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BUCKLING OF CIRCULAR, ANNULAR PLATES OF NON-UNIFORM THICKNESS

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Abstract—This paper deals with the solution of the title problem in the case where the outer boundary is subjected to uniform, hydrostatic pressure while the inner edge of the plate is free. It is assumed that the plate thickness varies (a) in a discontinuous fashion and (b) linearly.

An approximate approach is proposed using polynomial coordinate functions which identically satisfy the boundary conditions at the outer edge, only. The eigenvalues are determined using the optimized Rayleigh–Ritz method and good engineering agreement is shown to exist with buckling parameters obtained by means of a finite element code.

1. INTRODUCTION

The technical literature contains very limited information on critical, buckling loads of the structural systems shown in Fig. 1 (Timoshenko and Gere, 1961). Even in the case of the annular plate of uniform thickness the buckling parameter available in the open literature, has been determined for Poisson's ratio (ν) equal to 1/3.

The problem is of great practical significance in civil, mechanical, naval and ocean engineering applications.

This paper presents a very simple methodology to tackle this rather complex elastic stability problem. Numerical data is obtained for plates of uniform thickness, h_0, and plates of non-uniform thickness, Fig. 1a and Fig. 1b, for several values of Poisson's ratio.

2. APPROXIMATE SOLUTION BY MEANS OF THE OPTIMIZED RAYLEIGH–RITZ METHOD

Since the outer boundary of the plate is subjected to a uniformly applied pressure p_0 the radial stress resultant $N(\bar{r})$ is given by the classical Lamé solution

$$N(\bar{r}) = \frac{N_o}{1-(b/a)^2}\,[1-(\bar{r}/b)^2] \quad (1)$$

where $N_0 = ph_0$.

Determination of the critical buckling load is defined by minimization of the governing functional

$$J(W) = \int_a^b D(\bar{r})\left[\left(W'' + \frac{W'}{\bar{r}}\right)^2 - 2(1-\nu)\frac{W'W''}{\bar{r}}\right]\bar{r}d\bar{r} - \int_a^b N(\bar{r})W'^2\bar{r}\,d\bar{r} \quad (2)$$

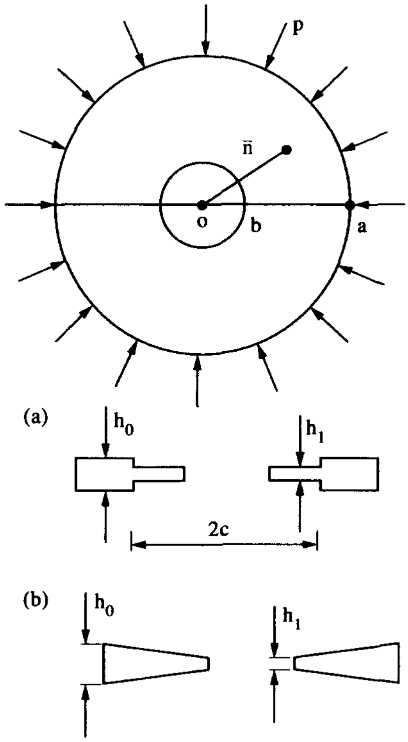

Fig. 1. Annular plate: elastic stability analysis.

subject to appropriate boundary conditions.

Introducing the dimensionless variable $r = \bar{r}/a$ and substituting in Equation (2) one obtains

$$\frac{a^2}{D_o} J(W) = \int_{r_b}^{1} g(r) \left[\left(W'' + \frac{W'}{r} \right)^2 - 2(1-\upsilon) \frac{W'W''}{r} \right] rdr - \frac{\lambda}{1-r_b^2} \int_{r_b}^{1} \left(1 - \frac{r_b^2}{r^2} \right) W'^2 \, rdr \tag{3}$$

where

$$r_b = \frac{b}{a},\ D_0 = D(1),\ D(r) = D_0 g(r),\ \lambda = \frac{N_0 a^2}{D_0}$$

