37) holds at every density point of μ (in the sense explained in Example 5.3.2).

5.4 Quasiasymptotic Behavior via multiresolution expansions

In this last Section we will study the quasiasymptotic behavior of a distribution at a point through multiresolution expansions. The quasiasymptotic behavior is a natural extension of Łojasiewicz' notion of distributional point values. It was introduced by Zav'yalov in connection with various problems from quantum field theory [113] and basically measures the pointwise scaling asymptotic properties of a distribution via comparison with Karamata regularly varying functions. We remark that the quasiasymptotic behavior is closely related to Meyer's pointwise weak scaling exponents [55]. For studies about wavelet analysis and quasiasymptotics, we refer to [70, 72, 86, 87, 95, 110].

In [70], Pilipović, Takači, and Teofanov studied the quasiasymptotic properties of a tempered distribution f in terms of its multiresolution expansion $\{q_j f\}$ with respect to an r-regular MRA. A similar study was carried out by Sohn [95] for distributions of exponential type. There, it was wrongly stated in [70, Thrm. 3] that if a tempered distribution $f \in \mathcal{S}'_r(\mathbb{R}^n)$ has quasiasymptotic behavior at the origin, then each of its projections $q_j f$, with respect to an r-regular MRA, has the same quasiasymptotic behavior as f. An analog result was claimed to hold in [95, Thrm. 3.2] for distributions of exponential type (i.e., elements of $\mathcal{K}'_M(\mathbb{R})$ with $M(x) = |x|$). Unfortunately, such results turn out to be false (see Example (5.4.1)). We will provide an appropriate characterization of the quasiasymptotic behavior in terms of multiresolution expansions. As an application, we give an MRA criterion for the determination of (symmetric) α-density points of measures.

Throughout the rest of this section, L always stands for an slowly varying function at the origin and α stands for a real number. Recall Definitions 2.7 and 2.9 for quasiasymptotic behavior (quasiasymptotic) of distributions.

Example 5.4.1. Consider $f = \delta$, the Dirac delta. Since δ is a homogeneous distribution of degree $-n$, the relation (2.9) trivially holds with $\mathbf{x}_0 = 0$, $g = \delta$, $\alpha = -n$, and L identically equal to 1. On the other hand,

$$q_j f(\mathbf{x}) = 2^{jn} \sum_{\mathbf{m} \in \mathbb{Z}^n} \overline{\phi(\mathbf{m})} \phi(2^j \mathbf{x} + \mathbf{m}).$$

So, $q_j f(\mathbf{0}) = 2^{jn} \sum_{\mathbf{m} \in \mathbb{Z}^n} (\widehat{\phi} * \widehat{\bar{\phi}})(2\pi \mathbf{m})$, as follows from the Poisson summation formula. If we assume that $\widehat{\phi}$ is positive and symmetric with respect to the origin, we get that $q_j f(\mathbf{0}) \geq 2^j ||\widehat{\phi}||_2^2 > 0$. So

$$(q_j f)(\varepsilon \mathbf{x}) = (q_j f)(\mathbf{0}) + o(1) \quad \text{as} \quad \varepsilon \to 0^+ \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n).$$

In particular, $q_j f$ cannot have the same quasiasymptotic behavior $\varepsilon^{-n}\delta(\mathbf{x}) + o(\varepsilon^{-n})$ as f, contrary to what was claimed in [70, 95].

The fact that each $q_j f$ is a continuous function prevents it to have quasiasymptotics of arbitrary degree. For instance, as the previous example, if $(q_j f)(\mathbf{0}) \neq 0$, the only quasiasymptotics at $\mathbf{0}$ that $q_j f$ could have is a distributional point value.

Moreover, if the MRA admits a scaling function from $\mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$, then each $q_j f \in C^\infty(\mathbb{R}^n)$; consequently, the only quasiasymptotics that $q_j f$ can have is of order $\alpha = k \in \mathbb{N}$ with respect to the constant slowly varying function $L = 1$, and the g in this case must be a homogeneous polynomial of degree k. Nevertheless, as shown below, the quasiasymptotics of distributions can still be studied via multiresolution expansions if one takes a different approach from that followed in [70, 95]. The next theorem is a version of Theorem 5.3.3 for quasiasymptotics.

Theorem 5.4.1. *Let $f \in \mathcal{K}'_M(\mathbb{R}^n)$ (resp. $f \in \mathcal{S}'(\mathbb{R}^n)$). If f has the quasiasymptotic behavior (2.9), then there is $r \in \mathbb{N}$ such that $f \in \mathcal{K}'_{M,r}(\mathbb{R}^n)$ (resp. $f \in \mathcal{S}'_r(\mathbb{R}^n)$) and*

$$\lim_{\varepsilon \to 0^+} \frac{f(\mathbf{x}_0 + \varepsilon \mathbf{x})}{\varepsilon^\alpha L(\varepsilon)} = g(\mathbf{x}) \quad \textit{strongly in } \mathcal{K}'_{M,r}(\mathbb{R}^n) \ (\textit{resp. } \mathcal{S}'_r(\mathbb{R}^n)). \tag{5.38}$$

In particular, the limit (2.7) is also valid for all $\varphi \in \mathcal{K}_M(\mathbb{R}^n)$ (resp. $\varphi \in \mathcal{S}(\mathbb{R}^n)$).

Proof. We actually show first the last assertion, i.e., that (**??**) is valid for all test functions from $\mathcal{K}_M(\mathbb{R}^n)$ (resp. $\in \mathcal{S}(\mathbb{R}^n)$). So, let $\varphi \in \mathcal{K}_M(\mathbb{R}^n)$ (resp. $\varphi \in \mathcal{S}(\mathbb{R}^n)$). Decompose $f = f_1 + f_2$ where $\mathbf{x}_0 \notin \operatorname{supp} f_1$ and f_2 has compact support. Clearly, f_2 has the same quasisymptotics at $\mathbf{x}_0$ as f. Furthermore, a theorem of Zav'yalov [119] (see also [71, Thrm. Cor. 7.3]) $\langle f_2(\mathbf{x}_0 + \varepsilon\mathbf{x}), \varphi(\mathbf{x})\rangle \sim \varepsilon^\alpha L(\varepsilon)\langle g(\mathbf{x}), \varphi(\mathbf{x})\rangle$ as $\varepsilon \to 0^+$. By the well-known properties of slowly varying functions [93], we have that $\varepsilon = o(L(\varepsilon))$ as $\varepsilon \to 0^+$ (indeed, $\varepsilon^\sigma = o(L(\varepsilon))$, for all $\sigma > 0$ citeSeneta). Take a positive integer $k > \alpha + 1$, then $\varepsilon^k = o(\varepsilon^\alpha L(\varepsilon))$ as $\varepsilon \to 0^+$. Applying Proposition 5.3.1,

$$\begin{aligned}
\langle f(\mathbf{x}_0 + \varepsilon\mathbf{x}), \varphi(\mathbf{x})\rangle &= \varepsilon^\alpha L(\varepsilon)\langle g(\mathbf{x}), \varphi(\mathbf{x})\rangle + o(\varepsilon^\alpha L(\varepsilon)) + \langle f_1(\mathbf{x}_0 + \varepsilon\mathbf{x}), \varphi(\mathbf{x})\rangle \\
&= \varepsilon^\alpha L(\varepsilon)\langle g(\mathbf{x}), \varphi(\mathbf{x})\rangle + o(\varepsilon^\alpha L(\varepsilon)) + O(\varepsilon^k) \\
&= \varepsilon^\alpha L(\varepsilon)\langle g(\mathbf{x}), \varphi(\mathbf{x})\rangle + o(\varepsilon^\alpha L(\varepsilon)) \quad \text{as } \varepsilon \to 0^+,
\end{aligned}$$

as asserted. Because of the Montel property of $\mathcal{K}_M(\mathbb{R}^n)$ (resp. $\mathcal{S}(\mathbb{R}^n)$),

$$\lim_{\varepsilon \to 0^+} \frac{f(\mathbf{x}_0 + \varepsilon \mathbf{x})}{\varepsilon^\alpha L(\varepsilon)} = g(\mathbf{x}) \quad \text{strongly in } \mathcal{K}'_M(\mathbb{R}^n) \ (\text{resp. } \mathcal{S}'(\mathbb{R}^n)). \tag{5.39}$$

The existence of r fulfilling (5.38) is a consequence of (5.39) and the representation (1.13) (resp. (1.14)) of $\mathcal{K}'_M(\mathbb{R}^n)$ (resp. $\mathcal{S}'(\mathbb{R}^n)$) as a regular inductive limit. □

We also have a version of Theorem 5.3.4 for quasiasymptotics.

Theorem 5.4.2. *Let $f \in \mathcal{K}'_{M,r}(\mathbb{R}^n)$ (resp. $f \in \mathcal{S}'_r(\mathbb{R}^n)$) satisfy (5.38). If $\{q_\lambda f\}_{\lambda \in \mathbb{R}}$ is the multiresolution expansion of f in an (M, r)-regular (resp. r-regular) MRA, then $\{(q_\lambda f)(\mathbf{x}_0)\}_\lambda$ has asymptotic behavior*

$$(q_\lambda f)(\mathbf{x}_0) = L(2^{-\lambda})(q_\lambda g_{\mathbf{x}_0})(\mathbf{x}_0) + o(2^{-\alpha\lambda} L(2^{-\lambda})) \quad \textit{as } \lambda \to \infty, \tag{5.40}$$

where $g_{\mathbf{x}_0}(\mathbf{y}) = g(\mathbf{y} - \mathbf{x}_0)$.

Proof. The proof is similar to that of Theorem 5.3.4. By (5.35), (5.38), the homogeneity of g, and the fact that the net $\{\varphi_\lambda\}_{\lambda\in\mathbb{R}}$ is bounded, we get

$$\begin{aligned}(q_\lambda f)(\mathbf{x}_0)\langle f(\mathbf{x}_0+2^{-\lambda}\mathbf{y}),\varphi_\lambda(\mathbf{y})\rangle\\ &= 2^{-\alpha\lambda}L(2^{-\lambda})\langle g,\varphi_\lambda\rangle + o(2^{-\alpha\lambda}L(2^{-\lambda}))\\ &= L(2^{-\lambda})(q_\lambda g_{\mathbf{x}_0})(\mathbf{x}_0) + o(2^{-\alpha\lambda}L(2^{-\lambda})) \quad \text{as } \lambda\to\infty.\end{aligned}$$

□

Corollary 5.4.1. *Let $f\in\mathcal{S}'(\mathbb{R}^n)$. Suppose that the MRA admits a scaling function $\phi\in\mathcal{S}(\mathbb{R}^n)$. Then (5.40) holds at every point where (2.9) is satisfied.*

Note that if $\alpha = k\in\mathbb{N}$, $k\le r$, and $g=P$ is a homogeneous polynomial of degree k, then (5.40) becomes $(q_\lambda f)(\mathbf{x}_0)\sim 2^{-k\lambda}L(2^{-\lambda})P(0)$ as $\lambda\to\infty$, as follows from (5.9); so that one recovers Theorem 5.3.4 if $k=0$. On the other hand, if $k>0$, we only get in this case the growth order relation $(q_\lambda f)(\mathbf{x}_0) = o(2^{-k\lambda}L(2^{-\lambda}))$ as $\lambda\to\infty$.

It was claimed in [95] and [70] that the quasiasymptotic properties of f at $\mathbf{x}_0=\mathbf{0}$ can be obtained from those of $\{q_j f\}_{j\in\mathbb{Z}}$. The theorems [**?**, Thrm. 4] and [95, Thrm. 3.2] also turn out to be false. The next theorem provides a characterization of quasiasymptotics in terms of slightly different asymptotic conditions on $\{q_\lambda f\}_{\lambda\in\mathbb{R}}$, which amend those from [70, Thrm. 4].

Theorem 5.4.3. *Suppose that the MRA is (M,r)-regular (resp. r-regular). Then, a distribution $f\in\mathcal{K}'_{M,r}(\mathbb{R}^n)$ (resp. $f\in\mathcal{S}'_r(\mathbb{R}^n)$) satisfies*

$$\lim_{\varepsilon\to0^+}\frac{f(\mathbf{x}_0+\varepsilon\mathbf{x})}{\varepsilon^\alpha L(\varepsilon)} = g(\mathbf{x}) \quad \textit{weakly}^* \textit{ in } \mathcal{K}'_{M,r}(\mathbb{R}^n)\ (\textit{resp. } \mathcal{S}'_r(\mathbb{R}^n)). \tag{5.41}$$

if and only if

$$\lim_{\varepsilon\to0^+}\frac{(q_{\frac{1}{\varepsilon}} f)(\mathbf{x}_0+\varepsilon\mathbf{x})}{\varepsilon^\alpha L(\varepsilon)} = g(\mathbf{x}) \quad \textit{weakly}^* \textit{ in } \mathcal{K}'_{M,r}(\mathbb{R}^n)\ (\textit{resp. } \mathcal{S}'_r(\mathbb{R}^n)) \tag{5.42}$$

and

$$f(\mathbf{x}_0+\varepsilon\mathbf{x}) = O(\varepsilon^\alpha L(\varepsilon)) \quad \textit{as } \varepsilon\to0^+ \ \textit{ in } \mathcal{K}'_{M,r}(\mathbb{R}^n)\ (\textit{resp. } \mathcal{S}'_r(\mathbb{R}^n)), \tag{5.43}$$

in the sense that $\langle f(\mathbf{x}_0+\varepsilon\mathbf{x}),\varphi(\mathbf{x})\rangle = O(\varepsilon^\alpha L(\varepsilon))$ for all $\varphi\in\mathcal{K}_{M,r}(\mathbb{R}^n)$ (resp. $\varphi\in\mathcal{S}_r(\mathbb{R}^n)$).

Remark 5.4.1. The relation (5.41) holds strongly in $\mathcal{K}'_{M,r+1}(\mathbb{R}^n)$ (resp. $\mathcal{S}'_{r+1}(\mathbb{R}^n)$).