In the case of a simply supported edge the governing boundary conditions are

$$\begin{cases} W(1) = 0 \\ W''(1) + \nu W'(1) = 0 \end{cases} \tag{4a,b}$$

while for a clamped edge one must comply the essencial boundary conditions

$$\begin{cases} W(1) = 0 \\ W'(1) = 0 \end{cases} \tag{5a,b}$$

Consider now the case of a circular, annular plate of discontinuously varying thickness, Fig. 1a. One has

$$g(r) = \begin{cases} \rho_h^3\,,\ r_b \leq r \leq r_c, (r_c = c/a) \\ 1\ r_c < r < 1 \end{cases} \tag{6}$$

On the other hand, when the thickness varies linearly with the radial variable $g(r)$ results

$$g(r) = \left[\rho_h + \frac{1 - \rho_h}{1 - r_b}(r - r_b)\right]^3 \tag{7}$$

It is quite convenient to approximate the displacement amplitude $W(r)$ by means of a summation of simple polynomial coordinate functions (Laura *et al.*, 1975).

$$W \cong W_\alpha = \sum_{j=1}^{J} C_j \psi_j(r) \tag{8}$$

where

$$\psi_j(r) = (\alpha_j r^P + \beta_j r^2 + 1) r^{j-1}$$

The α_js and β_js are determined substituting each coordinate function in the governing outer boundary conditions. The exponential parameter p allows for minimization of the calculated eigenvalue (Laura, 1995).

Substituting Equation (8) in Equation (3) and requiring that the functional be a minimum with respect to the C_is one obtains

$$\frac{a^2}{2D_o}\frac{\partial J}{\partial C_i} = \sum_J \left\{ \left[\int_{r_b}^{1} g(r) \left(\psi_j'' + \frac{\psi_j'}{r}\right)\left(\psi_i'' + \frac{\psi_i'}{r}\right) - \frac{1 - \nu}{r}(\psi_j''\psi_i' + \psi_j'\psi_i'') r\, dr - \frac{\lambda}{1 - r_b^2} \int_{r_b}^{1} \left(1 - \frac{r_b^2}{r^2}\right) \psi_j' \psi_i' r\, dr \right\} C_j = 0 ; \tag{9}$$

$i = 1,2,3$

The non-triviality condition leads to a secular determinant in λ and the lowest root is the desired critical buckling parameter.

Clearly one could construct polynomial coordinate functions, each one with two additional terms, in such a manner as to satisfy also the natural boundary conditions at the inner edge. However this will render the procedure lengthier and as it will shown in the next section the proposed approximate approach yields good engineering accuracy with a minimum amount of labour.

3. FINITE ELEMENT SOLUTION

The numerical results have been obtained using SAMCEF (1994) finite element code using hybrid elements of triangular and rectangular shape (elements type 55 and 56 of the SAMCEF Library). The number of elements varied in accordance with the ratio b/a (for b/a=0.1 the mesh of half of the plate contained 661 elements).

4. NUMERICAL RESULTS

All calculations have been performed making J=3 in Equation (8).

Fig. 2a and Fig. 2b depict comparisons of values of $N_0\ a^2/D_0$ for simply supported and clamped annular plates at the outer boundary, with results available in the literature for $\nu=1/3$. Good engineering agreement is shown to exist.

Table 1, Table 2, Table 3, Table 4, Table 5, Table 6, Table 7, Table 8, and Table 9 contain values of $N_0\ a^2/D_0$ for annular plates with outer edge simply supported while Table 10, Table 11, Table 12, Table 13, Table 14, Table 15, Table 16, Table 17 and Table 18 show values of the buckling coefficient for the clamped outer edge situation for the configuration corresponding to Fig. 1a, including the uniform thickness case (Table 1 and Table 10).