Proof. Observe that (5.41) trivially implies (5.43). Our problem is then to show the equivalence between (5.41) and (5.42) under the assumption (5.43). Define the kernel

$$J_\varepsilon(\mathbf{x},\mathbf{y}) = q_{2^{1/\varepsilon}\varepsilon,2^{1/\varepsilon}\mathbf{x}_0}(\mathbf{y},\mathbf{x}) = \varepsilon^n 2^{n/\varepsilon} q_0(\varepsilon 2^{1/\varepsilon}\mathbf{y}+2^{1/\varepsilon}\mathbf{x}_0, \varepsilon 2^{1/\varepsilon}\mathbf{x}+2^{1/\varepsilon}\mathbf{x}_0), \quad \mathbf{x},\mathbf{y}\in\mathbb{R}^n,$$

and the operator

$$(J_\varepsilon\varphi)(\mathbf{x}) = \int_{\mathbb{R}^n} J_\varepsilon(\mathbf{x},\mathbf{y})\varphi(\mathbf{y})d\mathbf{y}, \quad \varphi \in \mathcal{K}_{M,r}(\mathbb{R}^n) \text{ (resp. } \varphi \in \mathcal{S}_r(\mathbb{R}^n)).$$

Theorem 5.2.2 implies that J_ε is an approximation of the identity in these spaces, i.e., for every test function $\lim_{\varepsilon\to 0^+} J_\varepsilon\varphi = \varphi$ in $\mathcal{K}_{M,r}(\mathbb{R}^n)$ (resp. in $\mathcal{S}_r(\mathbb{R}^n)$). The Banach-Steinhaus theorem implies that

$$\left\{\frac{f(\mathbf{x}_0+\varepsilon\mathbf{x})}{\varepsilon^\alpha L(\varepsilon)} : \varepsilon \in (0,1)\right\}$$

is an equicontinuous family of linear functionals, hence

$$\lim_{\varepsilon\to 0^+}\left\langle \frac{f(\mathbf{x}_0+\varepsilon\mathbf{y})}{\varepsilon^\alpha L(\varepsilon)}, (J_\varepsilon\varphi)(\mathbf{y}) - \varphi(\mathbf{y})\right\rangle = 0,$$

for each test function φ. Notice that

$$\begin{aligned}\langle (q_{1/\varepsilon}f)(\mathbf{x}_0+\varepsilon\mathbf{x}), \varphi(\mathbf{x})\rangle &= \langle\langle f(\mathbf{y}), q_{1/\varepsilon}(\mathbf{x}_0+\varepsilon\mathbf{x},\mathbf{y})\rangle, \varphi(\mathbf{x})\rangle \\ &= \langle f(\mathbf{x}_0+\varepsilon\mathbf{y}), (J_\varepsilon\varphi)(\mathbf{y})\rangle,\end{aligned}$$

and so,

$$\left\langle \frac{(q_{1/\varepsilon}f)(\mathbf{x}_0+\varepsilon\mathbf{x})}{\varepsilon^\alpha L(\varepsilon)}, \varphi(\mathbf{x})\right\rangle = \left\langle \frac{f(\mathbf{x}_0+\varepsilon\mathbf{y})}{\varepsilon^\alpha L(\varepsilon)}, \varphi(\mathbf{y})\right\rangle + o(1) \quad \text{as } \varepsilon \to 0^+,$$

which yields the desired equivalence. □

We illustrate Theorem 5.4.3 with a application to the determination of (symmetric) α-dimensional densities of measures. Let $\alpha > 0$ and let μ be a Radon measure. Following [11, Def. 2.14], we say that $\mathbf{x}_0$ is an α-density point of μ if the limit

$$\theta^\alpha(\mu,\mathbf{x}_0) := \lim_{\varepsilon\to 0^+}\frac{\mu(B(\mathbf{x}_0,\varepsilon))}{\omega_\alpha\varepsilon^\alpha}$$

exists (and is finite), where the normalizing constant is $\omega_\alpha = \pi^{\alpha/2}\Gamma(\alpha+1/2)$. The number $\theta^\alpha(\mu,\mathbf{x}_0)$ is called the (symmetric) α-density of μ at $\mathbf{x}_0$. The ensuing proposition tells us that if a positive measure has a certain quasiasymptotic behavior at $\mathbf{x}_0$, then $\theta^\alpha(\mu,\mathbf{x}_0)$ exists.

Proposition 5.4.1. *Let μ be a positive Radon measure and let $\alpha > 0$. If μ has the quasiasymptotic behavior*

$$\mu(\mathbf{x}_0+\varepsilon\mathbf{x}) = \varepsilon^{\alpha-n}L(\varepsilon)\upsilon(\mathbf{x}) + o(\varepsilon^{\alpha-n}L(\varepsilon)) \quad \text{as } \varepsilon\to 0^+, \quad \text{in } \mathcal{D}'(\mathbb{R}^n), \tag{5.44}$$

then

$$\lim_{\varepsilon\to 0^+}\frac{\mu(\mathbf{x}_0+\varepsilon B)}{\varepsilon^\alpha L(\varepsilon)} = \upsilon(B), \quad \text{for every bounded open set } B. \tag{5.45}$$

In particular, if L is identically 1 and $d\upsilon(\mathbf{x}) = \ell|\mathbf{x}|^{\alpha-n}d\mathbf{x}$, then $\mathbf{x}_0$ is an α-density point of μ and in fact

$$\theta^\alpha(\mu,\mathbf{x}_0) = \frac{\omega_n\ell}{\alpha\omega_\alpha}. \tag{5.46}$$

Proof. By translating, we may assume that $\mathbf{x}_0 = 0$. The quasiasymptotic behavior (5.45) then means that

$$\int_{\mathbb{R}^n} \varphi\left(\frac{\mathbf{x}}{\varepsilon}\right) d\mu(\mathbf{x}) \sim \varepsilon^\alpha L(\varepsilon) \int_{\mathbb{R}^n} \varphi(\mathbf{x}) d\upsilon(\mathbf{x}) \quad \text{as } \varepsilon \to 0^+, \tag{5.47}$$

for each $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Let $\sigma > 0$ be arbitrary. Find open sets Ω_1 and Ω_2 such that $\overline{\Omega}_1 \subset B \subset \overline{B} \subset \Omega_2$ and $\upsilon(\Omega_2 \setminus \Omega_1) < \sigma$. We now select suitable test functions in (5.47). Find $\varphi_1, \varphi_2 \in \mathcal{D}(\mathbb{R}^n)$ such that $0 \leq \varphi_j \leq 1$, $j = 1, 2$, $\varphi_2(x) = 1$ for $x \in B$, $\operatorname{supp} \varphi_2 \subseteq \Omega_2$, $\varphi_1(x) = 1$ for $x \in \Omega_1$, and $\operatorname{supp} \varphi_1 \subseteq B$. Then,

$$\begin{aligned} \limsup_{\varepsilon \to 0^+} \frac{\mu(\varepsilon B)}{\varepsilon^\alpha L(\varepsilon)} &\leq \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^\alpha L(\varepsilon)} \int_{\mathbb{R}^n} \varphi_2\left(\frac{\mathbf{x}}{\varepsilon}\right) d\mu(\mathbf{x}) = \int_{\mathbb{R}^n} \varphi_2(\mathbf{x}) d\upsilon(\mathbf{x}) \leq \upsilon(\Omega_2) \\ &\leq \upsilon(B) + \sigma. \end{aligned}$$

Likewise, using φ_1 in (5.47), one concludes that

$$\liminf_{\varepsilon \to 0^+} \frac{\mu(\varepsilon B)}{\varepsilon^\alpha L(\varepsilon)} \geq \int_{\mathbb{R}^n} \varphi_1(\mathbf{x}) d\upsilon(\mathbf{x}) \geq \upsilon(B) - \sigma\,.$$

Since σ was arbitrary, we obtain (5.45). The last assertion follows by taking $B = B(\mathbf{0}, 1)$ and noticing that in this case $\upsilon(B(\mathbf{0}, 1)) = \ell \int_{|x|<1} |\mathbf{x}|^{\alpha-n} d\mathbf{x} = \ell\omega_n/\alpha$, which yields (5.46). □

We end with an MRA criterion for α-density points of positive measures.

Corollary 5.4.2. *Suppose that the MRA is (M, r)-regular (resp. r-regular) and let μ be a positive Radon measure that satisfies (5.36) (resp. (5.29)). If*

$$\mu(B(\mathbf{x}_0, \varepsilon)) = O(\varepsilon^\alpha) \quad \textit{as } \varepsilon \to 0^+, \tag{5.48}$$

and

$$\lim_{\varepsilon \to 0^+} \frac{(q_{\frac{1}{\varepsilon}}\mu)(\mathbf{x}_0 + \varepsilon\mathbf{x})}{\varepsilon^{\alpha-n}} = \ell|\mathbf{x}|^{\alpha-n} \quad \textit{weakly}^* \textit{ in } \mathcal{K}'_{M,r}(\mathbb{R}^n) \ (\textit{resp. } \mathcal{S}'_r(\mathbb{R}^n)), \tag{5.49}$$

then μ possesses an α-density at $\mathbf{x}_0$, given by (5.46).

Proof. Let us show that (5.48) leads to (5.43) with $f = \mu$ and α replaced by $\alpha - n$. Indeed, write $\mu = \mu_1 + \mu_2$, where $\mu_1(V) := \mu(V \cap B(\mathbf{x}_0, 1))$ for every Borel set V. Let $\varphi \in \mathcal{K}_{M,r}(\mathbb{R}^n)$ (resp. $\varphi \in \mathcal{S}_r(\mathbb{R}^n)$). Set $C = \sup_{x \in \mathbb{R}^n} |x|^\alpha |\varphi(x)| < \infty$. The condition (5.48) implies that $\int_{\mathbb{R}^n} |\mathbf{x} - \mathbf{x}_0|^{-\alpha} d\mu(\mathbf{x}) < \infty$. Using Proposition 5.3.1 and (5.48),

$$\begin{aligned} |\langle \mu(\mathbf{x}_0 + \varepsilon\mathbf{x}), \varphi(\mathbf{x})| &\leq \varepsilon^{-n} \int_{\mathbb{R}^n} \left|\varphi\left(\frac{\mathbf{x} - \mathbf{x}_0}{\varepsilon}\right)\right| d\mu_1(\mathbf{x}) + O(\varepsilon^\alpha) \\ &= \varepsilon^{-n} \int_{\varepsilon \leq |\mathbf{x} - \mathbf{x}_0|} \left|\varphi\left(\frac{\mathbf{x} - \mathbf{x}_0}{\varepsilon}\right)\right| d\mu_1(\mathbf{x}) + O(\varepsilon^{\alpha-n}) \\ &\leq C\varepsilon^{\alpha-n} \int_{\mathbb{R}^n} \frac{d\mu(\mathbf{x})}{|\mathbf{x} - \mathbf{x}_0|^\alpha} + O(\varepsilon^{\alpha-n}) = O(\varepsilon^{\alpha-n}). \end{aligned}$$

Theorem 5.4.3 implies that μ has the quasiasymptotic behavior

$$\mu(\mathbf{x}_0 + \varepsilon\mathbf{x}) = \ell|\varepsilon\mathbf{x}|^{\alpha-n} + o(\varepsilon^{\alpha-n}) \quad \text{as } \varepsilon \to 0^+, \quad \text{in } \mathcal{D}'(\mathbb{R}^n),$$

and the conclusion then follows from Proposition 5.4.1. □

Bibliography

[1] Beurling, A., *Sur les intégrales de Fourier absolument convergentes et leur application à une transformation fonctionelle,* in: IX Congr. Math. Scand., Helsingfors, (1938), 345–366.

[2] Bingham, N. H., Goldie, C. M., Teugels, J. L., *Regular Variation*, Cambridge University Press, Cambridge, 1987.

[3] Boggess, A., Narcowich, A. J., *A Firts Course in Wavelets with Fourier Analysis*, John Wiley and Sons, Hoboken, NJ, 2009

[4] Candès, E. J. *Ridgelet: theory and applications.* [Ph.D. thesis], Department of Statistics, Stanford University; 1998.

[5] Candès, E. J. *Harmonic analysis of neural networks.* Appl. Comput. Harmon. Anal., 6, (1999), 197–218.

[6] Candès, E. J., Donoho D. L. *Ridgelets: a key to higher-dimensional intermittency?*, R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci., 357, (1999), 2495–2509.

[7] Chung, S. Y., Kim, D., Lee, S., *Characterizations for Beurling-Björck space and Schwartz space*, Proc. Amer. Math. Soc., 125, (1997), 3229-3234.

[8] Cordero E., *Gelfand-Shilov window classes for weighted modulation spaces,* Integral Transforms Spec. Funct., 18, (2007), 829–837.

[9] Cordero, E., Pilipović, S., Rodino, L., Teofanov, N., *Localization operators and exponential weights for modulation spaces,* Mediterr. J. Math., 2, (2005), 381–394.

[10] I. Daubechies, *Ten lectures on wavelets*, SIAM, Phyladelphia, 1992.

[11] De Lellis, C., *Rectifiable sets, densities and tangent measures,* European Mathematical Society, Zürich, 2008.

[12] Donoghue, W. F., *Distributions and Fourier Transforms*, Academic Press, New York-London, 1969.

[13] Drozhzhinov, Y. N., Zav'yalov, B. I., *Asymptotically homogeneous generalized functions and boundary properties of functions holomorphic in tubular cones.* Izv. Math., 70, (2006), 1117–1164.

[14] Drozhzhinov, Y. N., Zav'yalov, B. I., *Tauberian theorems for generalized functions with values in Banach spaces*, (Russian) Izv. Ross. Akad. Nauk Ser. Mat., 66, (2002), 47–118; English translation in Izv. Math., 66, (2002), 701–769.

[15] Drozhzhinov, Y. N., Zav'yalov, B. I., *Multidimensional Tauberian theorems for generalized functions with values in Banach spaces*, (Russian) Mat. Sb., 194, (2003), 17–64; English translation in Sb. Math., 194, (2003), 1599–1646.

[16] Estrada, R., *Characterization of the Fourier series of a distribution having a value at a point,* Proc. Amer. Math. Soc. 124, (1996), 1205–1212.

[17] Estrada, R., Kanwal, R. P., *A distributional approach to Asymptotics: Theory and Application*, Birkhäuser, Boston, 2002.

[18] Estrada, R., Vindas, J., *On Tauber's second Tauberian theorem*, Tohoku Math. J. 64 (2012), 539-560.