The eigenvalues have been determined for Poisson's ratio equal to 0.2, 0.3, 0.33 and 0.40; $h_1/h_0=1$, 0.8 and 0.6; b/a=0.1, 0.2... 0.7 and c/a=0.2, 0.3... 0.8. They are computed using the optimized Rayleigh–Ritz method and in the cases of Table 1, Table 3, Table 7,

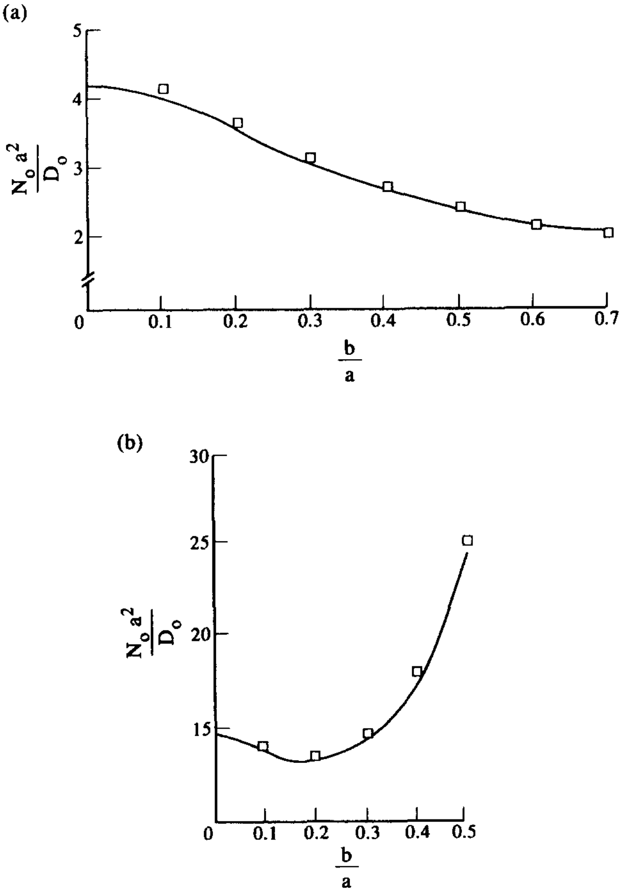

Fig. 2. Buckling Coefficients of Circular Annular Plates of Uniform Thickness; a) Outer Edge: Clamped Optimized Rayleigh–Ritz (Timoshenko and Gere, 1961). □ Optimized Rayleigh–Ritz. – (Timoshenko and Gere, 1961).

Table 1. Buckling coefficient, uniform thickness. External boundary simply supported. (1): Optimized Rayleigh–Ritz approach. (2): Finite elements method.

ν		Values of b/a						
		0.1	0.2	0.3	0.4	0.5	0.6	0.7
0.2	(1)	3.821	3.488	3.110	2.801	2.561	2.374	2.224
0.3	(1)	4.047	3.611	3.140	2.775	2.504	2.298	2.138
	(2)	3.997		3.102		2.497		2.134
0.33	(1)	4.111	3.639	3.137	2.755	2.475	2.265	2.103
0.4	(1)	4.254	3.685	3.105	2.685	2.387	2.168	2.002

Table 2. Buckling coefficient (discontinuously varying thickness, h_1/h_0=0.8). External boundary simply supported. ν=0.2

b/a	Values of c/a						
	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.1	3.624	3.343	3.107	2.860	2.610	2.409	2.253
0.2		3.155	2.906	2.664	2.432	2.246	2.091
0.3			2.793	2.515	2.281	2.081	1.903
0.4				2.485	2.184	1.975	1.775
0.5					2.239	1.954	1.713
0.6						2.028	1.720
0.7							1.825

Table 3. Buckling coefficient, discontinuously varying thickness h_1/h_0=0.8) External boundary simply supported. ν=0.3(1): Optimized Rayleigh–Ritz approach. (2): Finite elements method.