[19] Estrada, R., Vindas, J., *On Borel summability and analytic functionals,* Rocky Mountain J. Math., 43, (2013), 895–903.

[20] Estrada, R., Vindas, J., *On the point behavior of Fourier series and conjugate series,* Z. Anal. Anwend. 29, (2010), 487–504.

[21] Feichtinger, H. G., Gröchenig K., *Gabor wavelets and the Heisenberg group: Gabor expansions and short time Fourier transform from the group theoretical point of view*, In: Wavelets : a tutorial in theory and applications (C.K. Chui, eds.), pp. 359–397 Wavelet Anal. Appl., Academic Press, Boston, (2), 1992.

[22] Feichtinger, H. G., *Modulation spaces on locally compact Abelian groups*, Technical report, January, 1983.

[23] Feichtinger, H. G., *Atomic characterizations of modulation spaces through Gabor-type representations*, Proc. Conf. Constructive Function Theory, Rocky Mountain J. Math. 19, (1989), 113–126.

[24] Gelfand, I. M., Graev, M. I., Vilenkin, N. Ya. *Generalized functions.* Vol. 5: Integral geometry and representation theory. Academic Press, New York-London, 1966.

[25] Gelfand, I. M, Shilov, G. E., *Generalized functions II.*, Academic press, New York-London, 1968.

[26] Gröchenig K., *Foundations of Time-Frequency Analysis*, Appl. Numer. Harmon. Anal. Birkhäuser Boston, Boston, MA 2001.

[27] Gröchenig K., *Weight functions in time-frequency analysis*, In: Pseudodifferential Operators: Partial Differential Equations and Time-Frequency Analysis (L.Rodino and et al., eds.), Fields Inst. Commun., Amer. Math. Soc., Providence, RI, 52 (2007), 343–336.

[28] Feichtinger, H. G.,, *On a new Segal algebra,* Monatsh. Math., 92, (1981), 269–289.

[29] Gröchenig, K., Zimmermann, G., *Hardy's theorem and the short-time Fourier transform of Schwartz functions*, J. London Math. Soc., 63, (2001), 205–214.

[30] Gröchenig, K., Zimmermann, G., *Spaces of test functions via the STFT*, J. Funct. Spaces Appl., 2, (2004), 25–53.

[31] Hasumi, M., *Note on the n-dimensional tempered ultra-distributions*, Tôhoku Math. J.,13, (1961), 94–104.

[32] Hernández, E., Weiss, G., *A First Course of Wavelets*, CRC Press, Boca Raton, 1996.

[33] Helgason, S. *The Radon transform.* Second edition, Birkhäuser Boston, Inc., Boston, MA, 1999.

[34] Hertle, A. *Continuity of the Radon transform and its inverse on Euclidean spaces.* Math. Z, 184, (1983), 165–192.

[35] Hertle A. *On the Range of Radon transform and its dual.* Math. Ann., 267, (1984), 91–99.

[36] Hille, E., Phillips, R. S., *Functional analysis and semi-groups,* American Mathematical Society, Providence, R. I., 1957.

[37] Holschneider, M. *Wavelets. An analysis tool.* The Clarendon Press, Oxford University Press, New York, 1995.

[38] Hoskins, R. F., Sousa Pinto, J., *Theories of generalised functions. Distributions, ultradistributions and other generalised functions,* Horwood Publishing Limited, Chichester, 2005.

[39] Kelly, S. E., , Kon, M. A., Raphael, L. A., *Local convergence for wavelet expansions,* J. Funct. Anal., 126, (1994), 102–138.

[40] Kelly, S. E., , Kon, M. A., Raphael, L. A., *Pointwise convergence of wavelet expansions,* Bull. Amer. Math. Soc. (N.S.), 30, (1994), 87–94.

[41] Kelly, S. E., , Kon, M. A., Raphael, L. A., *Pointwise convergence of wavelet expansion*, Bull. Amer. Math. Soc. 30 (1994), 87–94.

[42] Komatsu, H., *Projective and injective limits of weakly compact sequence of locally convex spaces*, J. Math. Soc. Japan, 19(3), (1967), 366–383

[43] Kostadinova S., Pilipović, S., Saneva, K., Vindas, J., *The ridgelet transform of distributions*, Integral Transforms and Special Functions, in press (DOI:10.1080/10652469.2013.853057)"

[44] Kostadinova S., Pilipović, S., Saneva, K., Vindas, J., *The short-time Fourier transform of distributions of exponential type and Tauberian theorems for shift-asymptotics*, submited in Jornal od Approximation Theory.

[45] Kostadinova S., Vindas, J., *Multiresolution expansions and quasiasymptotic behavior of distributions*, submited.

[46] Kostadinova S., Pilipović, S., Saneva, K., Vindas, J., *The ridgelet transform and quasiasymptotic be- havior of distributions*, to appear in Proceedings of the 9-th ISAAC congress, Krakow, 2013

[47] Korevaar, J., *Tauberian Theory-A Century of developments*, Springer-Verlag, Berlin, 2004.

[48] Köthe, G., *Topological vector spaces. II,* Springer-Verlag, New York-Berlin, 1979.

[49] Łojasiewicz, S., *Sur la fixation des variables dans une distribution,* Studia Math., 17, (1958), 1–64.

[50] S. Łojasiewicz, *Sur la valeur et la limite d'une distribution en un point,* Studia Math., 16, (1957), 1–36.

[51] Ludwig, D. *The Radon transform on euclidean space.* Comm. Pure Appl. Math., 19, (1966), 49–81.

[52] Mallat, S., *Multiresolution approximation and wavelets*, Trans. Amer. Math. Soc., 315, 69-88 (1989).

[53] Mallat, S., *Multiresolution approximations and wavelet orthonormal bases of* $L^2(\mathbb{R})$, Trans. Amer. Math. Soc., 315, (1989), 69–87.

[54] Mallat, S., *A wavelet Tour of Signal Processing*, Acad. Press, San Diego, CA, 1998.

[55] Meyer, Y., *Wavelets, vibrations and scalings.* American Mathematical Society, Providence; 1998.

[56] Meyer, Y., *Wavelets and Operators*, Cambridge Univ.Press, Cambridge, 1992.

[57] Misra, O. P., Lavoine J. L., *Transform analysis of generalized functions,* North-Holland Publishing Co., Amsterdam, 1986.

[58] Morimoto, M., *Theory of tempered ultrahyperfunctions. I, II,* Proc. Japan Acad., 51, (1975), 87–91; 51, (1975), 213–218.

[59] Obiedat H. M., Mustafa, Z., Awawdeh, F., *Short-time Fourier transform over the Silva space*, Inter. J. Pure Appl. Math., 44, (2008), 755–764.

[60] Park, Y. S., Morimoto, M., *Fourier ultra-hyperfunctions in the Euclidean* $n-$*space*, J. Fac. Sci. Univ. Tokyo Sect. IA Math., 20, (1973), 121–127.

[61] Pilipović, S., *On the quasiasymptotic of Schwartz distributions*, Math. Nachr., 137, (1988), 1925.

[62] Pilipović, S., *Tempered ultradistributions,* Boll. Un. Mat. Ital. B (7)2, (1988), 235–251.

[63] Pilipović, S., Stanković, B., *S-asymptotic of a distribution*, Pliska Stud. Math. Bulgar., 10, (1989), 147156.

[64] Pilipović, S., *On the Behavior of a Distribution at the Origin*, Math. Nachr., 141 (1989), 2732.

[65] Pilipović, S., Stanković, B., Takači, A., *Asymptotic Behaviour and Stieltjes Transformation of Distribution*, Taubner-Texte zur Mathematik, band 116, 1990.

[66] Pilipović, S., Stanković, B., *Structural theorems for the S-asymptotic and quasiasymptotic of distributions*, Math. Pannon., 4, (1993), 2335.

[67] Pilipović, S., Stanković, B., *Wiener Tauberian theorems for distributions,* J. London Math. Soc. (2)47, (1993), 507–515.

[68] Pilipović, S., *Quasiasymptotics and S-asymptotics in $\mathcal{S}'$ and $\mathcal{D}'$*, Publ. Inst. Math. (Beograd), 72 (1995), 1320.

[69] Pilipović, S., *Quasiasymptotic Expansion and the Laplace Transformation*, Applicable Analysis, 35, (1996), 243–261.

[70] Pilipović, S., Takači, A., Teofanov, N., *Wavelets and quasiasymptotics at a point*, J. Approx. Theory, 97, (1999), 40–52.

[71] Pilipović, S., Vindas, J. *Multidimensional Tauberian theorems for vector-valued distributions.* Publ. Inst. Math. (Beograd), in press.

[72] Pilipović, S., Teofanov, N., *Multiresolution expansion, approximation order and quasiasymptotic behavior of tempered distributions*, J. Math. Anal. Appl., 331, (2007), 455–471.

[73] Pilipović, S., Stanković, B., Vindas, J., *Asymptotic behavior of generalized functions*, Series on Analysis, Applications and Computation, 5, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012.

[74] Pilipović, S., Rakić, D., Vindas, J. *New classes of weighted Hölder-Zygmund spaces and the wavelet transform.* J. Funct. Spaces Appl.; 2012, Article ID 815475, 18 pp.

[75] Rakić, D., *Multiresolution expansion in $\mathcal{D}'_{L^p}(\mathbb{R}^n)$,* Integral Transforms Spec. Funct.,20, (2009), 231–238.

[76] Ramm, A. G. *The Radon transform on distributions.* Proc. Japan Acad. Ser. A Math. Sci., 71, (1995), 202–206.

[77] Robertson A. P., Robertson W., *Topological vector spaces,* Cambridge University Press, London-New York, 1973.

[78] Roopkumar, R. *Ridgelet transform on tempered distributions*, Comment. Math. Univ. Carolin, 51, (2010), 431–439.

[79] Roopkumar, R. *Extended ridgelet transform on distributions and Boehmians*, Asian-Eur, J. Math., 4, (2011), 507–521.

[80] Rudin, W., *Real and complex analysis,* McGraw-Hill Book Co., New York, 1987.

[81] Sampson, G., Zielezny, Z., *Hypoelliptic convolution equations in $\mathcal{K}_p, p > 1$*, Transactions of the AMS, 223, (1976), 133–154.

[82] Saneva, K., Bučkovska, A., *Asymptotic behaviour of the distributional wavelet transform at 0*, Math. Balkanica (N.S.) 18, (2004), 437441.

[83] Saneva, K., Bučkovska, A., *Asymptotic expansion of distributional wavelet transform,* Integral Transforms Spec. Funct. 17, (2006), 8591.

[84] Saneva, K., Bučkovska, A., *Tauberian theorems for distributional wavelet transform,* Integral Transforms Spec. Funct. 18, (2007), 359368.

[85] Saneva, K., Анализа на асимптотското однесување и асимптотскиот развој на вејвлет трансформацијата, [Ph. D. thesis], University Ss. Ciril and Methodius, Skopje, 2008.

[86] Saneva, K., *Asymptotic behaviour of wavelet coefficients*, Integral Transform Spec. Funct. 20 (3-4), (2009), 333-339.

[87] Saneva, K., Vindas, J., *Wavelet expansions and asymptotic behavior of distributions*, J. Math. Anal. Appl. 370, (2010), 543–554.

[88] Saneva, K., Aceska, R., Kostadinova, S., *Some Abelian and Tauberian results for the short-time Fourier transform*, Novi Sad J. Math., 43, (2013), 81–89.

[89] Schwartz, L., *Théory des Distributions I.*, 2nd ed. Hermann, Paris, 1957.

[90] Sebastião e Silva, J., *Les fonctions analytiques comme ultra-distributions dans le calcul opérationnel,* Math. Ann., 136, (1958), 58–96.

[91] Sebastião e Silva, J., *Les séries de multipôles des physiciens et la théorie des ultradistributions,* Math. Ann., 174, (1967), 109–142.

[92] Sebastião e Silva, J. *Sur la définition et la structure des distributions vectorielles*. Portugal. Math., 19, (1960), 1–80.

[93] Seneta, E., *Regularly Varying Functions*, Berlin: Springer Verlag, 1976.

[94] Shaefer, H. H., *Topologycal Vector Spaces*, Springer-Verlag, 1986.

[95] Sohn, B. K., *Quasiasymptotics in exponential distributions by wavelet analysis*, Nihonkai Math. J., 23, (2012), 21–42.

[96] Sohn, B. K., *Multiresolution expansion and approximation order of generalized tempered distributions,* Int. J. Math. Math. (2013), Art. ID 190981, 8 pp.

[97] Sohn, B. K., Pahk, D. H., *Pointwise convergence of wavelet expansion of* $\mathcal{K}_M^r{}'(\mathbb{R})$, Bull. Korean Math. Soc. 38, (2001), 81–91.

[98] Stanković, B., *Fourier hyperfunctions having the S-asymptotics*, Bull. Cl. Sci. Math. Nat. Sci. Math., 24, (1999), 6775.

[99] Sznajder, S., Zieleźny, Z., *Solvability of convolution equations in* $\mathcal{K}_p'$*,* $p > 1$*,* Pacific J. Math., 63,(1976), 539–544.

[100] Tao, T., *On the almost everywhere convergence of wavelet summation methods,* Appl. Comput. Harmon. Anal. **3** (1996) 384–387.

[101] Teofanov, N., *Convergence of multiresolution expansion in the Schwartz class*, Math. Balcanica, 20, (2006), 101-111.

[102] Toft, J., *The Bargmann transform on modulation and Gelfand-Shilov spaces, with applications to Toeplitz and pseudo-differential operators,* J. Pseudo-Differ. Oper. Appl., 3, (2012), 145–227.

[103] Trèves, F., *Topological vector spaces, distributions and kernel*, Academic Press, New York, 1967.

[104] Vindas, J., Estrada, R., *Distributional point values and convergence of Fourier series and integrals,* J. Fourier Anal. Appl., 13, (2007), 551–576.

[105] Vindas, J., *Structural Theorems for Quasiasymptotics of Distributions at Infinity*, Pub. Inst. Math. Beograd, N.S., 84(98), (2008), 159–174.

[106] Vindas, J., Pilipović, S., *Structural theorems for quasiasymptotics of distributions at the origin*, Math. Nachr. 282(2.11), (2009), 1584–1599.