b/a		Values of c/a						
		0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.1	(1)	3.806	3.485	3.224	2.964	2.706	2.505	2.351
	(2)	3.750			2.963			2.345
0.2			3.220	2.946	2.696	2.465	2.284	2.136
0.3				2.786	2.493	2.258	2.064	1.896
0.4	(1)				2.438	2.131	1.927	1.737
	(2)				2.481			1.778
0.5						2.171	1.888	1.656
0.6							1.951	1.651
0.7	(1)							1.745
	(2)							1.807

Table 10, Table 12 and Table 16 some values have been determined using the finite element method. Since the analytical formulation yields upper bounds it is concluded that in the case of Table 3 they are more accurate than those obtained by means of the finite element method for b/a=0.4 and c/a=0.5 and 0.8; and for b/a=0.7 and c/a=0.8. Similarly:

Table 4. Buckling coefficient (discontinuously varying thickness, h_1/h_0=0.8). External boundary simply supported. ν=0.33

b/a	Values of c/a						
	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.1	3.856	3.522	3.254	2.990	2.732	2.531	2.378
0.2		3.229	2.948	2.697	2.468	2.290	2.143
0.3			2.772	2.476	2.242	2.051	1.886
0.4				2.413	2.106	1.904	1.718
0.5					2.141	1.860	1.631
0.6						1.919	1.623
0.7							1.713

Table 5. Buckling coefficient (discontinuously varying thickness, h_1/h_0=0.8). External boundary simply supported. ν=0.4

b/a	Values of c/a						
	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.1	3.960	3.596	3.312	3.042	2.783	2.585	2.437
0.2		3.228	2.934	2.682	2.460	2.290	2.150
0.3			2.717	2.416	2.187	2.004	1.850
0.4				2.333	2.030	1.835	1.659
0.5					2.053	1.779	1.561
0.6						1.829	1.544
0.7							1.625

Table 6. Buckling coefficient (discontinuously varying thickness, h_1/h_0=0.6). External boundary simply supported. ν=0.2

b/a	Values of c/a						
	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.1	3.450	2.923	2.525	2.134	1.766	1.482	1.260
0.2		2.922	2.481	2.068	1.696	1.415	1.196
0.3			2.585	2.114	1.726	1.405	1.136
0.4				2.284	1.786	1.446	1.130
0.5					2.037	1.573	1.184
0.6						1.815	1.316
0.7							1.580

for b/a=0.4, c/a=0.5 and 0.8 (Table 7); b/a=0.3 and c/a=0.4 (Table 12); and b/a=0.3 and c/a=0.4 (Table 16).

In the case of Table 16 the analytical approach yields a value of $N_0\, a^2/D_0$ which is, apparently, extremely high for b/a=0.5 and c/a=0.8.

The agreement between both sets of values is, in general, quite good for the remaining situations.

Table 7. Buckling coefficient (discontinuously varying thickness, $h_1/h_0=0.6$). External boundary simply supported. $\nu=0.3$ (1) : Optimized Rayleigh–Ritz approach. (2) : Finite elements method.

b/a		Values of c/a						
		0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.1	(1)	3.588	2.976	2.524	2.137	1.770	1.491	1.279
	(2)	3.553			2.221			1.324
0.2			2.944	2.459	2.031	1.663	1.395	1.190
0.3				2.554	2.058	1.667	1.357	1.105
0.4	(1)				2.224	1.717	1.384	1.083
	(2)				2.329			1.169
0.5						1.964	1.502	1.128
0.6							1.737	1.251
0.7	(1)							1.505
	(2)							1.638

Table 8. Buckling coefficient, discontinuously varying thickness. $h_1/h_0=0.6$. External boundary simply supported. $\nu=0.33$.

b/a	Values of c/a						
	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.1	3.622	2.984	2.540	2.132	1.766	1.492	1.283
0.2		2.939	2.442	2.012	1.648	1.385	1.185
0.3			2.533	2.032	1.643	1.338	1.091
0.4				2.196	1.689	1.360	1.065
0.5					1.933	1.475	1.106
0.6						1.706	1.226
0.7							1.475

Table 9. Buckling coefficient, discontinuously varying thickness. $h_1/h_0=0.6$. External boundary simply supported. $\nu=0.4$

b/a	Values of c/a						
	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.1	3.685	2.983	2.520	2.111	1.752	1.487	1.289
0.2		2.906	2.384	1.953	1.601	1.353	1.167
0.3			2.463	1.954	1.573	1.282	1.052
0.4				2.111	1.608	1.292	1.013
0.5					1.845	1.398	1.046
0.6						1.619	1.159
0.7							1.395

Table 10. Buckling coefficient, uniform thickness. External boundary clamped. (1): Optimized Rayleigh approach. (2): Finite elements method.

ν		Values of b/a				
		0.1	0.2	0.3	0.4	0.5
0.2		14.320	14.268	15.769	19.491	26.864
0.3	(1)	14.131	13.755	15.037	18.593	25.787
	(2)	13.868		14.889		25.560
0.33		14.067	13.586	14.800	18.308	25.450
0.4		13.902	13.159	14.214	17.611	24.638

Table 11. Buckling coefficient, discontinuously varying thickness. h_1/h_0=0.8. External boundary clamped. ν=0.2.

b/a	Values of c/a						
	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.1	13.303	12.300	11.692	11.195	10.684	10.070	9.299
0.2		13.194	12.423	11.773	11.099	10.327	9.464
0.3			14.865	14.078	13.337	12.392	11.093
0.4				18.661	17.756	16.729	14.859
0.5					26.005	24.952	22.506

Table 12. Buckling coefficient, discontinuously varying thickness. h_1/h_0=0.8. External boundary clamped. ν=0.3.(1): Optimized Rayleigh approach. (2): Finite elements method.

b/a		Values of c/a						
		0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.1	(1)	12.951	11.879	11.275	10.808	10.355	9.810	9.111
	(2)	12.972			11.127			8.901
0.2			12.572	11.788	11.162	10.536	9.819	9.042
0.3	(1)			14.084	13.304	12.598	11.712	10.525
	(2)			14.527				10.492
0.4					17.747	16.869	15.897	14.143
0.5	(1)					24.934	23.921	21.593
	(2)					24.003		19.173

Table 19 presents buckling coefficients for an annular plate which is simply supported at the outer boundary when the thickness varies linearly, Fig. 1b, while Table 20 deals with the case of an outer clamped edge.

It is important to emphasize the fact that Poisson's ratio has considerable weight upon the values of the buckling coefficient, specially when the outer edge is simply supported, i.e. in the case of Table 1 the buckling coefficient increases in about 10% when ν varies from 0.20 to 0.40 for b/a=0.1 while it decreases in 10% for b/a=0.7 for the same variation of ν.

Table 13. Buckling coefficient, discontinuously varying thickness. h_1/h_0=0.8. External boundary clamped. ν=0.33.

b/a	Values of c/a						
	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.1	12.835	11.743	11.143	10.685	10.250	9.726	9.049
0.2		12.369	11.585	10.967	10.357	9.656	8.904
0.3			13.834	13.061	12.366	11.498	10.345
0.4				17.460	16.594	15.638	13.919
0.5					24.602	23.604	21.313

Table 14. Buckling coefficient, discontinuously varying thickness. h_1/h_0=0.8. External boundary clamped. ν=0.4.

b/a	Values of c/a						
	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.1	12.821	11.404	10.816	10.387	9.992	9.515	8.891
0.2		11.863	11.085	10.493	9.917	9.319	8.659
0.3			13.224	12.470	11.805	10.980	9.902
0.4				16.769	15.934	15.019	13.380
0.5					23.864	22.851	20.644

Table 15. Buckling coefficient, discontinuously varying thickness. h_1/h_0=0.6. External boundary clamped. ν=0.2.

b/a	Values of c/a						
	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.1	12.665	10.683	9.528	8.450	7.240	6.137	5.097
0.2		12.364	11.016	9.703	8.086	6.546	5.316
0.3			14.231	12.855	11.231	9.023	6.813
0.4				18.096	16.415	14.004	10.351
0.5					25.464	23.280	17.959

The proposed approach is also applicable when the outer edge is elastically restrained against rotation. In this case condition 4(b) is replaced by

$$\frac{dW}{dr}(1) = -\varnothing D_o[W''(1) + \nu W'(1)] \quad (10)$$

where $\varnothing$: edge flexibility coefficient. When $\varnothing \to 0$ the edge is rigidly clamped and when $\varnothing \to \infty$ one has the simply supported condition.

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Table 16. Buckling coefficient, discontinuously varying thickness. $h_1/h_0=0.6$. External boundary clamped. $\nu=0.3$. (1): Rayleigh–Ritz approach. (2): Finite elements method.

b/a		Values of c/a						
		0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.1	(1)	12.201	10.094	8.970	7.977	6.872	5.863	4.945
	(2)	12.325			8.527			4.536
0.2			11.664	10.303	9.048	7.545	6.111	5.005
0.3	(1)			13.421	12.066	10.524	8.438	6.387
	(2)			14.647				5.852
0.4					17.175	15.551	13.253	9.801
0.5	(1)					24.398	22.300	17.234
	(2)					22.269		12.111

Table 17. Buckling coefficient, discontinuously varying thickness. $h_1/h_0=0.6$. External boundary clamped. $\nu=0.33$.

b/a	Values of c/a						
	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.1	12.047	9.908	8.798	7.832	6.759	5.777	4.894
0.2		11.439	10.081	8.847	7.379	5.977	4.905
0.3			13.166	11.823	10.309	8.259	6.253
0.4				16.889	15.288	13.025	9.629
0.5					24.118	22.003	17.013

Table 18. Buckling coefficient, discontinuously varying thickness. $h_1/h_0=0.6$. External boundary clamped. $\nu=0.4$

b/a	Values of c/a						
	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.1	11.656	9.457	8.389	7.490	6.493	5.568	4.764
0.2		10.885	9.545	8.368	6.983	5.660	4.660
0.3			12.547	11.243	9.798	7.837	5.930
0.4				16.205	14.665	12.488	9.216
0.5					23.339	21.304	16.491

Table 19. Buckling coefficient, continuous varying thickness. External boundary simply supported.

	Values of b/a						
	0.1	0.2	0.3	0.4	0.5	0.6	0.7
h_1/h_0							ν=0.2
0.8	2.737	2.542	2.177	1.968	1.812	1.694	1.601
0.6	1.885	1.660	1.480	1.354	1.263	1.196	1.145
							ν=0.3
0.8	2.860	2.502	2.169	1.929	1.757	1.630	1.533
0.6	1.927	1.657	1.450	1.310	1.213	1.143	1.090
							ν=0.33
0.8	2.892	2.509	2.158	1.909	1.733	1.604	1.506
0.6	1.936	1.651	1.435	1.291	1.193	1.122	1.069
							ν=0.4
0.8	2.962	2.512	2.116	1.847	1.662	1.529	1.429
0.6	1.950	1.626	1.388	1.236	1.135	1.064	1.012

Table 20. Buckling coefficient, continuous varying thickness. External boundary clamped.

	Values of b/a				
	0.1	0.2	0.3	0.4	0.5
h_1/h_0					ν=0.2
0.8	10.710	10.842	12.308	15.662	22.159
0.6	7.524	7.744	9.184	12.150	17.680
					ν=0.3
0.8	10.419	10.345	11.667	14.924	21.260
0.6	7.191	7.295	8.655	11.541	16.962
					ν=0.33
0.8	10.325	10.185	11.464	14.687	20.982
0.6	7.086	7.154	8.490	11.353	16.724
					ν=0.4
0.8	10.091	9.787	10.967	14.115	20.318
0.6	6.829	6.812	8.095	10.907	16.223

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