[107] Vindas, J., *Local behavior of distributions and applications*, [Ph. D. thesis], Louisiana State University, Baton Rouge, 2009.

[108] Vindas, J., Estrada, R., *On the support of tempered distributions,* Proc. Edinb. Math. Soc. (2), 53, (2010), 255–270.

[109] Vindas, J., Estrada, R., *On the order of summability of the Fourier inversion formula,* Anal. Theory Appl. 26, (2010), 13–42.

[110] Vindas, J., Pilipović, S., Rakic, D., *Tauberian theorems for the wavelet transform*, J. Fourier Anal. Appl. 17, (2011), 65–69.

[111] Vindas, J., *The structure of quasiasymptotics of Schwartz distributions*, Banach Center Publ. 88 (2010), 297-314.

[112] Vladimirov, V.S., *Generalized Functions in Mathematical Physics*, Nauka, Moscow, 1976.

[113] Vladimirov, V.S., Drožinov, Ju., Zavialov, B. I., *Tauberian Theorems for Generalized Functions*, Kluwer Academic Publishers Group, Dordrecht, 1988.

[114] Walter, G. G., *Pointwise Convergence of Wavelet Expansions*, Journal of Approxiamtion Theory, 80, (1995), 108–118.

[115] Walter, G. G., *Pointwise convergence of distribution expansions,* Studia Math., 26, (1966), 143–154.

[116] Walter, G. G., Shen, X., *Wavelets and Other Orthogonal Systems With Application*, CRS Press, Secon Editiotn, 2000.

[117] Walter, G. G., Shen, X., *Wavelets and other orthogonal systems,* Chapman & Hall/CRC, Boca Raton, FL, 2001.

[118] Wiener, N., *Tauberian theorems,* Ann. of Math., (2)33, (1932), 1–100.

[119] Zav'yalov, B. I., *Asymptotic properties of functions that are holomorphic in tubular cones,* (Russian) Mat. Sb. (N.S.) 136(178), (1988), 97–114; translation in Math. USSR-Sb., 64, (1989), 97–113.

[120] Zayed, A. I., *Pointwise convergence of a class of non-orthogonal wavelet expansions,* Proc. Amer. Math. Soc., 128, (2000), 3629–3637.

[121] Zieleźny, Z., *Hypoelliptic and entire elliptic convolution equations in subspaces of the space of distributions. II,* Studia Math., 32, (1969), 47–59.

Short Biography

Sanja Kostadin Kostadinova

Address: "Timo Trencev" 17, Strumica, Macedonia

Telephone number: +389 70 612 698

e-mail: sanja.kostadinova@gmail.com

Date and place of birth: 15.11.1985, Strumica, Macedonia

Citizenship: Macedonia

Education: 2010

MSc degree in applied mathematics in electrical engineering and information technologies,
with grade-point average 10.0,
Faculty of Electrical Engineering and Information Technologies,
University Ss. Cyril and Methodius - Skopje,
MSc thesis: "*Qualitative analysis od dynamical systems and their application*"
(macedonian)

2007

Bachelor degree in educational mathematics,
with grade-point average 9.87,
Faculty of Natural Sciences and Mathematics,
University Ss. Cyril and Methodius - Skopje

Work experience: 2009-today

teaching assistant,
Faculty of Electrical Engineering and Information Technologies,
University Ss. Cyril and Methodius - Skopje

2007-2009

demonstrator,
Faculty of Natural Sciences and Mathematics,
University Ss. Cyril and Methodius - Skopje

Notable Accomplishments: 2009

member of the
Organization Committee
for MICOOM 2009, september, 2009, Ohird, Macedonia

2007

participated
14-th International Mathematics Competition for University Students,
Blagoevgrad, Bulgaria

Bibliography-papers:

K. Saneva, R. Aceska, S. Kostadinova, *Some Abelian and Tauberian results for STFT*, Novi Sad J. Math. Vol. 43, No. 2, 2013, 81–89

S. Kostadinova, S. Pilipovic, K. Saneva, J. Vindas, *The ridgelet transform of distributions*, Integral Transforms and Special Functions, in press (DOI:10.1080/10652469.2013.853057)

S. Kostadinova, S. Pilipovic, K. Saneva, J. Vindas, *The short-time Fourier transform of distributions of exponential type and Tauberian theorems for shift-asymptotics*, submited

S. Kostadinova, J. Vindas, *Multiresolution expansions and quasiasymptotic behavior of distributions*, submited

S. Kostadinova, S. Pilipovic, K. Saneva, J. Vindas, *The ridgelet transform and quasiasymptotic be- havior of distributions*, to appear in Proceedings of the 9-th ISAAC congress, Krakow, 2013

Novi Sad, 2013 Sanja Kostadinova

____

Кратка Биографија

Сања Костадин Костадинова

Адреса: ”Тимо Тренчев” 17, Струмица, Македонија

Телефонски број: +389 70 612 698

e-mail: sanja.kostadinova@gmail.com

Датум и место рођења: 15.11.1985, Струмица, Македонија

Држављанство: Република Македонија

Образовање: 2010

Магистер електротехнике и информационих технологија,
са просеком 10.0,
Факултет електротехнике и инфомационих технологија,
Универзитет Св. Кирил и Методиј - Скопље,
Магистерски рад:
”*Квалитаттивна анализа динамичких система*
и њихова примена”
(македонски)

2007

Дипломирани професор по математика,
са просеком 9.87,
Природно-математички факултет,
Универзитет Св. Кирил и Методиј - Скопље

Радно искуство: 2009-данас

асистент,
Факултет електротехнике и инфомационих технологија,
Универзитет Св. Кирил и Методиј - Скопље

2007-2009

демонстратор,
Природно-математички факултет,
Универзитет Св. Кирил и Методиј - Скопље

Посебна достигнућа: 2009

члан организационих одбор
конференција МИЦООМ, септември 2009, Охрид, Македониаја

2007

учешће
14-тг Интернационално Математицко Такмицење за Универзитетке студенте, Благоевград, Бугариа

Библиографија-радови:

K. Saneva, R. Aceska, S. Kostadinova, *Some Abelian and Tauberian results for STFT*, Novi Sad J. Math. Vol. 43, No. 2, 2013, 81–89

S. Kostadinova, S. Pilipovic, K. Saneva, J. Vindas, *The ridgelet transform of distributions*, Integral Transforms and Special Functions, in press (DOI:10.1080/10652469.2013.853057)

S. Kostadinova, S. Pilipovic, K. Saneva, J. Vindas, *The short-time Fourier transform of distributions of exponential type and Tauberian theorems for shift-asymptotics*, submited

S. Kostadinova, J. Vindas, *Multiresolution expansions and quasiasymptotic behavior of distributions*, submited

S. Kostadinova, S. Pilipovic, K. Saneva, J. Vindas, *The ridgelet transform and quasiasymptotic be- havior of distributions*, to appear in Proceedings of the 9-th ISAAC congress, Krakow, 2013

Нови Сад, 2013 Сања Костадинова

University of Novi Sad
Faculty of Sciences
Key Words Documentation

Accession number:
ANO
Identification number:
INO
Document type: Monograph type
DT
Type of record: Printed text
TR
Content Code: PhD Dissertation
CC
Author: Sanja Kostadinova
AU
Mentor: Academician Professor Stevan Pilipović, PhD and Professor Jasson Vindas, PhD
MN
Title: Some classes of integral transforms on distribution spaces and generalized asymptotics
XI
Language of text: English
LT
Language of abstract: English/Serbian
LA
Country of publication: Serbia
CP
Locality of publication: Vojvodina
LP
Publication year: 2014
PY
Publisher: Author's reprint
PU
Publ. place: University of Novi Sad, Faculty of Sciences, Department of mathematics and informatics, Trg Dositeja Obradovića 4
PP

Physical description: (chapters/pages/literature/tables/images/graphs/appendices)
(6, 140, 110, 0, 0, 0, 0)
PD
Scientific field: Mathematics
SF
Scientific discipline: Functional analysis
SD
Key words: distributions, asymptotics, ridgelet transform, Radon transform, STFT, Multiresolution analysis
KW
Holding data: Library of Department of Mathematics and Informatics, Faculty of Sciences, University of Novi Sad
HD
Note:

Abstract: In this doctoral dissertation several integral transforms are discussed. The first one is the Short time Fourier transform (STFT). We present continuity theorems for the STFT and its adjoint on the test function space $\mathcal{K}_1(\mathbb{R}^n)$ and the topological tensor product $\mathcal{K}_1(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}(\mathbb{C}^n)$, where $\mathcal{U}(\mathbb{C}^n)$ is the space of entire rapidly decreasing functions in any horizontal band of $\mathbb{C}^n$. We then use such continuity results to develop a framework for the STFT on $\mathcal{K}_1'(\mathbb{R}^n)$. Also, we devote one section to the characterization of $\mathcal{K}_1'(\mathbb{R}^n)$ and related spaces via modulation spaces. We also obtain various Tauberian theorems for the short-time Fourier transform.
Part of the thesis is dedicated to the ridgelet and the Radon transform. We define and study the ridgelet transform of (Lizorkin) distributions and we show that the ridgelet transform and the ridgelet synthesis operator can be extended as continuous mappings $\mathcal{R}_\psi : \mathcal{S}_0'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{Y}^{n+1})$ and $\mathcal{R}_\psi^t : \mathcal{S}'(\mathbb{Y}^{n+1}) \to \mathcal{S}_0'(\mathbb{R}^n)$. We then use our results to develop a distributional framework for the ridgelet transform that is, we treat the ridgelet transform on $\mathcal{S}_0'(\mathbb{R}^n)$ via a duality approach. Then, the continuity theorems for the ridgelet transform are applied to discuss the continuity of the Radon transform on these spaces and their duals. Finally, we deal with some Abelian and Tauberian theorems relating the quasiasymptotic behavior of distributions with the quasiasymptotics of the its Radon and ridgelet transform.
The last chapter is dedicated to the MRA of M-exponential distributions. We study the convergence of multiresolution expansions in various test function and distribution spaces and we discuss the pointwise convergence of multiresolution expansions to the distributional point values of a distribution. We also provide a characterization of the quasiasymptotic behavior in terms of multiresolution expansions and give an MRA sufficient condition for the existence of α-density points of positive measures.

AB

Accepted by the Scientific Board on:

Defended:
Thesis defend board:

President:

Member:

Member:

Универзитет у Новом Саду Природно-Математички факултет Кључна Документацијска Информација

Редни број:
Идентификациони број:
ИБР
Тип документације: Монографска документација
ТД
Тип записа: Текстуални штампани материјал
ТЗ
Врста рада: Докторска дисертација
ВР
Аутор: Сања Костадинова
АУ
Ментор: Академик Професор др Стеван Пилиповић, Професор Јасон Виндас
МН
Наслов рада: Неке класе интегралних трансформација на простору дистрибуција и уопштена асимптотика
МР
Језик публикације: Енглески
ЈП
Језике извода: Енглески/Српски
ЈИ
Земља публиковања: Република Србија
ЗП
Уже географско подручје: Војводина
УГП
Година: 2014
ГО
Издавач: Ауторски репринт
ИЗ
Место и адреса: Универзитет у Новом Саду, Природно-математички факултет, Департман за математику и информатику, Трг Доситеја Обрадовића

4
МА
Физички опис рада: (број поглавља, број страна, број лит. цитата, број табела, број слика, број графика, број прилога)
(6, 140, 110, 0, 0, 0, 0)
ФО
Научна област: Математика
НО
Научна дисциплина: Функционална анализа
НД
Кључне речи: дистрибуција, асимптотика, ridglet трансформација, Радон трансформација, STFT, Мултирезолуциска анализа
ПО
УДК:
Чува се: Библиотека Департмана за математику и информатику, Природно-математички факултет, Универзитет у Новом Саду
ЧУ
Важна напомена:
ВН

Извод: У овој докторској дисертацији размотрени су неколико интегралне трансформације. Прва је short time Fourier transform (STFT). Дате су и доказане теореме о непрекидности STFT и њена синтеза на простору тест функције $\mathcal{K}_1(\mathbb{R}^n)$ и на простору $\mathcal{K}_1(\mathbb{R}^n)\widehat{\otimes}\mathcal{U}(\mathbb{C}^n)$ где $\mathcal{U}(\mathbb{C}^n)$ је простор од целих брзо опадајуче функције у произвољни хоризонтални опсег на $\mathbb{C}^n$. Онда, ове резултате непрекидности су искористени за равијање теорије STFT на простору $\mathcal{K}_1'(\mathbb{R}^n)$. Једно поглавје је посвечено карактеризације $\mathcal{K}_1'(\mathbb{R}^n)$ са сродних модулациских простора. Доказани су и различитих Тауберови резултата за STFT.
Део тезе је посвечен на ridglet и Радон трансформације. Ridgelet трансформација је дефинисана на (Лизоркин) дистрибуције и показано је да ridgelet трансформација и њен оператор синтезе могу да се прошире како непрекидна пресликавања $\mathcal{R}_\psi : \mathcal{S}_0'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{Y}^{n+1})$ и $\mathcal{R}_\psi^t : \mathcal{S}'(\mathbb{Y}^{n+1}) \to \mathcal{S}_0'(\mathbb{R}^n)$. Ridgelet трансформација на $\mathcal{S}_0'(\mathbb{R}^n)$ је дата преко дуалног приступа. Наше теореме непрекидности ридгелет трансформације су примењени у доказивању непрекидности Радонове трансформације на Лизоркин тест просторима и њиховим дуалима. На крају, дајемо Абелових и Тауберових теорема кои дају везе између квазиасимптотике дистрибуција и квазиасимптотике rigdelet и Радоновог трансфомацију.
Задње поглавје је посвечену мултирезолуцијског анализу М - експоненцијалних диструбуције. Проучавамо конвергенцију мултирезолуцијког развоју у различитих простори тест функције и дистрибуције и размотрена је точкаста конвергенција мултирезолуцијког развоју у тачку у дистрибутивног смислу. Обезбеђена је и карактеризација квазиасимптотике у поглед мултирезолуцијског развоју и дајемо довољни услов за постојење α -тачка густине за позитивне мере.

ИЗ

Датум прихватања теме од стране НН већа:
ДП
Датум одбране:
ДО
Чланови комисије:
КО

Председник:

Члан:

Члан: